Multigrid Solution of Discrete Convection-Diffusion Equations

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Convection-Diffusion in 2D

$$
\begin{aligned}
-\epsilon\nabla^2 u(x,y) + \mathbf{w}.\nabla u(x,y) &= f(x,y) \quad \text{in} \quad \Omega \in \mathbb{R}^2 \\
u(x,y) &= g \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

divergence-free convective velocity ('wind') $\mathbf{w}$

diffusion parameter $\epsilon << 1$

discretisation parameter h

mesh Péclet number $P_h = \dfrac{\|\mathbf{w}\| h}{2\epsilon}$

Boundary Layers and Oscillations

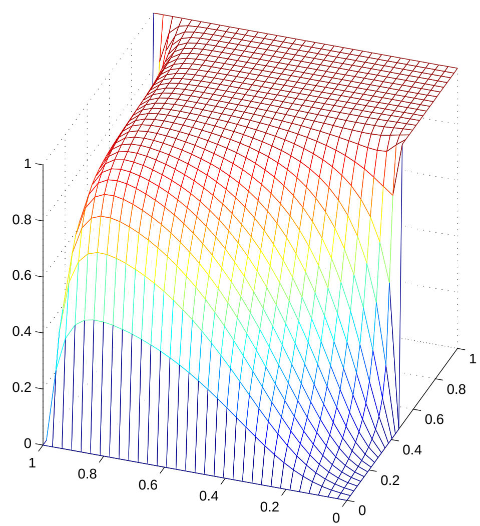
- Galerkin finite element method

$$\epsilon(\nabla u_h, \nabla v_h) + (\mathbf{w} \cdot \nabla u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

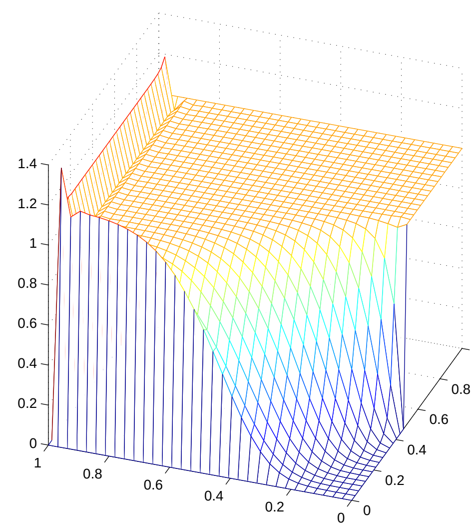
- solution features:

  exponential and characteristic boundary layers

- oscillations observed in discrete solutions for $P_h > 1$



$P_h = 0.5$



$P_h = 2$

Streamline Diffusion Method

streamline diffusion FEM, square bilinear elements

$$\begin{aligned} \epsilon(\nabla u_h, \nabla v_h) &+ (\mathbf{w}\cdot\nabla u_h, v_h) + \frac{\delta h}{||\mathbf{w}||}(\mathbf{w}\cdot\nabla u_h, \mathbf{w}\cdot\nabla v_h) \\ &= (f, v_h) + \frac{\delta h}{||\mathbf{w}||}(f, \mathbf{w}\cdot\nabla v_h) \quad \forall v_h \in V_h \end{aligned}$$

- $P_h \leq 1:$ $\delta = 0$ Galerkin FEM
- $P_h > 1:$ $\delta = \frac{1}{2} - \frac{\epsilon}{h}$ Streamline Diffusion

Multigrid Ideas

- fine grid (h),   coarse grid ($2h$)

- decompose a grid function into components in two subspaces

  approximate inverse operator
  for components in subspace 1

  smoothing iteration
  rapidly reduces error components in subspace 2

- recursive process on nested grids

- optimal in the sense of obtaining convergence rate independent of h

Issues for Convection-Diffusion

- approximation: choice of discretisation
  - oscillations on coarser grids?
  - grid transfer operators?
- smoothing: choice of relaxation method
  - direction of flow?
  - circular flows?
- multigrid can be implemented effectively for convection-diffusion problems

Convergence Analysis

- standard Poisson-type convergence analysis fails
- ideas for convection-diffusion less well-developed
- various successful approaches
  - perturbation arguments
    Bank (1981), Bramble, Pasciak and Xu (1988), Mandel (1986), Wang (1993)
  - matrix-based methods
    Reusken (2002), Olishanskii and Reusken (2002)

