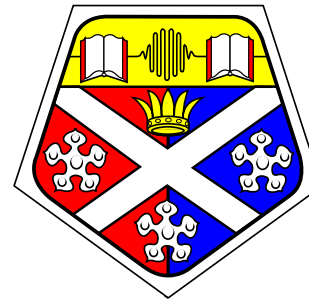


Multigrid Solution of Discrete Convection-Diffusion Equations

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Overview

- background
 - convection-diffusion problems
 - multigrid methods
- practical multigrid issues
 - **approximation** and **smoothing** properties
 - convergence analysis
- model problem Fourier analysis
 - matrix transformation
 - comparison with **semiperiodic** problem
 - implications for **Dirichlet** problems

Convection-Diffusion in 2D

$$\begin{aligned} -\epsilon \nabla^2 u(x, y) + \mathbf{w} \cdot \nabla u(x, y) &= f(x, y) \quad \text{in } \Omega \in \mathbb{R}^2 \\ u(x, y) &= g \quad \text{on } \partial\Omega \end{aligned}$$

divergence-free convective velocity ('wind') \mathbf{w}

diffusion parameter $\epsilon \ll 1$

discretisation parameter h

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mesh Péclet number $P_h = \frac{\|\mathbf{w}\| h}{2\epsilon}$

Boundary Layers and Oscillations

- Galerkin finite element method

$$\epsilon(\nabla u_h, \nabla v_h) + (\mathbf{w} \cdot \nabla u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

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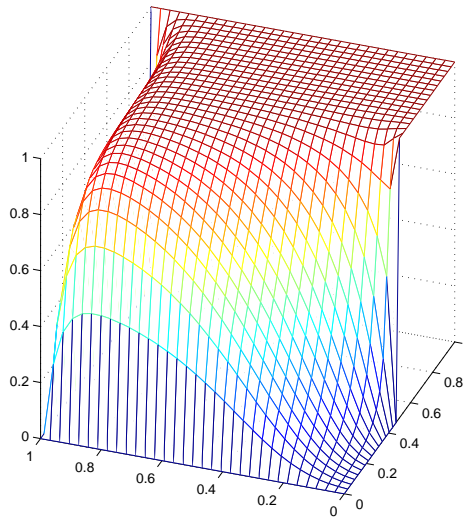
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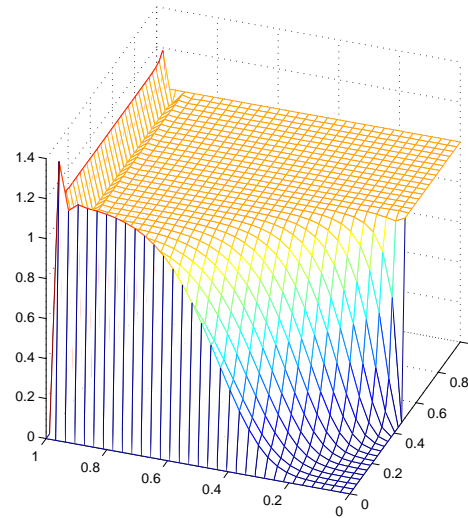
- solution features:

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- oscillations observed in discrete solutions for $P_h > 1$



$P_e = 0.5$



$P_e = 2$

Streamline Diffusion Method

streamline diffusion FEM, square bilinear elements

$$\begin{aligned} \epsilon(\nabla u_h, \nabla v_h) &+ (\mathbf{w} \cdot \nabla u_h, v_h) + \frac{\delta h}{\|\mathbf{w}\|} (\mathbf{w} \cdot \nabla u_h, \mathbf{w} \cdot \nabla v_h) \\ &= (f, v_h) + \frac{\delta h}{\|\mathbf{w}\|} (f, \mathbf{w} \cdot \nabla v_h) \quad \forall v_h \in V_h \end{aligned}$$

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- $P_h \leq 1$: $\delta = 0$

Galerkin FEM

- $P_h > 1$: $\delta = \frac{1}{2} - \frac{\epsilon}{h}$

Streamline Diffusion

Multigrid Ideas

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- recursive process on nested grids
- **optimal** in the sense of obtaining convergence rate independent of h

Issues for Convection-Diffusion

- **approximation**: choice of discretisation
 - oscillations on coarser grids?
 - grid transfer operators?

