Element-based preconditioners for problems in geomechanics

Alison Ramage
Dept of Mathematics
University of Strathclyde
Glasgow, Scotland

University of Strathclyde Glasgow

- Charles Augarde
  School of Engineering, University of Durham, England
- Jochen Staudacher

Simulations in Geomechanics

- soil-structure interaction problems
- large 3D simulations of complicated geometries
- soil behaviour dominated by irrecoverable deformations: elasto-plastic models
- saturated soils, undrained (incompressible)
- previous work in structural engineering: elastic models
- most soil models assume elastic behaviour at small strains
- AIM: study effects of adding plasticity

Linear elasticity: Lamé equation

$$-(\lambda+\mu)\nabla\left(\nabla\cdot\mathbf{u}\right)-\mu\nabla^2\mathbf{u}=\mathbf{f}\quad \text{in}\quad \Omega$$

- displacement $\mathbf{u}(x,y)=[u_1,u_2]^T$, body force $\mathbf{f}(x,y)$
- Lamé constants λ and μ

$$-\nabla\cdot\mathbf{S}(\mathbf{u})=\mathbf{f}\quad \text{in}\quad \Omega$$

- linearised strain $\mathbf{E}(\mathbf{u})=\frac{1}{2}\begin{bmatrix} 2\frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y} & 2\frac{\partial u_2}{\partial y}\end{bmatrix}$
- stress $\mathbf{S}(\mathbf{u})=2\mu\mathbf{E}(\mathbf{u})+\lambda\,\mathrm{tr}(\mathbf{E}(\mathbf{u}))\mathbf{I}$

Finite Element Approximation

- Dirichlet boundary value problem

$$-\nabla \cdot \mathbf{S}(\mathbf{u}) = \mathbf{f} \text{ in } \Omega, \qquad \mathbf{u} = \mathbf{0} \text{ on } \Gamma$$

- total of M nodes, n degrees of freedom
- Galerkin finite elements: shape functions $\phi_1, \ldots, \phi_M$
- shape derivative matrix

$$\mathbf{B} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial x} & \cdots & \frac{\partial \phi_M}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_2}{\partial y} & \cdots & \frac{\partial \phi_M}{\partial y} \\ \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_2}{\partial y} & \cdots & \frac{\partial \phi_M}{\partial y} & \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial x} & \cdots & \frac{\partial \phi_M}{\partial x} \end{bmatrix}$$

Global Stiffness Matrix

- constitutive matrix

$$\mathbf{E}^{el} = \begin{bmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Young's modulus E, Poisson's ratio ν

$$\mathbf{E}^{el} = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix}$$

- global stiffness matrix

$$\mathbf{K} = \int_\Omega \mathbf{B}^T \mathbf{E}^{el} \mathbf{B} \, d\Omega$$

Element Stiffness Matrix

- $n_e \times n_e$ element matrix $\mathbf{K}_e$, e.g. linear triangles

$$\mathbf{K}_e = \bar{E} \begin{bmatrix} \frac{3}{2} - 2\nu & \frac{1}{2} & \nu - 1 & \nu - \frac{1}{2} & \nu - \frac{1}{2} & -\nu \\ \frac{1}{2} & \frac{3}{2} - 2\nu & -\nu & \nu - \frac{1}{2} & \nu - \frac{1}{2} & \nu - 1 \\ \nu - 1 & -\nu & 1 - \nu & 0 & 0 & v \\ \nu - \frac{1}{2} & \nu - \frac{1}{2} & 0 & \frac{1}{2} - \nu & \frac{1}{2} - \nu & 0 \\ \nu - \frac{1}{2} & \nu - \frac{1}{2} & 0 & \frac{1}{2} - \nu & \frac{1}{2} - \nu & 0 \\ -\nu & \nu - 1 & \nu & 0 & 0 & 1 - \nu \end{bmatrix}$$

$$\bar{E} = \frac{E}{2(1 - 2\nu)(1 + \nu)}$$

- eigenvalues

$$\frac{E}{h^2(1 + \nu)} \left\{ 0, 0, 0, 1, \frac{2(\nu - 1) \pm \sqrt{1 - 2\nu + 4\nu^2}}{2(2\nu - 1)} \right\}$$

Stiffness Matrix Assembly

- $n_e \times n$ Boolean connectivity matrix $\mathbf{C}_e$

$$\bar{\mathbf{K}}_\mathbf{e} = \mathbf{C}_e^T \mathbf{K}_e \mathbf{C}_e \text{ for } e = 1, \ldots, E, \qquad \mathbf{K} = \sum_{e=1}^{E} \bar{\mathbf{K}}_\mathbf{e}$$