Multigrid Method

- two-grid method: N_f (fine grid), N_c (coarse grid)
- coefficient matrices: A_f (fine grid), A_c (coarse grid)

direct discretisation on coarse grid

- prolongation: bilinear interpolation P
- restriction: transpose of prolongation P^T
- smoothing: line Gauss-Seidel S_A
- ν steps of pre-smoothing, no post-smoothing

Multigrid Convergence

- algebraic error $\mathbf{e}_k = \hat{\mathbf{u}} - \mathbf{u}_k$
- two-grid iteration matrix $M = (I - PA_c^{-1}P^TA_f)S_A^\nu$
- error equation $\mathbf{e}_k = M\mathbf{e}_{k-1} = M^k\mathbf{e}_0$
- convergence?

$$\|\mathbf{e}_k\| \leq \|M\|^k \|\mathbf{e}_0\|$$

convergence if $\|M\| < 1$

Two-Grid Convergence Analysis

AIM: find an upper bound for

$$\|M\|_2 = \|(I - PA_c^{-1}P^TA_f)S_A^\nu\|_2$$

- Approach 1: write

$$M = (A_f^{-1} - PA_c^{-1}P^T)(A_fS_A^\nu) = M_AM_S$$

and bound $\|M_A\|_2$, $\|M_S\|_2$ separately

- Approach 2: bound $\|M\|_2$ directly

Computational Molecule

parameters $h,\ \epsilon,\ \delta$

$$
\begin{array}{lccccc}
M_2: & -\frac{1}{12}[(2\delta-1)h+4\epsilon] & & -\frac{1}{3}[(2\delta-1)h+\epsilon] & & -\frac{1}{12}[(2\delta-1)h+4\epsilon] \\
 & & \nwarrow & \uparrow & \nearrow & \\
M_1: & \frac{1}{3}(\delta h-\epsilon) & \leftarrow & \frac{4}{3}(\delta h+2\epsilon) & \rightarrow & \frac{1}{3}(\delta h-\epsilon) \\
 & & \swarrow & \downarrow & \searrow & \\
M_3: & -\frac{1}{12}[(2\delta+1)h+4\epsilon] & & -\frac{1}{3}[(2\delta+1)h+\epsilon] & & -\frac{1}{12}[(2\delta+1)h+4\epsilon]
\end{array}
$$

symmetric stencil

Coefficient Matrix

$$A = \begin{bmatrix} M_1 & M_2 & & & 0 \\ M_3 & M_1 & M_2 & & \\ & \ddots & \ddots & \ddots & \\ & & M_3 & M_1 & M_2 \\ 0 & & & M_3 & M_1 \end{bmatrix}$$

eigenvectors and eigenvalues:

$$M_1 \mathbf{v}_j = \lambda_j \mathbf{v}_j, \quad \lambda_j = m_{1c} + 2m_{1r} \cos \frac{j\pi}{N_f}$$
$$M_2 \mathbf{v}_j = \sigma_j \mathbf{v}_j, \quad \sigma_j = m_{2c} + 2m_{2r} \cos \frac{j\pi}{N_f}$$
$$M_3 \mathbf{v}_j = \gamma_j \mathbf{v}_j, \quad \gamma_j = m_{3c} + 2m_{3r} \cos \frac{j\pi}{N_f}$$

$$\mathbf{v}_j = \sqrt{\frac{2}{N_f}} \left[\sin \frac{j\pi}{N_f}, \quad \sin \frac{2j\pi}{N_f}, \quad \ldots, \sin \frac{(N_f - 1)j\pi}{N_f} \right]^T$$

Transformation: Coefficient Matrix (1)

N_f^2 elements, n_f^2 unknowns $(n_f = N_f - 1)$

$$\hat{V}_f = \left[\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_{n_f}\right], \qquad V_f = \texttt{diag}(\hat{V}_f, \ldots, \hat{V}_f)$$

$$M_1 \hat{V}_f = \hat{V}_f \Lambda, \qquad M_2 \hat{V}_f = \hat{V}_f \Sigma, \qquad M_3 \hat{V}_f = \hat{V}_f \Gamma$$

$$V_f^T A_f V_f = \hat{T}_f = \begin{bmatrix} \Lambda & \Sigma & & & 0 \\ \Gamma & \Lambda & \Sigma & & \\ & \ddots & \ddots & \ddots & \\ & & \Gamma & \Lambda & \Sigma \\ 0 & & & \Gamma & \Lambda \end{bmatrix}$$