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- multigrid can be implemented effectively for convection-diffusion problems

Convergence Analysis

- standard Poisson-type convergence analysis fails
- ideas for convection-diffusion less well-developed

Convergence Analysis

- standard Poisson-type convergence analysis fails
- ideas for convection-diffusion less well-developed
- various successful approaches
 - **perturbation arguments**
Bank (1981), Bramble, Pasciak and Xu (1988), Mandel (1986), Wang (1993)
 - **matrix-based methods**
Reusken (2002), Olishanskii and Reusken (2002)

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direct discretisation on coarse grid

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- restriction: transpose of prolongation P^T

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- restriction: transpose of prolongation P^T
- smoothing: line Gauss-Seidel S_A
- ν steps of pre-smoothing, no post-smoothing

Multigrid Convergence

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- error equation $\mathbf{e}_k = M\mathbf{e}_{k-1} = M^k\mathbf{e}_0$
- convergence?

$$\|\mathbf{e}_k\| \leq \|M\|^k \|\mathbf{e}_0\|$$

convergence if $\|M\| < 1$

Two-Grid Convergence Analysis

AIM: find an upper bound for

$$\|M\|_2 = \|(I - PA_c^{-1}P^T A_f)S_A^\nu\|_2$$

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- Approach 1: write

$$M = (A_f^{-1} - PA_c^{-1}P^T)(A_f S_A^\nu) = M_A M_S$$

and bound $\|M_A\|_2$, $\|M_S\|_2$ separately

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- Approach 2: bound $\|M\|_2$ directly

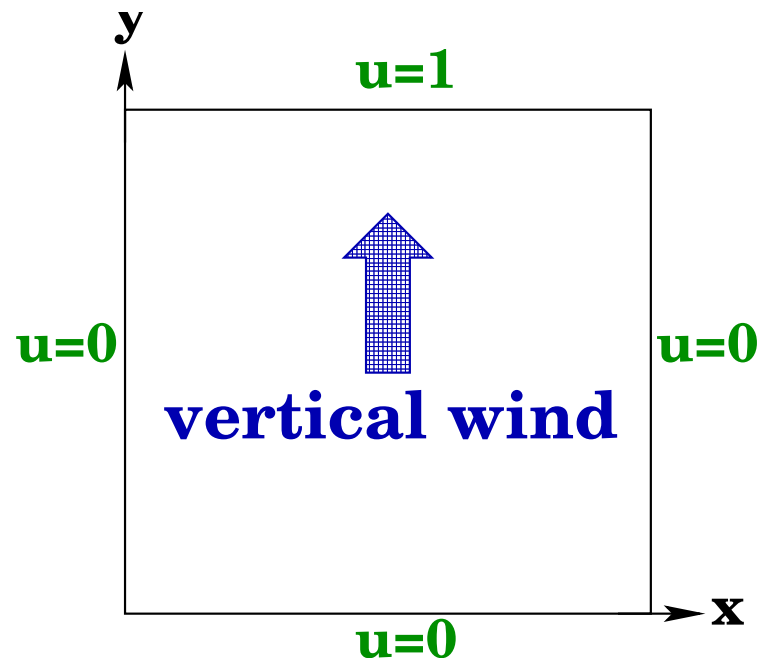
Model Problem

grid-aligned flow with vertical wind and $f = 0$

$$-\epsilon \nabla^2 u(x, y) + (0, 1) \cdot \nabla u(x, y) = 0$$

Dirichlet boundary conditions

square bilinear elements



Computational Molecule

parameters h, ϵ, δ

$$M_2 : \quad -\frac{1}{12}[(2\delta-1)h+4\epsilon] \quad -\frac{1}{3}[(2\delta-1)h+\epsilon] \quad -\frac{1}{12}[(2\delta-1)h+4\epsilon]$$



$$M_1 : \quad \frac{1}{3}(\delta h - \epsilon) \quad \leftarrow \quad \frac{4}{3}(\delta h + 2\epsilon) \quad \rightarrow \quad \frac{1}{3}(\delta h - \epsilon)$$



$$M_3 : \quad -\frac{1}{12}[(2\delta+1)h+4\epsilon] \quad -\frac{1}{3}[(2\delta+1)h+\epsilon] \quad -\frac{1}{12}[(2\delta+1)h+4\epsilon]$$

symmetric stencil

Coefficient Matrix

$$A = \begin{bmatrix} M_1 & M_2 & & & 0 \\ M_3 & M_1 & M_2 & & \\ & \ddots & \ddots & \ddots & \\ & & M_3 & M_1 & M_2 \\ 0 & & & M_3 & M_1 \end{bmatrix}$$

