Some Characteristics of Multigrid Performance for the Two-Dimensional Convection-Diffusion Equation

Alison Ramage
Dept of Mathematics
University of Strathclyde
Glasgow, Scotland

UNIVERSITY OF STRATHCLYDE

UNIVERSITY OF MARYLAND

Howard Elman
Dept of Computer Science
University of Maryland
College Park, MD, USA

Convection-Diffusion in 2D

$$
\begin{aligned}
-\epsilon\nabla^2 u(x,y) + \mathbf{w}.\nabla u(x,y) &= f(x,y) \quad \text{in} \quad \Omega \in \mathbb{R}^2 \\
u(x,y) &= g \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

convective velocity ('wind') $\mathbf{w}$

diffusion parameter $\epsilon << 1$

discretisation parameter h

mesh Péclet number $P_h = \frac{\|\mathbf{w}\| h}{2\epsilon}$

Streamline Diffusion Method

streamline diffusion FEM, square bilinear elements

$$\begin{aligned}\epsilon(\nabla u_h, \nabla v_h) \quad &+ \quad (\mathbf{w}\cdot\nabla u_h, v_h) + \frac{\delta h}{||\mathbf{w}||}(\mathbf{w}\cdot\nabla u_h, \mathbf{w}\cdot\nabla v_h) \\ &= \quad (f, v_h) + \frac{\delta h}{||\mathbf{w}||}(f, \mathbf{w}\cdot\nabla v_h) \quad \forall v_h \in V_h\end{aligned}$$

- $P_h \leq 1: \quad \delta = 0$ Galerkin FEM
- $P_h > 1: \quad \delta = \frac{1}{2} - \frac{\epsilon}{h}$

Model Problem

grid-aligned flow with vertical wind and $f = 0$

$$-\epsilon\nabla^2 u(x,y) + (0,1).\nabla u(x,y) = 0$$

Dirichlet boundary conditions

computational molecule:

M_2 :	$-\frac{1}{12}[(2\delta-1)h+4\epsilon]$		$-\frac{1}{3}[(2\delta-1)h+\epsilon]$		$-\frac{1}{12}[(2\delta-1)h+4\epsilon]$
		$\nwarrow$	$\uparrow$	$\nearrow$	
M_1 :	$\frac{1}{3}(\delta h-\epsilon)$	$\leftarrow$	$\frac{4}{3}(\delta h+2\epsilon)$	$\rightarrow$	$\frac{1}{3}(\delta h-\epsilon)$
		$\swarrow$	$\downarrow$	$\searrow$	
M_3 :	$-\frac{1}{12}[(2\delta+1)h+4\epsilon]$		$-\frac{1}{3}[(2\delta+1)h+\epsilon]$		$-\frac{1}{12}[(2\delta+1)h+4\epsilon]$

Coefficient Matrix

$$A = \begin{bmatrix} M_1 & M_2 & & & 0 \\ M_3 & M_1 & M_2 & & \\ & \ddots & \ddots & \ddots & \\ & & M_3 & M_1 & M_2 \\ 0 & & & M_3 & M_1 \end{bmatrix}$$

eigenvectors and eigenvalues:

$$\begin{aligned} M_1 \mathbf{v}_j &= \lambda_j \mathbf{v}_j, \quad \lambda_j = m_{1c} + 2m_{1r} \cos \tfrac{j\pi}{N} \\ M_2 \mathbf{v}_j &= \sigma_j \mathbf{v}_j, \quad \sigma_j = m_{2c} + 2m_{2r} \cos \tfrac{j\pi}{N} \\ M_3 \mathbf{v}_j &= \gamma_j \mathbf{v}_j, \quad \gamma_j = m_{3c} + 2m_{3r} \cos \tfrac{j\pi}{N} \end{aligned}$$

$$\mathbf{v}_j = \sqrt{\frac{2}{N}} \left[\sin \frac{j\pi}{N}, \quad \sin \frac{2j\pi}{N}, \quad \ldots, \sin \frac{(N-1)j\pi}{N} \right]^T$$

Multigrid Method

- two-grid method

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- coefficient matrices: A_f (fine grid), A_c (coarse grid)

direct discretisation on coarse grid

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- restriction: transpose of prolongation P^T

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- ν steps of pre-smoothing, no post-smoothing