Transformation: Coefficient Matrix (2)

permute into tridiagonal form:

$$\Pi_f^T \hat{T}_f \Pi_f = T_f = \begin{bmatrix} T_1 & & & & 0 \\ & T_2 & & & \\ & & \ddots & & \\ & & & T_{n_f-1} & \\ 0 & & & & T_{n_f} \end{bmatrix}$$

$$T_j = \texttt{tridiag}(\gamma_j, \lambda_j, \sigma_j)$$

$$A_f = Q_f T_f Q_f^T \qquad Q_f = V_f \Pi_f$$

coarse grid: $A_c = Q_c T_c Q_c^T \qquad Q_c = V_c \Pi_c$

Transformation: Smoothing Matrix

block matrix splitting: $A_f = D_A - L_A - U_A$

Gauss-Seidel smoothing matrix:

$$S_A = (D_A - L_A)^{-1} U_A = I - (D_A - L_A)^{-1} A_f$$

transformation:

$$S_A = Q_f S_T Q_f^T$$

where $S_T = I - (D_T - L_T)^{-1} T_f$ is block-diagonal

Transformation: Prolongation Matrix

2D prolongation matrix: $P = L \otimes L$

$$L^T = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & & & & & & \\ & & \frac{1}{2} & 1 & \frac{1}{2} & & & & \\ & & & & & \ddots & & & \\ & & & & & & \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

transformation: $Q_f = (I_f \otimes \hat{V}_f)\Pi_f, \quad Q_c = (I_c \otimes \hat{V}_c)\Pi_c$

$$\bar{P} = Q_f^T P Q_c = \mathcal{A}^T \otimes L$$

$$\mathcal{A} = \begin{bmatrix} \alpha_1 & & & & & & \alpha_{N_f-1} \\ & \alpha_2 & & & & \alpha_{N_f-2} & \\ & & \ddots & & \iddots & & \\ & & & \alpha_{n_c} \quad 0 \quad \alpha_{N_f-n_c} & & & \end{bmatrix}$$

Transformation: Iteration Matrix (1)

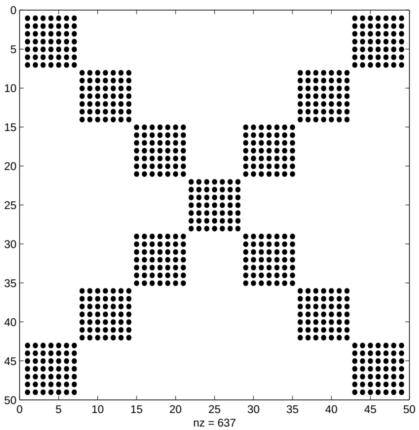
$$
\begin{aligned}
M &= (I - PA_c^{-1}P^TA_f)S_A^\nu \\
&= (I - PQ_cT_c^{-1}Q_c^TP^TQ_fT_fQ_f^T)S_A^\nu \\
&= Q_f(I - \bar{P}T_c^{-1}\bar{P}^TT_f)Q_f^T(Q_fS_TQ_f^T)^\nu \\
&= Q_f\left(I - \bar{P}T_c^{-1}\bar{P}^TT_f\right)S_T^\nu Q_f^T \\
\Rightarrow M &= Q_f\bar{M}Q_f^T
\end{aligned}
$$

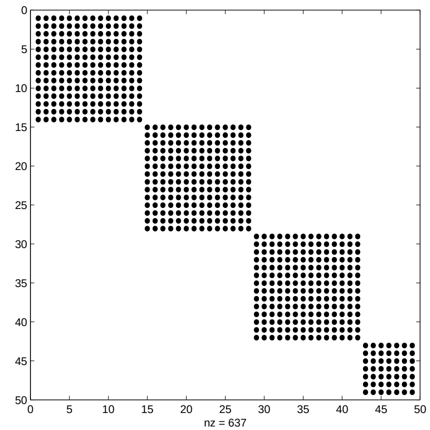
where $\bar{M} = \left(I - \bar{P}T_c^{-1}\bar{P}^TT_f\right)S_T^\nu$