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eigenvectors and eigenvalues:

$$\begin{aligned} M_1 \mathbf{v}_j &= \lambda_j \mathbf{v}_j, & \lambda_j &= m_{1c} + 2m_{1r} \cos \frac{j\pi}{N} \\ M_2 \mathbf{v}_j &= \sigma_j \mathbf{v}_j, & \sigma_j &= m_{2c} + 2m_{2r} \cos \frac{j\pi}{N} \\ M_3 \mathbf{v}_j &= \gamma_j \mathbf{v}_j, & \gamma_j &= m_{3c} + 2m_{3r} \cos \frac{j\pi}{N} \end{aligned}$$

$$\mathbf{v}_j = \sqrt{\frac{2}{N}} \left[\sin \frac{j\pi}{N}, \quad \sin \frac{2j\pi}{N}, \quad \dots, \quad \sin \frac{(N-1)j\pi}{N} \right]^T$$

Transformation: Coefficient Matrix (1)

N_f^2 elements, n_f^2 unknowns ($n_f = N_f - 1$)

$$\hat{V}_f = [\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_{n_f}], \quad V_f = \text{diag}(\hat{V}_f, \dots, \hat{V}_f)$$

$$M_1 \hat{V}_f = \hat{V}_f \Lambda, \quad M_2 \hat{V}_f = \hat{V}_f \Sigma, \quad M_3 \hat{V}_f = \hat{V}_f \Gamma$$

$$V_f^T A_f V_f = \hat{T}_f = \begin{bmatrix} \Lambda & \Sigma & & & 0 \\ \Gamma & \Lambda & \Sigma & & \\ & \ddots & \ddots & \ddots & \\ & & \Gamma & \Lambda & \Sigma \\ 0 & & & \Gamma & \Lambda \end{bmatrix}$$

Transformation: Coefficient Matrix (2)

permute into tridiagonal form:

$$\Pi_f^T \hat{T}_f \Pi_f = T_f = \begin{bmatrix} T_1 & & & & 0 \\ & T_2 & & & \\ & & \ddots & & \\ & & & T_{n_f-1} & \\ 0 & & & & T_{n_f} \end{bmatrix}$$

$$T_j = \text{tridiag}(\gamma_j, \lambda_j, \sigma_j)$$

$$A_f = Q_f T_f Q_f^T \quad Q_f = V_f \Pi_f$$

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$$\text{coarse grid: } A_c = Q_c T_c Q_c^T \quad Q_c = V_c \Pi_c$$

Transformation: Smoothing Matrix

block matrix splitting: $A_f = D_A - L_A - U_A$

Gauss-Seidel smoothing matrix:

$$S_A = (D_A - L_A)^{-1}U_A = I - (D_A - L_A)^{-1}A_f$$

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transformation:

$$S_A = Q_f S_T Q_f^T$$

where $S_T = I - (D_T - L_T)^{-1}T_f$ is block-diagonal

Transformation: Iteration Matrix (1)

$$\begin{aligned}M &= (I - PA_c^{-1}P^T A_f)S_A^\nu \\&= (I - PQ_c T_c^{-1}Q_c^T P^T Q_f T_f Q_f^T)S_A^\nu \\&= Q_f(I - \bar{P}T_c^{-1}\bar{P}^T T_f)Q_f^T(Q_f S_T Q_f^T)^\nu \\&= Q_f \left(I - \bar{P}T_c^{-1}\bar{P}^T T_f \right) S_T^\nu Q_f^T\end{aligned}$$

$$\Rightarrow M = Q_f \bar{M} Q_f^T$$

where $\bar{M} = (I - \bar{P}T_c^{-1}\bar{P}^T T_f) S_T^\nu$

Transformation: Iteration Matrix (1)