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- smoothing: line Gauss-Seidel S_A
- ν steps of pre-smoothing, no post-smoothing
- two-grid iteration matrix $M = (I - PA_c^{-1}P^TA_f)S_A^\nu$
- error equation

$$\mathbf{e}_k = M\mathbf{e}_{k-1} = M^k\mathbf{e}_0$$

Two-Grid Convergence Analysis

AIM: find an upper bound for

$$\|M\|_2 = \|(I - PA_c^{-1}P^TA_f)S_A^{\nu}\|_2$$

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$$\|M\|_2 = \|(I - PA_c^{-1}P^TA_f)S_A^{\nu}\|_2$$

- Approach 1: bound $\|M\|_2$ directly
- Approach 2: write

$$M = (A_f^{-1} - PA_c^{-1}P^T)(A_fS_A^{\nu}) = M_AM_S$$

and bound $\|M_A\|_2$, $\|M_S\|_2$ separately

Transformation: Coefficient Matrix (1)

N_f^2 elements, n_f^2 unknowns $(n_f = N_f - 1)$

$$\hat{V}_f = \left[\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_{n_f}\right], \qquad V_f = \texttt{diag}(\hat{V}_f, \ldots, \hat{V}_f)$$

$$M_1 \hat{V}_f = \hat{V}_f \Lambda, \qquad M_2 \hat{V}_f = \hat{V}_f \Sigma, \qquad M_3 \hat{V}_f = \hat{V}_f \Gamma$$

$$V_f^T A_f V_f = \hat{T}_f = \begin{bmatrix} \Lambda & \Sigma & & & 0 \\ \Gamma & \Lambda & \Sigma & & \\ & \ddots & \ddots & \ddots & \\ & & \Gamma & \Lambda & \Sigma \\ 0 & & & \Gamma & \Lambda \end{bmatrix}$$

Transformation: Coefficient Matrix (2)

permute into tridiagonal form:

$$\Pi_f^T \hat{T}_f \Pi_f = T_f = \begin{bmatrix} T_1 & & & & 0 \\ & T_2 & & & \\ & & \ddots & & \\ & & & T_{n_f-1} & \\ 0 & & & & T_{n_f} \end{bmatrix}$$

$$T_j = \mathtt{tridiag}(\gamma_j, \lambda_j, \sigma_j)$$

$$A_f = Q_f T_f Q_f^T \qquad Q_f = V_f \Pi_f$$

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coarse grid: $A_c = Q_c T_c Q_c^T \qquad Q_c = V_c \Pi_c$

Transformation: Smoothing Matrix

block matrix splitting: $A_f = D_A - L_A - U_A$

Gauss-Seidel smoothing matrix:

$$S_A = (D_A - L_A)^{-1} U_A = I - (D_A - L_A)^{-1} A_f$$

transformation:

$$S_A = Q_f S_T Q_f^T$$

where $S_T = I - (D_T - L_T)^{-1} T_f$ is block-diagonal

Transformation: Prolongation Matrix

2D prolongation matrix: $P = L \otimes L$

$$L^T = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & & & & & \\ & & \frac{1}{2} & 1 & \frac{1}{2} & & & \\ & & & & & \ddots & & \\ & & & & & \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

transformation: $Q_f = (I_f \otimes \hat{V}_f)\Pi_f, \quad Q_c = (I_c \otimes \hat{V}_c)\Pi_c$

$$\bar{P} = Q_f^T P Q_c = \mathcal{A}^T \otimes L$$

$$\mathcal{A} = \begin{bmatrix} \alpha_1 & & & & & & \alpha_{N_f-1} \\ & \alpha_2 & & & & \alpha_{N_f-2} & \\ & & \ddots & & \iddots & & \\ & & \alpha_{n_c} & 0 & \alpha_{N_f-n_c} & & \end{bmatrix}$$

Transformation: Iteration Matrix (1)

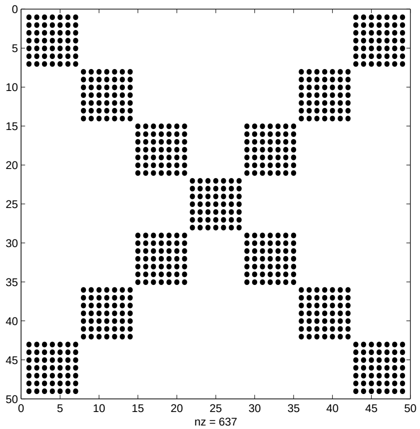
$$
\begin{aligned}
M &= (I - PA_c^{-1}P^TA_f)S_A^\nu \\
&= (I - PQ_cT_c^{-1}Q_c^TP^TQ_fT_fQ_f^T)S_A^\nu \\
&= Q_f(I - \bar{P}T_c^{-1}\bar{P}^TT_f)Q_f^T(Q_fS_TQ_f^T)^\nu \\
&= Q_f\left(I - \bar{P}T_c^{-1}\bar{P}^TT_f\right)S_T^\nu Q_f^T \\
\Rightarrow M &= Q_f\bar{M}Q_f^T
\end{aligned}
$$