Q_f is orthogonal:

$$\|M\|_2 = \|\bar{M}\|_2$$

Transformed Iteration Matrix (2)





$$\begin{bmatrix} \mathbf{B_1} & & & & & & \mathbf{C_1} \\ & \mathbf{B_2} & & & & \mathbf{C_2} & \\ & & \mathbf{B_3} & & \mathbf{C_3} & & \\ & & & \mathbf{B_4} & & & \\ & & \mathbf{C_5} & & \mathbf{B_5} & & \\ & \mathbf{C_6} & & & & \mathbf{B_6} & \\ \mathbf{C_7} & & & & & & \mathbf{B_7} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{B_1} & \mathbf{C_1} & & & & & \\ \mathbf{C_7} & \mathbf{B_7} & & & & & \\ & & \mathbf{B_2} & \mathbf{C_2} & & & \\ & & \mathbf{C_6} & \mathbf{B_6} & & & \\ & & & & \mathbf{B_3} & \mathbf{C_3} & \\ & & & & \mathbf{C_5} & \mathbf{B_5} & \\ & & & & & & \mathbf{B_4} \end{bmatrix}$$

$$\|\bar{M}\|_2 = \max\left\{ \max_{j=1,\ldots,n_c} \left\| \begin{bmatrix} B_j & C_j \\ C_k & B_k \end{bmatrix} \right\|_2, \|B_{N_c}\|_2 \right\}, \quad k = N_f - j$$

The Story So Far...

- $n_f^2 \times n_f^2$ two-grid iteration matrix M
- Fourier transformation converts 2D problem to a set of n_f problems with 1D structure
- $\|M\|_2$ can be found from norms of N_c smaller problems

  n_c of size $2n_f \times 2n_f$, 1 of size $n_f \times n_f$

- IDEA: analyse semiperiodic version of the problem

  n_c of size $2N_f \times 2N_f$, 1 of size $N_f \times N_f$

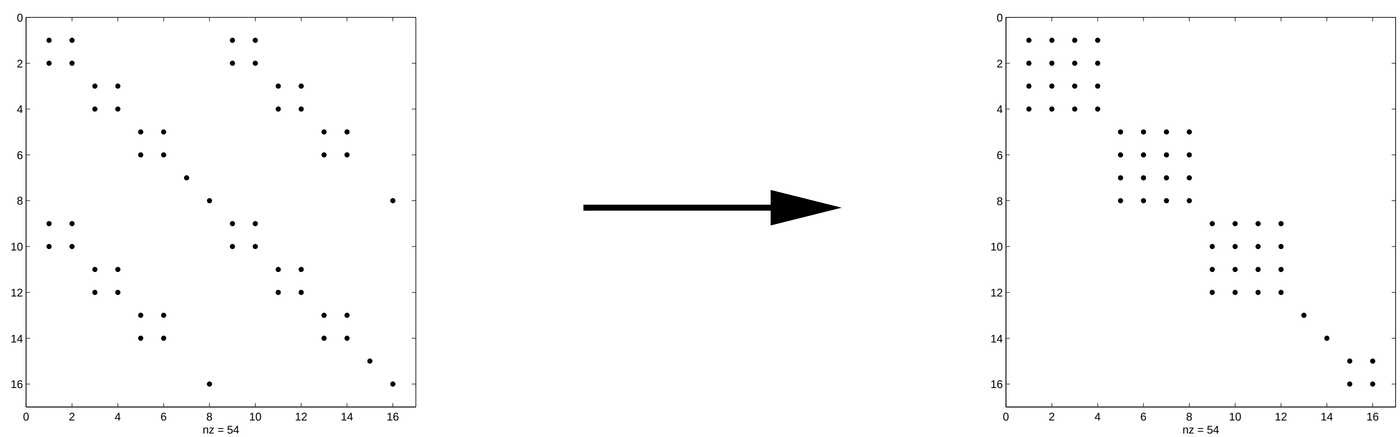
- gain insight into Dirichlet problem behaviour?