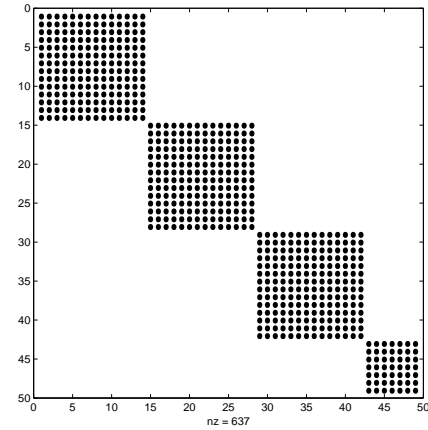
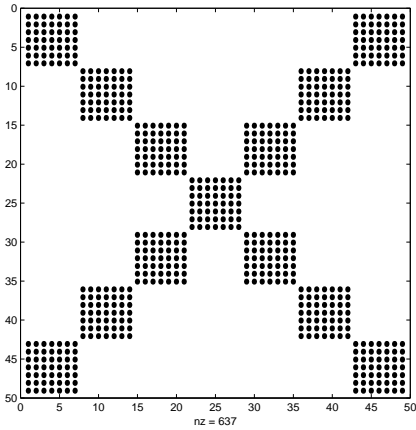
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where $\bar{M} = (I - \bar{P}T_c^{-1}\bar{P}^T T_f) S_T^\nu$

Q_f is orthogonal:

$$\|M\|_2 = \|\bar{M}\|_2$$

Transformed Iteration Matrix (2)

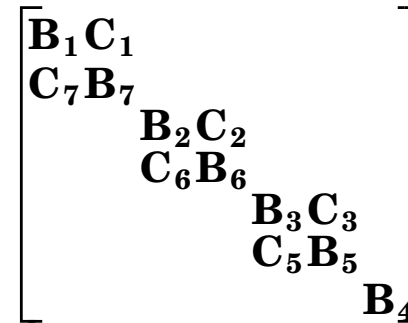
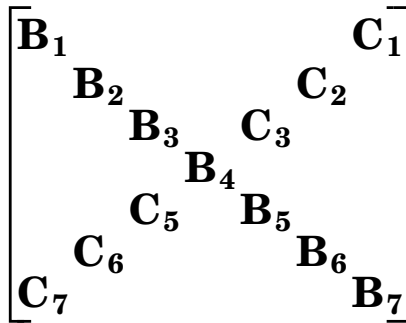
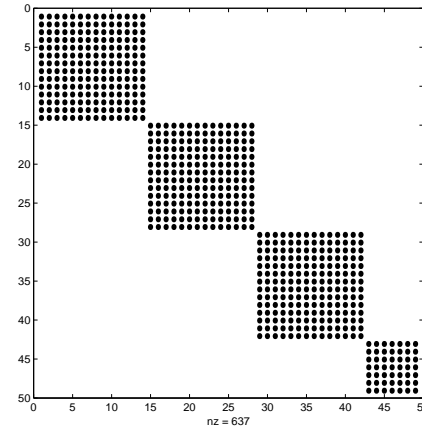
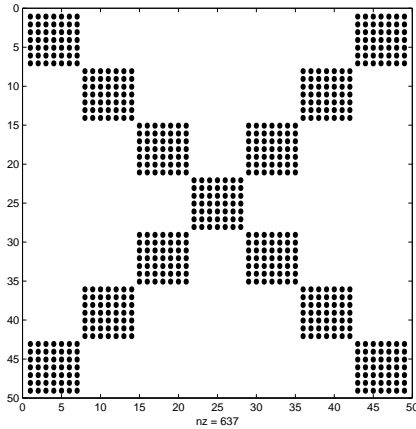


$$\begin{bmatrix} B_1 & & & & & & C_1 \\ & B_2 & & & & & \\ & & B_3 & & & & C_2 \\ & & & B_4 & & & C_3 \\ & & & & B_5 & & \\ & & & & & B_6 & \\ C_7 & & & & & & B_7 \end{bmatrix}$$



$$\begin{bmatrix} B_1 C_1 & & & & & & \\ C_7 B_7 & & & & & & \\ & B_2 C_2 & & & & & \\ & & C_6 B_6 & & & & \\ & & & B_3 C_3 & & & \\ & & & & C_5 B_5 & & \\ & & & & & B_4 & \end{bmatrix}$$

Transformed Iteration Matrix (2)



$$\|\bar{M}\|_2 = \max \left\{ \max_{j=1, \dots, n_c} \left\| \begin{bmatrix} B_j & C_j \\ C_k & B_k \end{bmatrix} \right\|_2, \|B_{N_c}\|_2 \right\}, \quad k = N_f - j$$

The Story So Far...