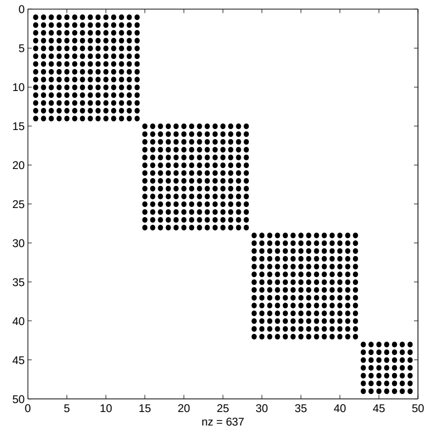
where $\bar{M} = \left(I - \bar{P}T_c^{-1}\bar{P}^TT_f\right)S_T^\nu$

Q_f is orthogonal:

$$\|M\|_2 = \|\bar{M}\|_2$$

Transformed Iteration Matrix (2)





$$\begin{bmatrix} \mathbf{B_1} & & & & & & \mathbf{C_1} \\ & \mathbf{B_2} & & & & \mathbf{C_2} & \\ & & \mathbf{B_3} & & \mathbf{C_3} & & \\ & & & \mathbf{B_4} & & & \\ & & \mathbf{C_5} & & \mathbf{B_5} & & \\ & \mathbf{C_6} & & & & \mathbf{B_6} & \\ \mathbf{C_7} & & & & & & \mathbf{B_7} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{B_1} & \mathbf{C_1} & & & & & \\ \mathbf{C_7} & \mathbf{B_7} & & & & & \\ & & \mathbf{B_2} & \mathbf{C_2} & & & \\ & & \mathbf{C_6} & \mathbf{B_6} & & & \\ & & & & \mathbf{B_3} & \mathbf{C_3} & \\ & & & & \mathbf{C_5} & \mathbf{B_5} & \\ & & & & & & \mathbf{B_4} \end{bmatrix}$$

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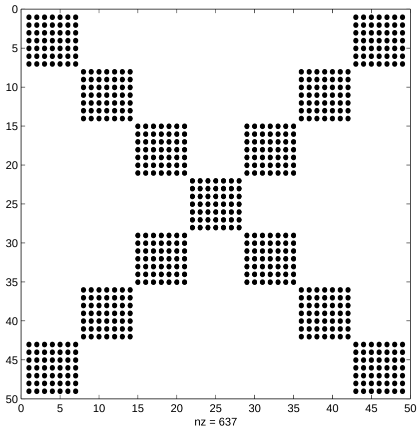

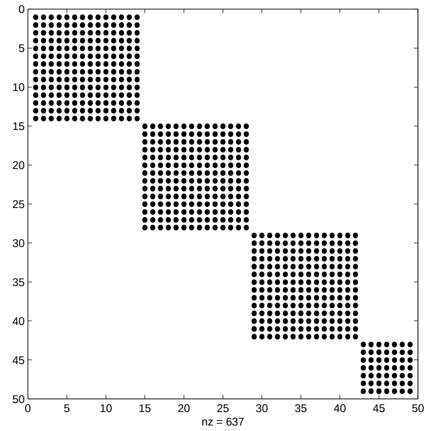

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$$\|\bar{M}\|_2 = \max\left\{ \max_{j=1,\dots,n_c} \left\| \begin{bmatrix} B_j & C_j \\ C_k & B_k \end{bmatrix} \right\|_2, \|B_{N_c}\|_2 \right\}, \quad k = N_f - j$$

The Story So Far...