- two observations:
  - order nodal displacements

$$\mathbf{u} = [u_1, u_2, \ldots, u_M, v_1, v_2, \ldots, v_M]^T$$

block stiffness matrix $\mathbf{K} = \begin{bmatrix} \mathbf{K}_{xx} & \mathbf{K}_{xy} \\ \mathbf{K}_{xy}^T & \mathbf{K}_{yy} \end{bmatrix}$

same block structure applies to each $\mathbf{K}_e$

  - for linear elasticity $\kappa(\mathbf{K}) = O(h^{-2})$

Element-based Preconditioners

- connectivity and element matrices stored
- global stiffness matrix (preconditioner) never assembled
- preconditioning matrix $\mathbf{P} = \sum_{e=1}^{E} \mathbf{C}_e^T \, \mathbf{P}_e \, \mathbf{C}_e$
- diagonal scaling (DIAG)

$$\mathbf{P}_{DIAG} = \mathrm{diag}(\mathbf{K}) = \sum_{\mathrm{e}=1}^{\mathrm{E}} \mathbf{C}_{\mathrm{e}}^{\mathrm{T}} \, \mathrm{diag}(\mathbf{K}_{\mathrm{e}}) \, \mathbf{C}_{\mathrm{e}}$$

$$\kappa(\mathbf{P}_{DIAG}^{-1}\mathbf{K}) = O(h^{-2})$$

- true for any preconditioner of the form $\sum_{e=1}^{E} \mathbf{C}_e^T \, \mathbf{Q}_e \, \mathbf{C}_e$ for some $n_e \times n_e$ matrices $\mathbf{Q}_e$

Element-by-element methods (EBE)

- Hughes, Levit and Winget (1983)
- regularise assembly of each element $\bar{\mathbf{K}}_e$

$$\tilde{\mathbf{K}}_e = \mathbf{I}_n + \mathbf{D}^{-1/2}\left(\bar{\mathbf{K}}_e - \bar{\mathbf{D}}_e\right)\mathbf{D}^{-1/2}$$

$$\mathbf{D} = \mathrm{diag}(\mathbf{K}),\ \bar{\mathbf{D}}_e = \mathrm{diag}(\bar{\mathbf{K}}_\mathrm{e})$$

- factorise $\tilde{\mathbf{K}}_e = \mathbf{L}_e\mathbf{D}_e\mathbf{L}_e^T$

$$\mathbf{P}_{EBE} = \mathbf{D}^{1/2}\left[\prod_{e=1}^{E}\mathbf{L}_e\right]\left[\prod_{e=1}^{E}\mathbf{D}_e\right]\left[\prod_{e=E}^{1}\mathbf{L}_e^T\right]\mathbf{D}^{1/2}$$

$$\kappa(\mathbf{P}_{EBE}^{-1}\mathbf{K}) = O(h^{-2})$$

- can be applied directly to elasticity problems

Element-based Symmetric Gauss-Seidel (SGS)

- EBE requires additional storage for factorisations
- split

$$\tilde{\mathbf{K}}_e = \mathbf{I}_n - \mathbf{L}_e - \mathbf{L}_e^T$$

$\mathbf{L}_e =$ strict lower triangle of $\tilde{\mathbf{K}}_e$, $\mathbf{D} = \mathrm{diag}(\mathbf{K})$

$$\mathbf{P}_{SGS} = \mathbf{D}^{1/2} \left[\prod_{e=1}^{E} (\mathbf{I}_n - \mathbf{L}_e) \right] \left[\prod_{e=E}^{1} \left(\mathbf{I}_n - \mathbf{L}_e^T \right) \right] \mathbf{D}^{1/2}$$

$$\kappa(\mathbf{P}_{SGS}^{-1}\mathbf{K}) = O(h^{-2})$$

- other matrix splittings can be applied at an element level

Element matrix factorisation (EMF)

- Gustafsson and Lindskog (1986)
- Cholesky factorisation $\mathbf{K}_e = \bar{\mathbf{L}}_e \bar{\mathbf{L}}_e^T$
- assemble factors to form $\mathbf{L}$ and $\mathbf{D}$ (requires a particular global and local numbering of unknowns)

$$\mathbf{P}(\eta) = [\mathbf{L}(1+\eta h)^{-1} + \mathbf{D}(1+\eta h)][\mathbf{L}(1+\eta h)^{-1} + \mathbf{D}(1+\eta h)]^T$$

$$\kappa(\mathbf{P}_{EMF}^{-1}\mathbf{K}) = O(h^{-1})$$

- $\mu = 0$ here
- can break down for linear elasticity
- similar method by Kaasschieter (1989)