- $n_f^2 \times n_f^2$ two-grid iteration matrix M
- Fourier transformation converts 2D problem to a set of n_f problems with 1D structure
- $\|M\|_2$ can be found from norms of N_c smaller problems

n_c of size $2n_f \times 2n_f$, 1 of size $n_f \times n_f$

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- **IDEA:** analyse semiperiodic version of the problem
 n_c of size $2N_f \times 2N_f$, 1 of size $N_f \times N_f$
- gain insight into Dirichlet problem behaviour?

Semiperiodic problem

- B_j, C_j are replaced by periodic versions, e.g.

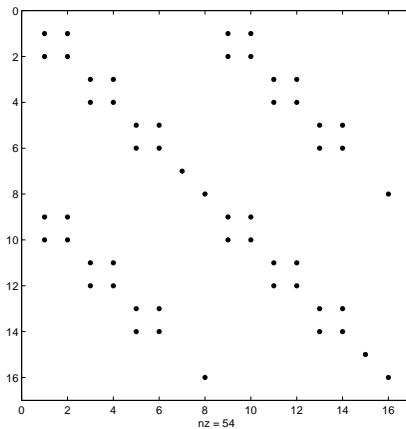
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- B_j^{per}, C_j^{per} become block diagonal with 2×2 blocks

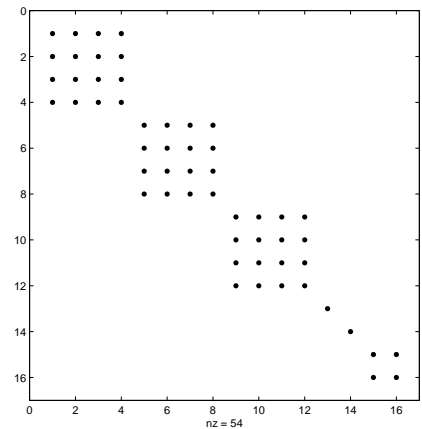
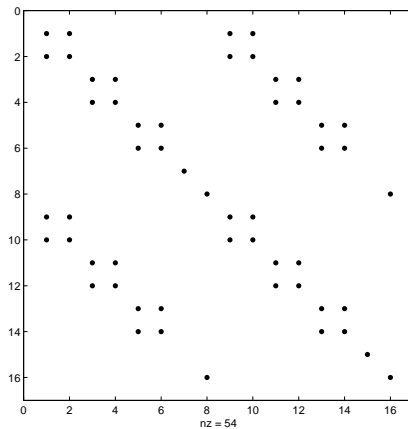


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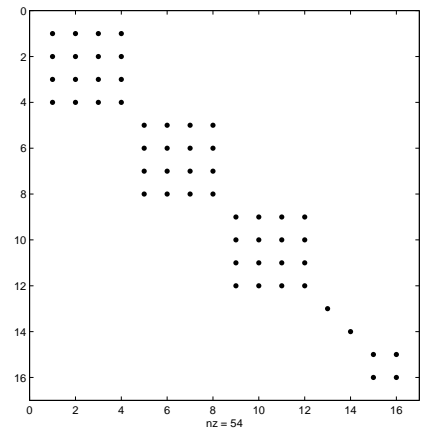
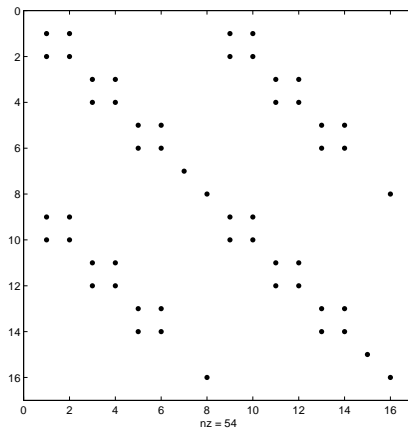