- $n_f^2 \times n_f^2$ two-grid iteration matrix M

- Fourier transformation converts 2D problem to a set of n_f problems with 1D structure

- $\|M\|_2$ can be found from norms of N_c smaller problems

  n_c of size $2n_f \times 2n_f$, 1 of size $n_f \times n_f$

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- IDEA: analyse periodic versions of these new problems

  n_c of size $2N_f \times 2N_f$, 1 of size $N_f \times N_f$

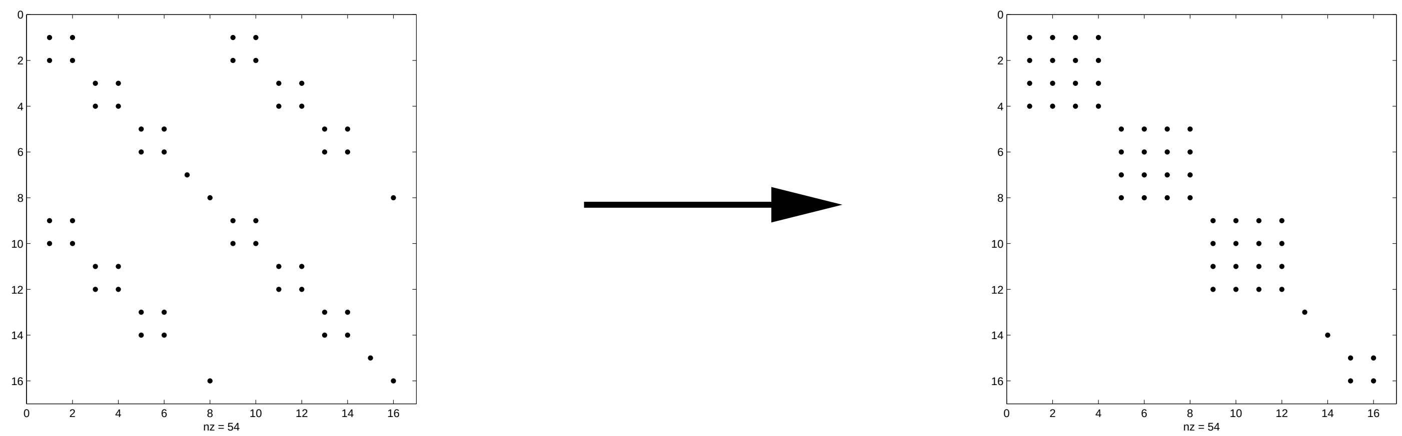
- gain insight into Dirichlet problem behaviour?

Periodic version

- replace B_j, C_j by periodic versions, e.g.

$$B_j^{per} = [I - \bar{P}_j^{per} (T_c^{per})_j^{-1} (\bar{P}_j^{per})^T (T_f^{per})_j]\, S_j^{per}$$

- transform using coarse grid periodic eigenvectors
- each B_j, C_j becomes block diagonal with 2×2 blocks
- permute into block diagonal form



- 2-norm given by maximum 2-norm of the 4×4 blocks

Analytic result

- with periodic approximation, when $P_h > 1$

$$\|M^{per}\|_2 = \frac{\sqrt{3 + \cos(2\pi h)}}{\sqrt{2}(5^{\nu})}$$

independent of ϵ

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$$\|M^{per}\|_2 = \frac{\sqrt{3 + \cos(2\pi h)}}{\sqrt{2}(5^\nu)}$$