Semiperiodic problem

- B_j, C_j are replaced by periodic versions, e.g.

$$B_j^{per} = [I - \bar{P}_j^{per} (T_c^{per})_j^{-1} (\bar{P}_j^{per})^T (T_f^{per})_j] \, S_j^{per}$$

- transform using coarse grid periodic eigenvectors
- B_j^{per}, C_j^{per} become block diagonal with 2×2 blocks
- permute into block diagonal form



- 2-norm given by maximum 2-norm of the 4×4 blocks

Analytic result

- with semiperiodic approximation, when $P_h > 1$

$$\|M^{per}\|_2 = \frac{\sqrt{3 + \cos(2\pi h)}}{\sqrt{2}(5^{\nu})}$$

independent of ϵ

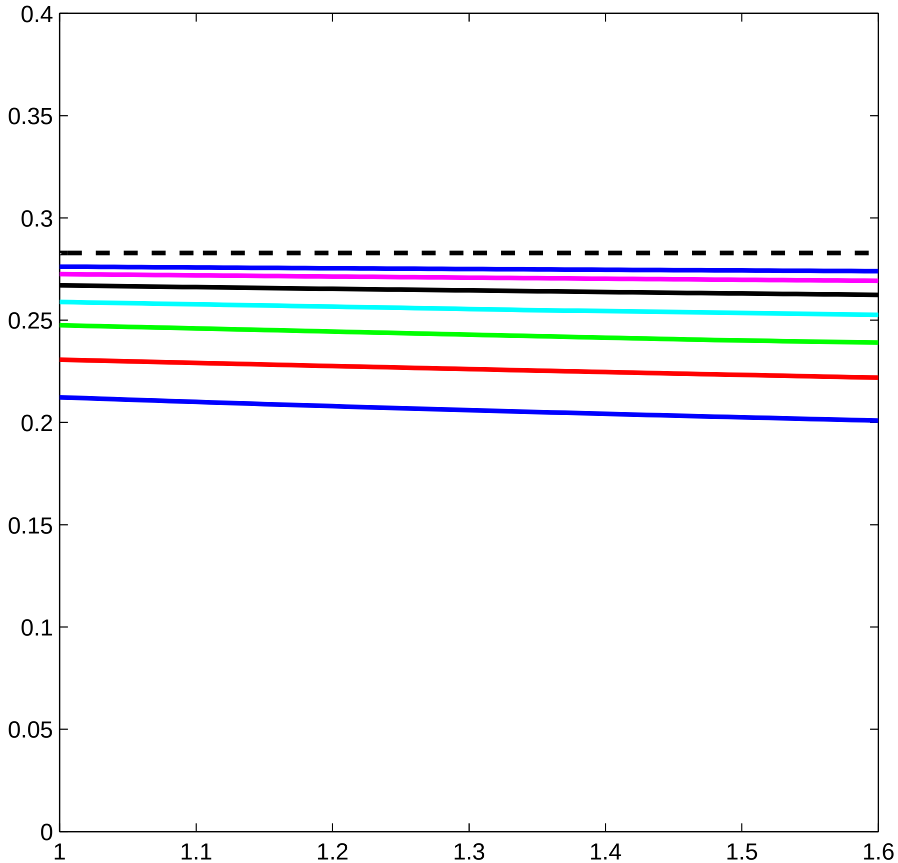
- as h is small in practice,

$$\|M^{per}\|_2 \simeq \frac{\sqrt{2}}{5^{\nu}}$$

- when $P_h < 1$, analysis is more detailed but good approximations to $\|M^{per}\|_2$ can be derived