Matrix Reduction Techniques

- notation of Saint-Georges et al. (1996)
- AIM: make $\mathbf{K}$ a *Stieltjes matrix*

  $\mathbf{K} = \{k_{ij}\}$ is SPD with $k_{ij} \leq 0$ for $i \neq j$

- C-reduction: lump positive off-diagonal entries in a row of $\mathbf{K}$ onto the diagonal $\Rightarrow$ Stieltjes matrix
- D-reduction: neglect any connections between degrees of freedom of different types (take block diagonal part or separate displacement component of $\mathbf{K}$)
- DC-reduction: perform the two reductions in sequence $\Rightarrow$ Stieltjes matrix

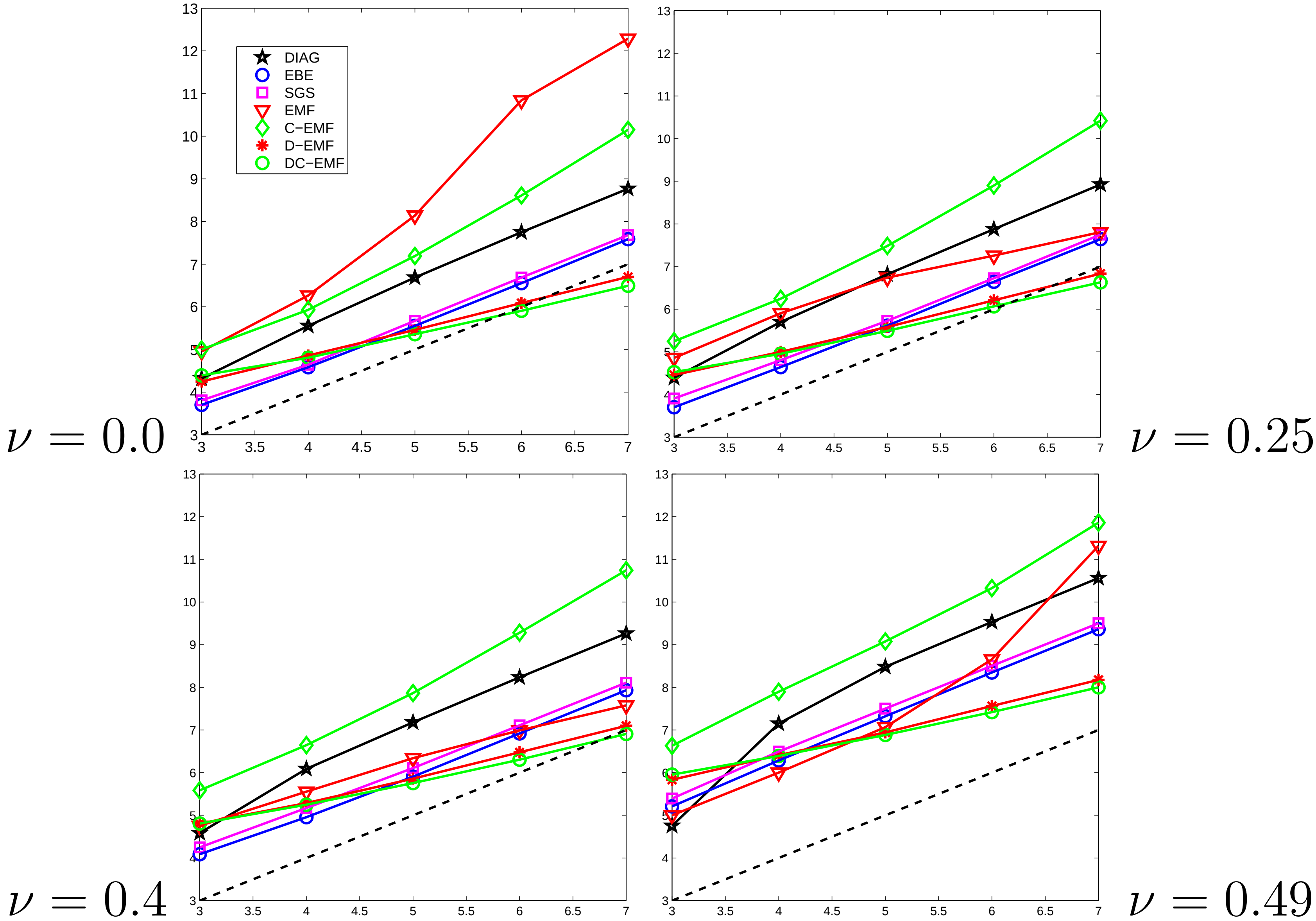
Matrix reduction on an element level

IDEA: combine matrix reduction with EMF factorisation

- new methods:
  - C-EMF, DC-EMF: reduced element matrix is Stieltjes $\Rightarrow$ element Cholesky factors for EMF can be computed
  - D-EMF: reduced element matrix block diagonal, each block has 1D nullspace $\Rightarrow$ element Cholesky factors for EMF can be computed
- theoretical results:

$$\lambda_{\min}(\mathbf{P}_{DC-EMF}^{-1}\mathbf{K}) = O(1), \quad \lambda_{\min}(\mathbf{P}_{D-EMF}^{-1}\mathbf{K}) = O(1)$$

Iteration Counts: Elasticity



Adding Plasticity

- elasto-plastic constitutive model: yield function F, plastic potential function P, hardening/softening rule
- stress-strain relationship

$$\boldsymbol{\sigma} = \mathbf{E}^{ep}\boldsymbol{\epsilon}, \qquad \mathbf{E}^{ep} = \mathbf{E}^{el} - \mathbf{E}^{pl}$$

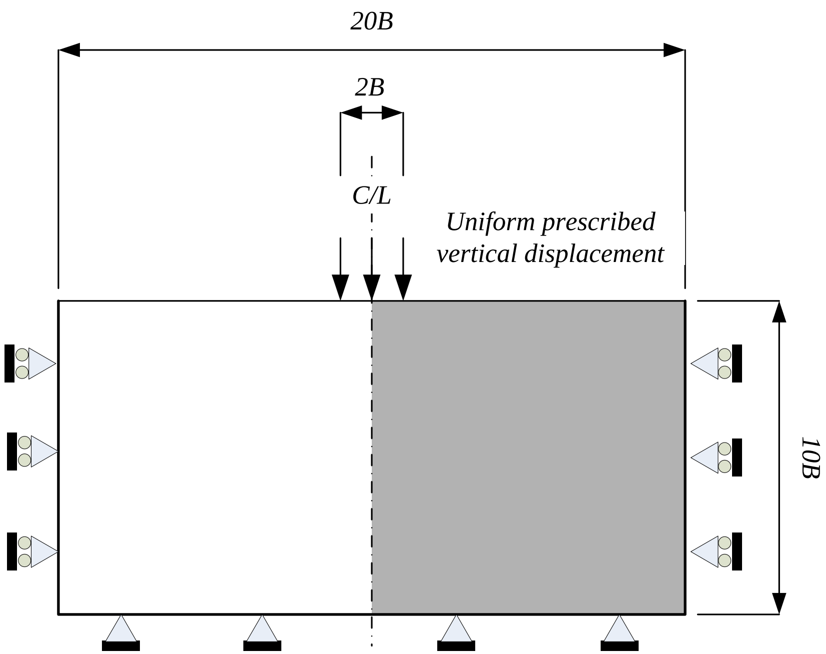
- assume perfect plasticity (zero hardening/softening), associated plastic flow (yield $=$ plastic potential)

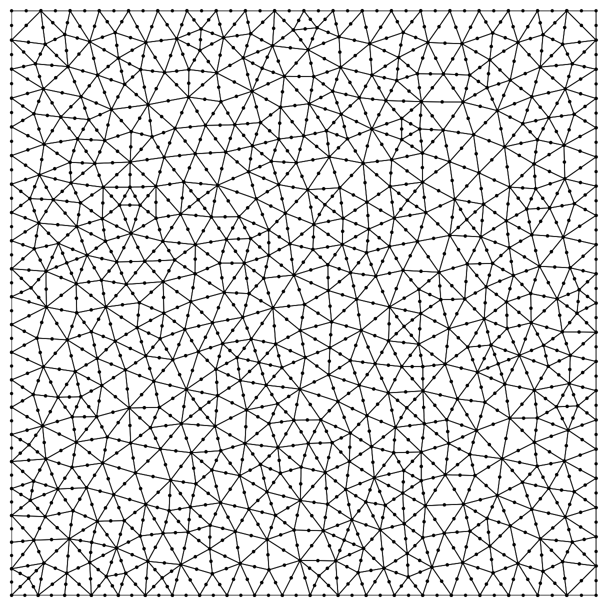
$$\mathbf{E}^{pl} = \frac{\mathbf{E}^{el}\dfrac{\partial F}{\partial \boldsymbol{\sigma}}\dfrac{\partial F}{\partial \boldsymbol{\sigma}}^{T}\mathbf{E}^{el}}{\dfrac{\partial F}{\partial \boldsymbol{\sigma}}^{T}\mathbf{E}^{el}\dfrac{\partial F}{\partial \boldsymbol{\sigma}}}$$