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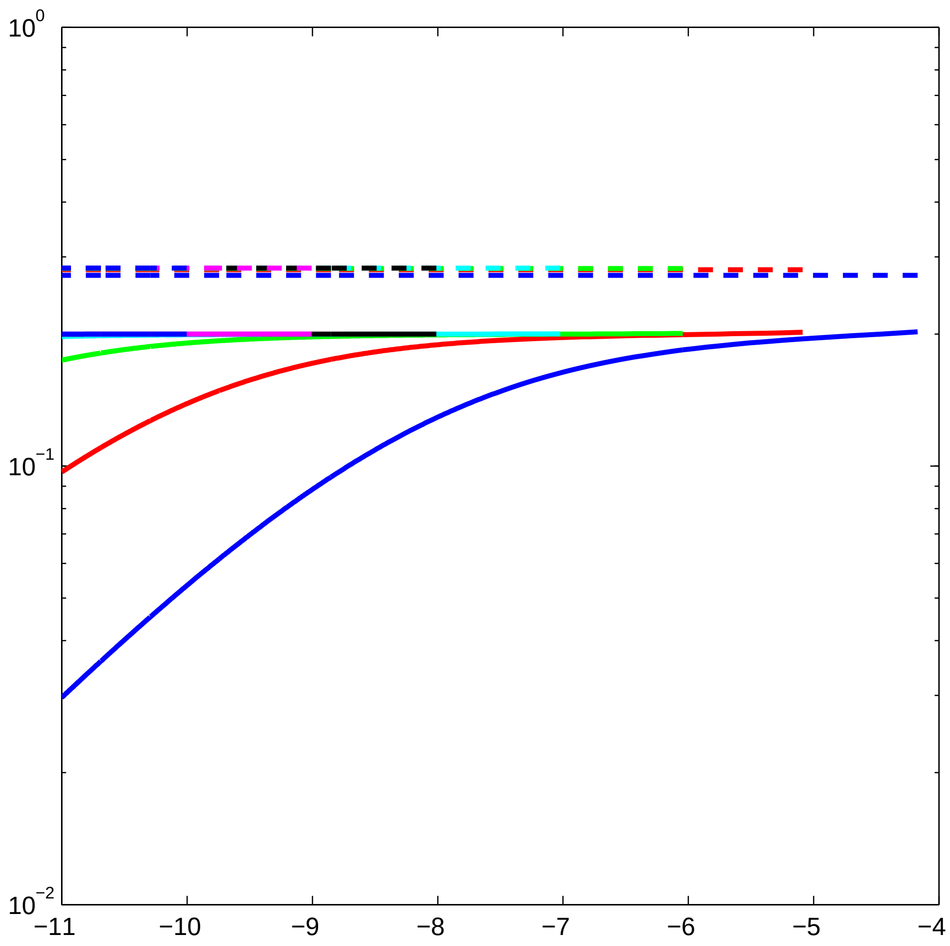
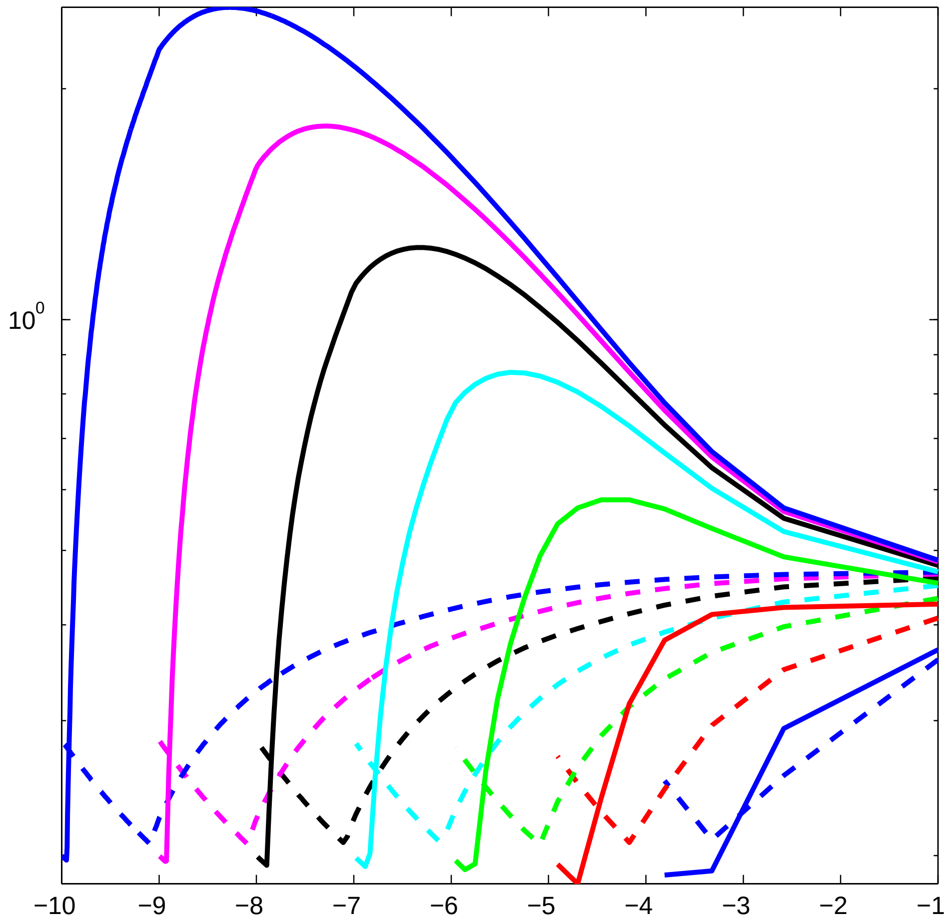
Does this periodic-type analysis correctly predict Dirichlet problem behaviour?