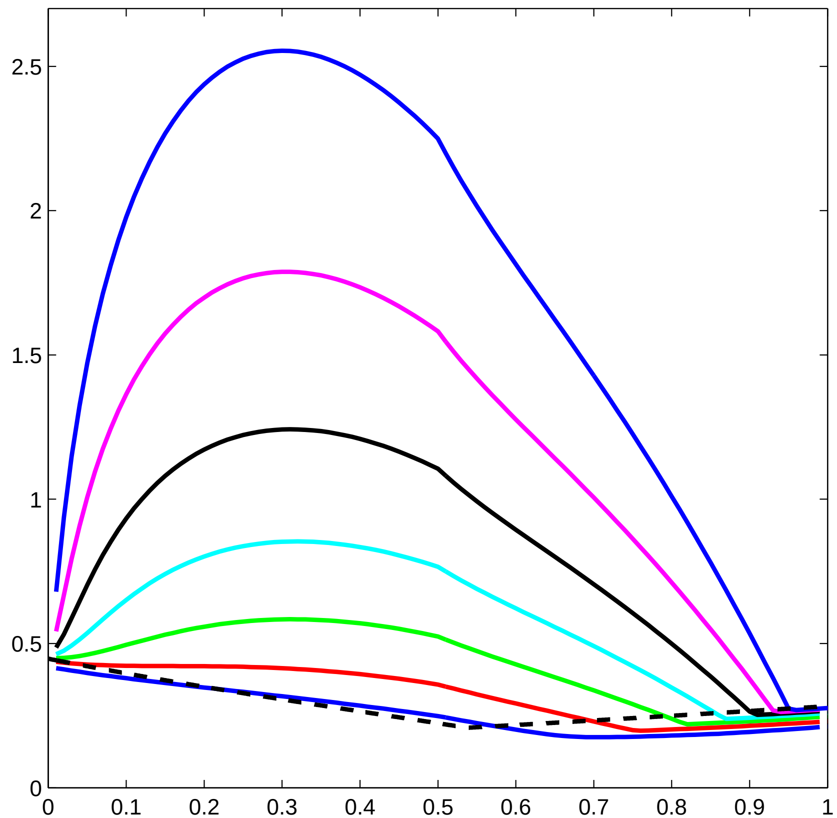
Question: Does this semiperiodic analysis correctly predict Dirichlet problem behaviour?

Model Problem Results (1)



- $\|M\|_2$ vs P_h
- $P_h \geq 1$ only
- semiperiodic: dashed line
- Dirichlet: solid lines
- h fixed for each line
- $h = \frac{1}{8}$ to $h = \frac{1}{512}$
- $\nu = 1$
- semiperiodic: $\frac{\sqrt{2}}{5} \simeq 0.28$
- Dirichlet $\rightarrow \frac{\sqrt{2}}{5}$

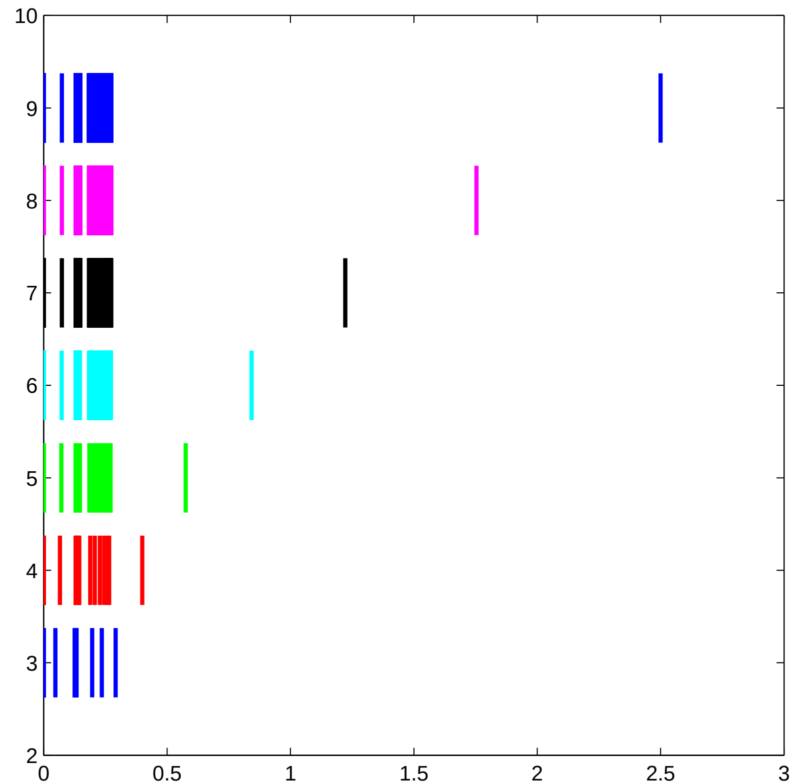
Model Problem Results (2)



- $\|M\|_2$ vs P_h
- $P_h < 1$ only
- semiperiodic: dashed line
- Dirichlet: solid lines
- h fixed for each line
- $h = \frac{1}{8}$ to $h = \frac{1}{512}$
- $\nu = 1$
- not a good match
- MG may diverge!