- $\mathbf{E}^{ep}$ is a rank-one update of $\mathbf{E}^{el}$

Footing test problem

- plane strain rigid footing modelled by prescribing vertically downwards displacements on selected surface nodes
- unstructured mesh of linear strain triangles





- load applied over a number of equal incremental steps

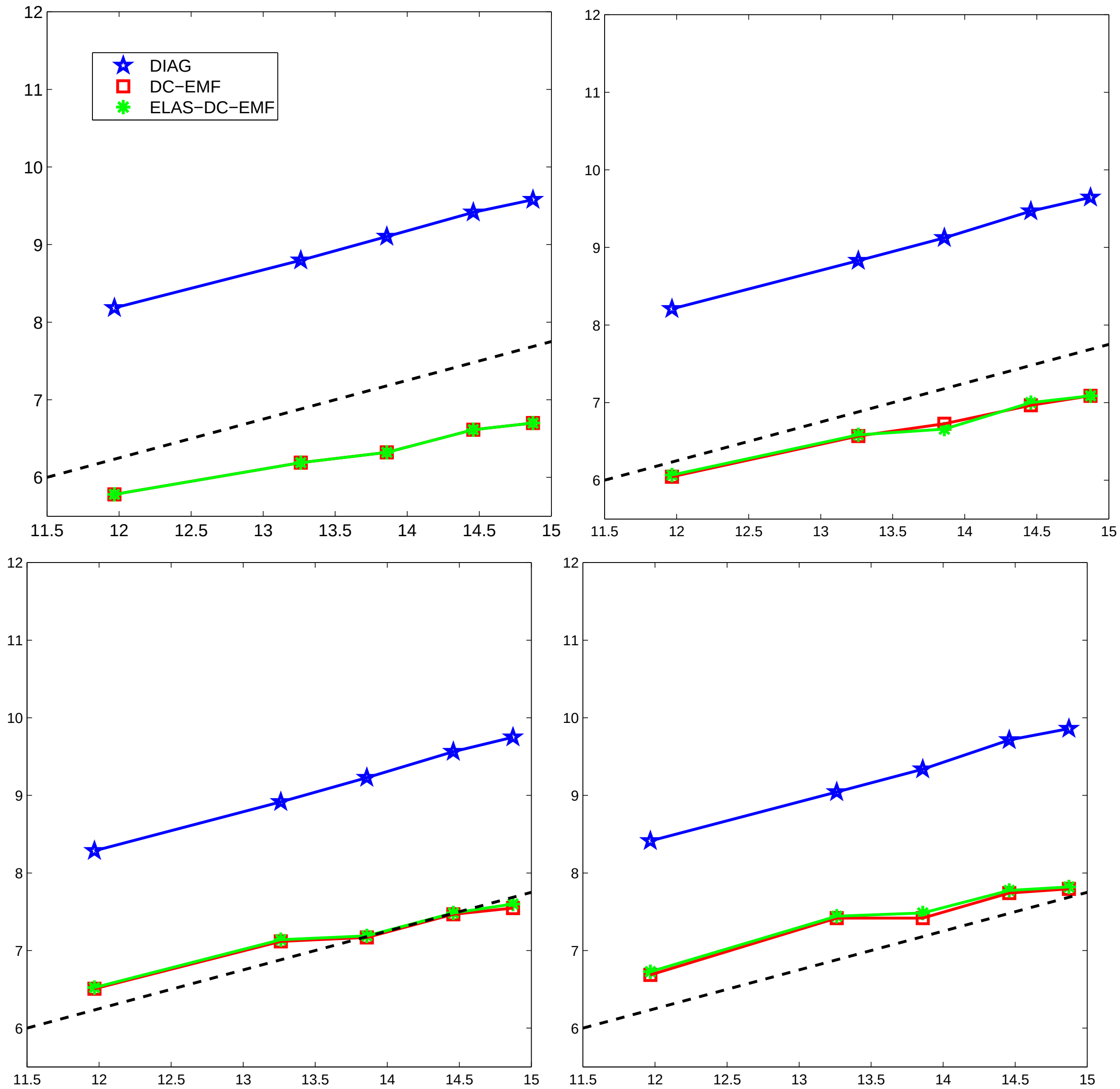
Comparison of Methods

- diagonal scaling DIAG
- EMF with DC-reduction DC-EMF
  - method of choice for elastic problems
- DC-EMF applied to the elastic part ELAS-DC-EMF
  - plasticity is ‘simple update’ of elasticity
  - IDEA: base preconditioner on the elastic part only
  - $\mathbf{E}^{el}$ does not change from load step to load step
  - preconditioner $\mathbf{P}$ need only be calculated once at the beginning of each simulation
- snapshot at four load steps $s = 1, 25, 50, 100$