Model Problem Results (1)



- $\log_{10}(\|M\|_2)$ vs $\log_2(\epsilon)$
- $P_h \geq 1$ only
- periodic: dotted lines
- Dirichlet: solid lines
- h fixed for each line
- $h = \frac{1}{8}$ to $h = \frac{1}{512}$
- $\nu = 1$
- periodic $\longrightarrow \frac{\sqrt{2}}{5} \simeq 0.2828$
- Dirichlet $\longrightarrow 0.2$

Model Problem Results (2)



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- $h = \frac{1}{8}$ to $h = \frac{1}{512}$
- $\nu = 1$
- not a good match!

MG Iteration Counts

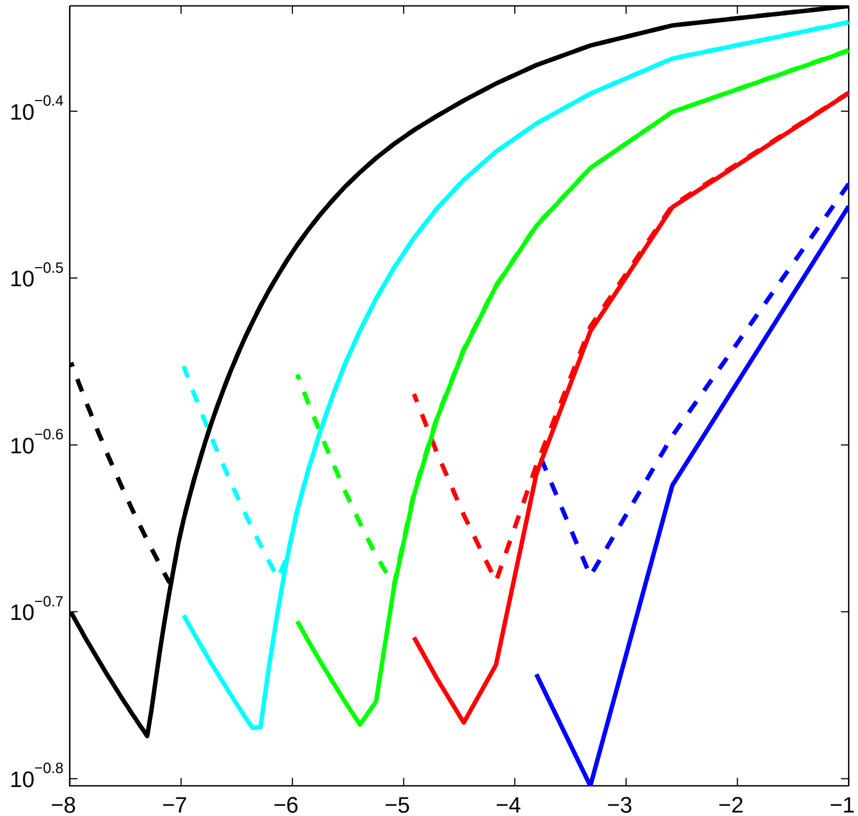
	ϵ										
h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$
$\frac{1}{4}$	5	5	5	5	5	4	4	3	2	2	2
$\frac{1}{8}$	7	7	6	6	5	5	4	4	3	2	2
$\frac{1}{16}$	7	7	7	6	5	5	5	4	4	3	2
$\frac{1}{32}$	7	7	7	7	6	5	5	4	4	3	3
$\frac{1}{64}$	7	7	7	7	6	5	5	4	4	4	3
$\frac{1}{128}$	7	6	6	6	6	6	5	4	4	4	3

$P_h < 1$ $\quad\quad$ $P_h \geq 1$

MG-like convergence for any value of P_h

Explanation?

- for $P_h < 1$, iteration matrix M has one 'bad' eigenvalue
- artificially 'removing' this eigenvalue gives



- $P_h < 1$ only
- periodic: $\|M^{per}\|_2$
- Dirichlet: $\sqrt{\lambda_2(M^T M)}$

- periodic-type analysis does not capture the effect of this eigenvalue

Remarks

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  - periodic-type analysis for smoothing matrix norm is representative of Dirichlet problem behaviour for all values of P_h,
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- Linear algebra gives useful insight into convergence of two-grid iteration.
- Separate approximation and smoothing matrices:
  - periodic-type analysis for smoothing matrix norm is representative of Dirichlet problem behaviour for all values of P_h,
  - periodic-type analysis for approximation matrix norm is representative of Dirichlet problem behaviour for $P_h \geq 1$: for $P_h < 1$, one 'bad' eigenvalue again causes trouble.
- Replacing the Dirichlet condition by a Neumann condition on the outflow boundary leads to similar computational results.