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- 2-norm given by maximum 2-norm of the 4×4 blocks

Analytic result

- with semiperiodic approximation, when $P_h > 1$

$$\|M^{per}\|_2 = \frac{\sqrt{3 + \cos(2\pi h)}}{\sqrt{2}(5^\nu)}$$

independent of ϵ

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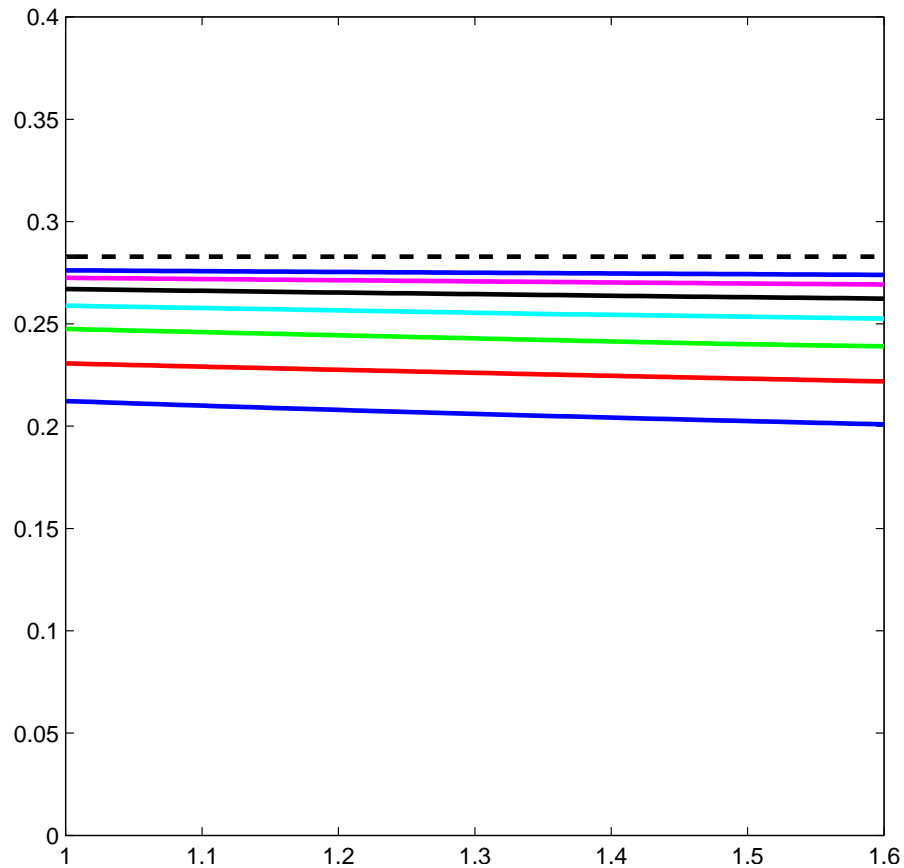
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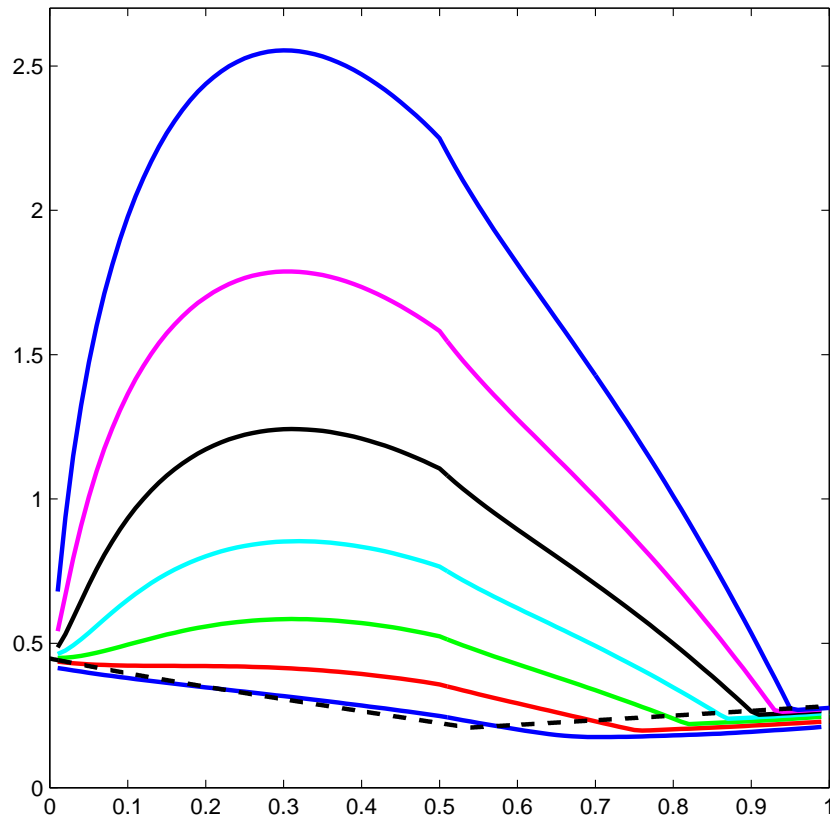
Question: Does this **semiperiodic** analysis correctly predict **Dirichlet** problem behaviour?