Iteration Counts: Plasticity

$\nu = 0.25$

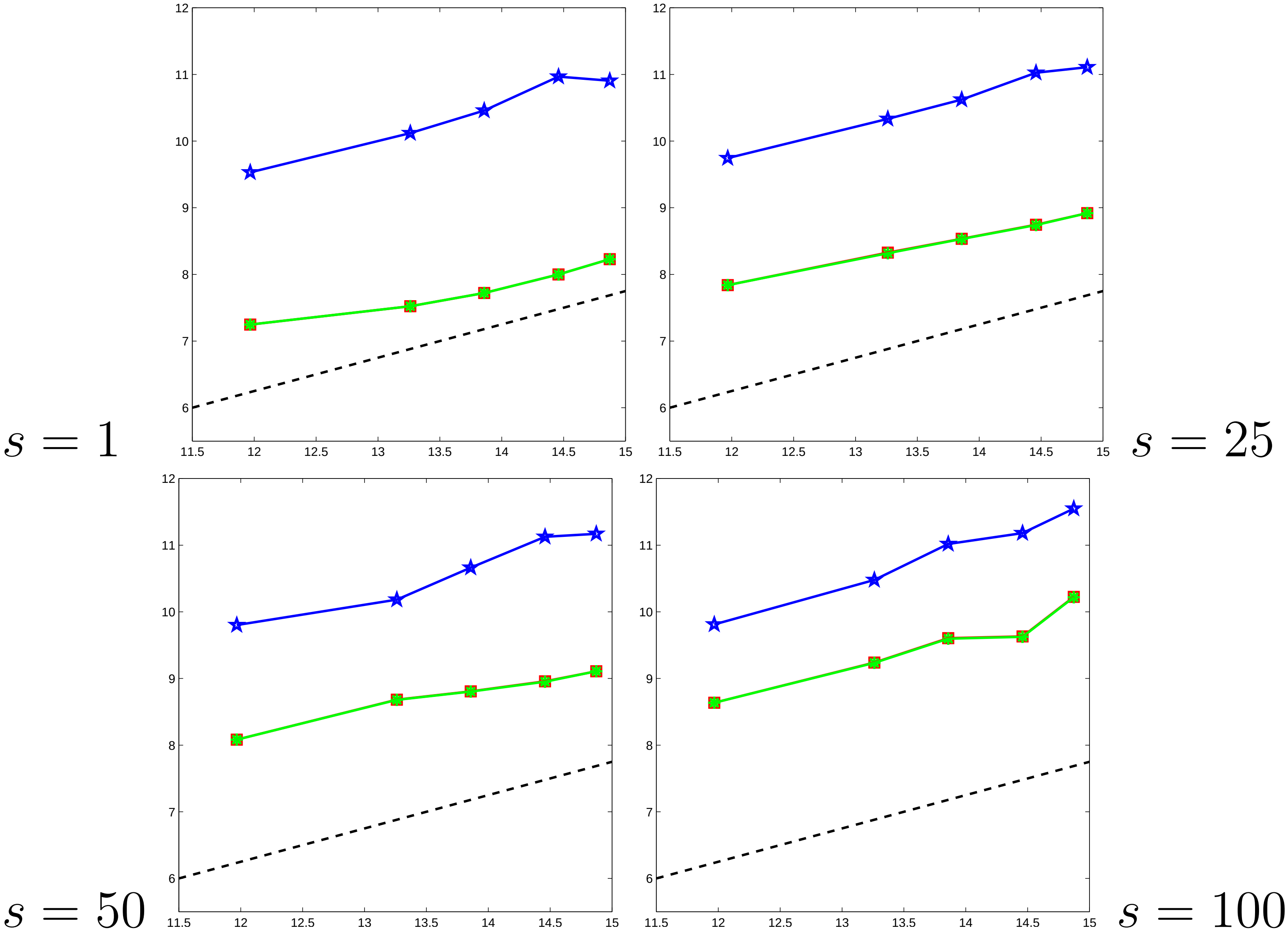
$s = 1$ $s = 25$



$s = 50$ $s = 100$

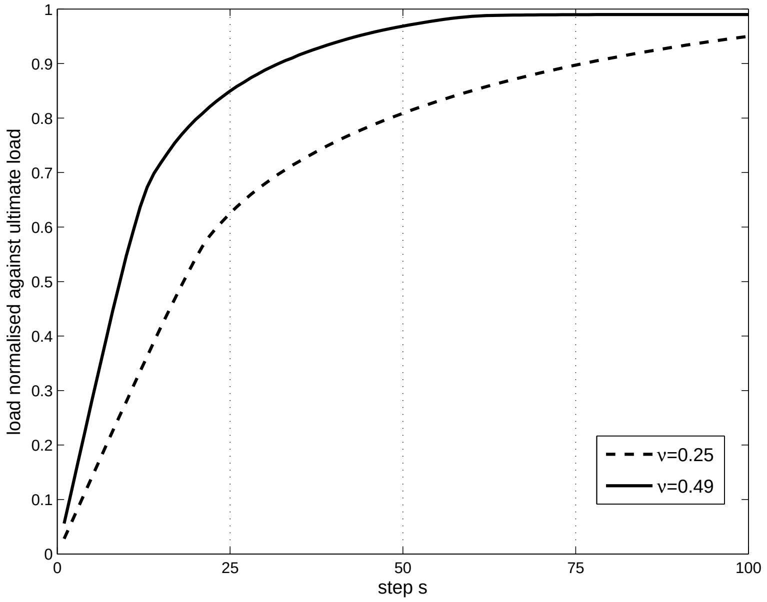
Iteration Counts: Plasticity

$\nu = 0.49$



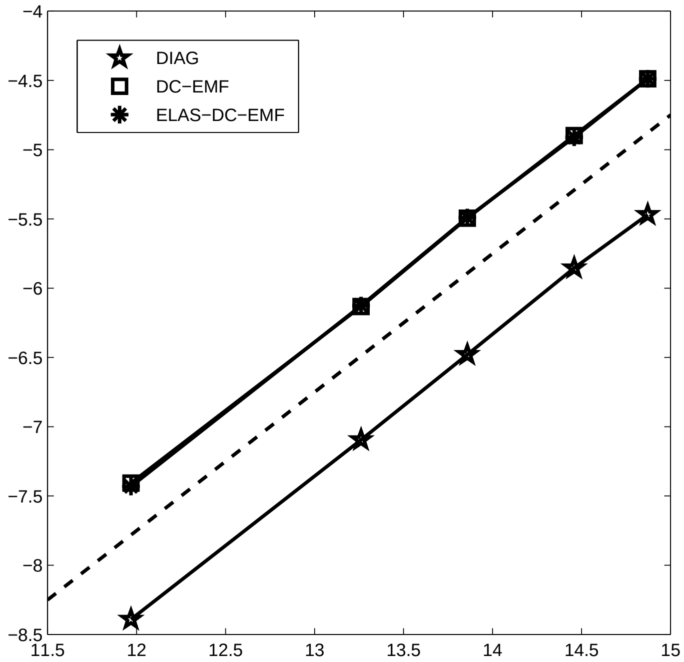