Model Problem Results (1)



- $\|M\|_2$ vs P_h
- $P_h \geq 1$ only
- semiperiodic: dashed line
- Dirichlet: solid lines
- h fixed for each line
- $h = \frac{1}{8}$ to $h = \frac{1}{512}$
- $\nu = 1$
- semiperiodic: $\frac{\sqrt{2}}{5} \simeq 0.28$
- Dirichlet $\rightarrow \frac{\sqrt{2}}{5}$

Model Problem Results (2)



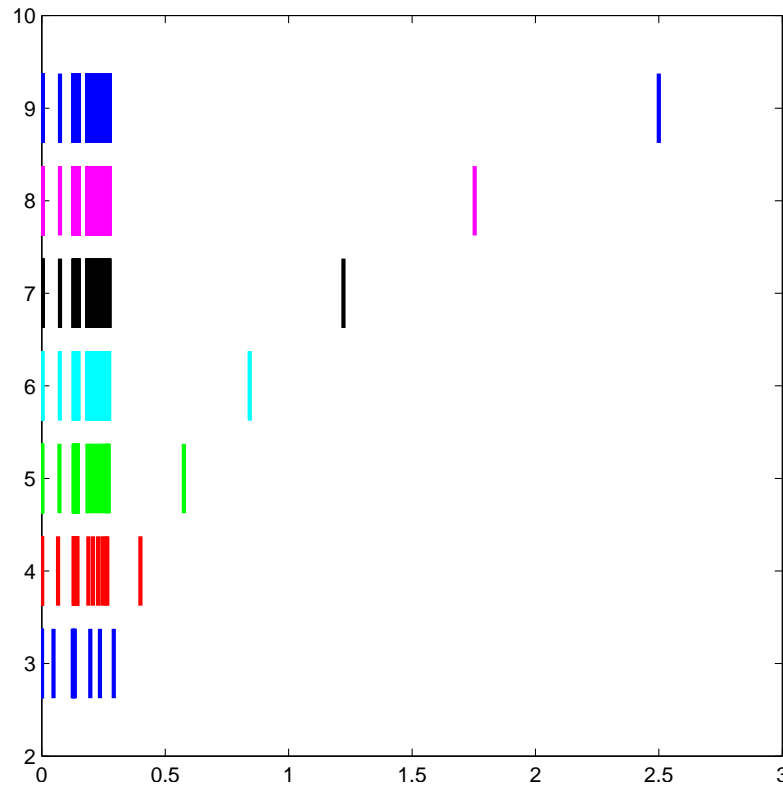
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- $\nu = 1$
- not a good match
- MG may diverge!