Full Simulation

- F90 geotechnical finite element code OXFEM
- modified Euler method
- one load stage comprising 100 load steps



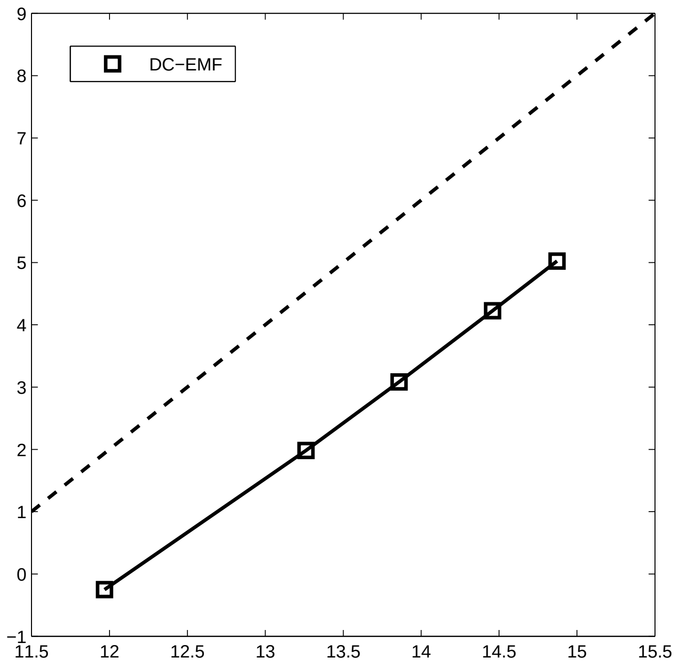
- 5 unstructured grids: 4000→30000 unknowns