Observations

- $\|M\|_2 = \sqrt{|\lambda_1(M^*M)|}$

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- for $P_h < 1$, matrix blocks have one 'bad' eigenvalue



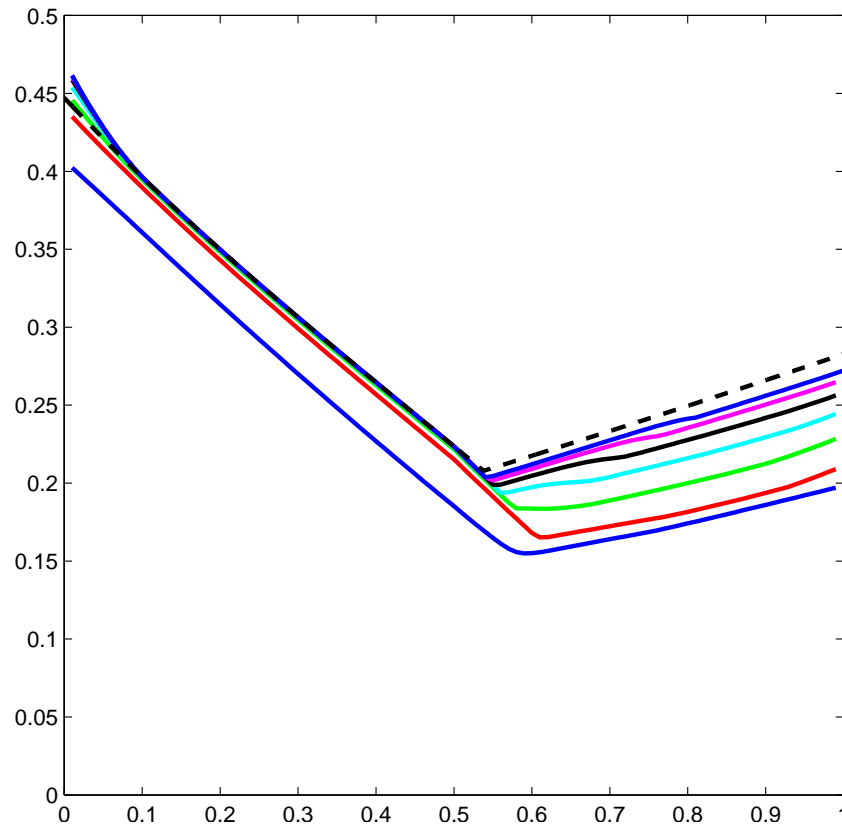
$$\sqrt{|\lambda_1(\mathcal{M}_1^* \mathcal{M}_1)|} \text{ for fixed } P_h = 0.38$$

Alternative Bound?

- artificially 'remove' this eigenvalue: use $\sqrt{|\lambda_2(M^*M)|}$

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- $P_h < 1$ only
- semiperiodic: $\|M^{per}\|_2$
- Dirichlet: $\sqrt{|\lambda_2(M^*M)|}$

MG Iteration Counts

- in practice, effect of outlying eigenvalue is **transient**

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| | ϵ | | | | | | | | | | |
|-----------------|---------------|---------------|---------------|----------------|----------------|----------------|-----------------|-----------------|-----------------|------------------|------------------|
| h | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ | $\frac{1}{64}$ | $\frac{1}{128}$ | $\frac{1}{256}$ | $\frac{1}{512}$ | $\frac{1}{1024}$ | $\frac{1}{2048}$ |
| $\frac{1}{4}$ | 5 | 5 | 5 | 5 | 5 | 4 | 4 | 3 | 2 | 2 | 2 |
| $\frac{1}{8}$ | 7 | 7 | 6 | 6 | 5 | 5 | 4 | 4 | 3 | 2 | 2 |
| $\frac{1}{16}$ | 7 | 7 | 7 | 6 | 5 | 5 | 5 | 4 | 4 | 3 | 2 |
| $\frac{1}{32}$ | 7 | 7 | 7 | 7 | 6 | 5 | 5 | 4 | 4 | 3 | 3 |
| $\frac{1}{64}$ | 7 | 7 | 7 | 7 | 6 | 5 | 5 | 4 | 4 | 4 | 3 |
| $\frac{1}{128}$ | 7 | 6 | 6 | 6 | 6 | 6 | 5 | 4 | 4 | 4 | 3 |

$P_h < 1$

$P_h \geq 1$

MG-like convergence for any value of P_h

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- Replacing the Dirichlet condition by a Neumann condition on the outflow boundary leads to similar computational results.