CPU times (1)



average CPU time per iteration

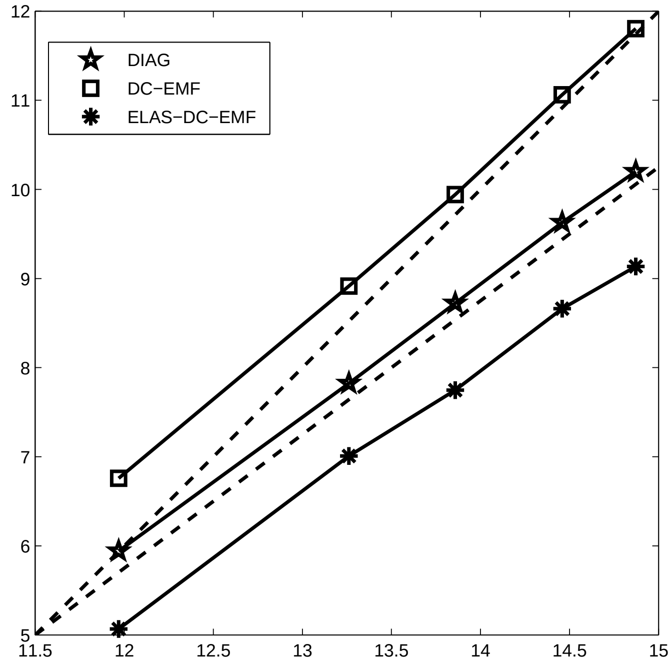
$$\bar{t}_k = cn$$



reduction/factorisation time

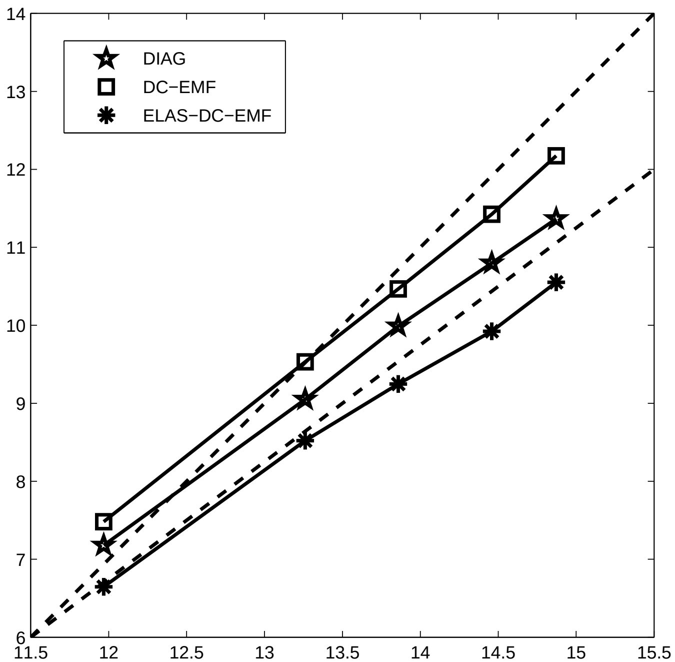
$$t_s = cn^2$$

CPU times (2)



total time, $\nu = 0.25$

$$t_{DIAG} = cn^{1.5}$$
$$t_{DC-EMF} = cn^{2}$$
$$t_{ELAS-DC-EMF} = cn^{1.5}$$



total time, $\nu = 0.49$

Summary

- For purely elastic problems, D-EMF and DC-EMF offer an improvement in terms of asymptotic behaviour over traditional methods such as diagonal scaling.
- Applying DC-EMF to the elastic part of the matrix provides a promising new element-based preconditioner.
- Future research
  - materials with non-associated flow rules ($\Rightarrow$ nonsymmetric systems)
  - consolidation problems ($\Rightarrow$ saddle-point systems)
  - unsaturated soils ($\Rightarrow$ extra degrees of freedom)
- collaboration with OASYS Ltd

Relevant Publications

- **Augarde, Ramage and Staudacher**
  On Element-based Preconditioners for Linear Elasticity Problems
  Computers and Structures **84**, pp. 2306-2315, 2006.

- **Augarde, Ramage and Staudacher**
  Element-based Preconditioners for Elasto-Plastic Problems
  International Journal of Numerical Methods in Engineering doi:10.1002/nme.1947, 2006.