Observations

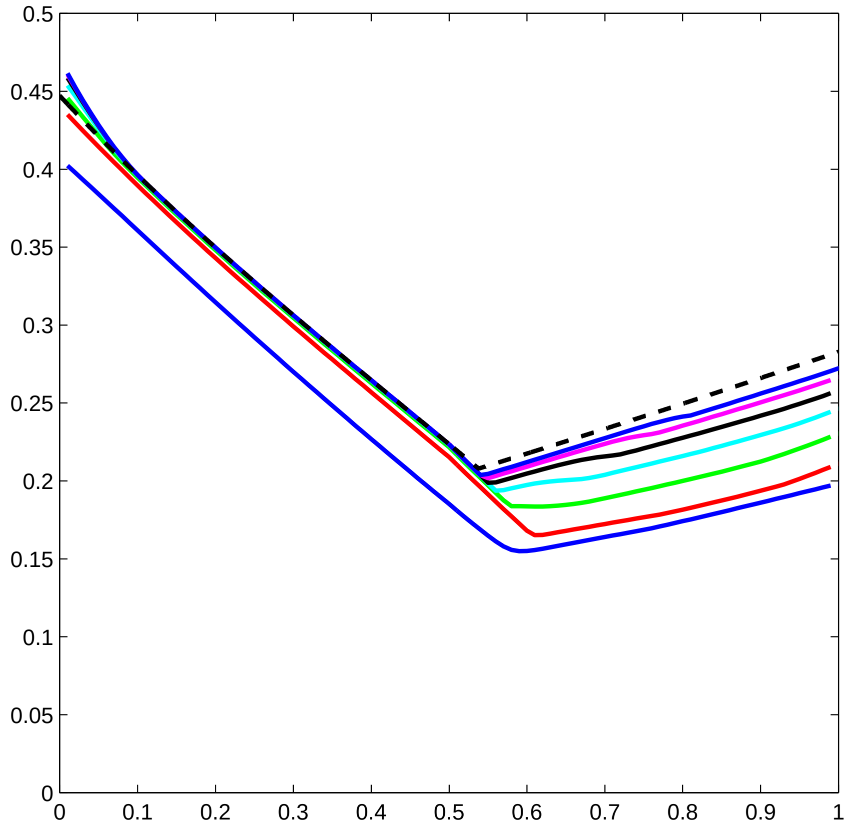
- $\|M\|_2 = \sqrt{|\lambda_1(M^*M)|}$
- for $P_h < 1$, matrix blocks have one ‘bad’ eigenvalue



$\sqrt{|\lambda(\mathcal{M}_1^*\mathcal{M}_1)|}$ for fixed $P_h = 0.38$

Alternative Bound?

- artificially 'remove' this eigenvalue: use $\sqrt{|\lambda_2(M^*M)|}$



- $P_h < 1$ only
- semiperiodic: $\|M^{per}\|_2$
- Dirichlet: $\sqrt{|\lambda_2(M^*M)|}$

Outlying eigenvalue

- in practice, the effect of this outlying eigenvalue is transient
- the eigenvector corresponding to the outlying eigenvalue is large only on grid lines very close to the inflow boundary
- after a few MG iterations, it is smooth and so is easily eliminated by coarse grid correction
- these effects do not have an impact on practical MG performance

MG Iteration Counts

- MG-like convergence for any value of P_h

	ϵ										
h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$
$\frac{1}{4}$	5	5	5	5	5	4	4	3	2	2	2
$\frac{1}{8}$	7	7	6	6	5	5	4	4	3	2	2
$\frac{1}{16}$	7	7	7	6	5	5	5	4	4	3	2
$\frac{1}{32}$	7	7	7	7	6	5	5	4	4	3	3
$\frac{1}{64}$	7	7	7	7	6	5	5	4	4	4	3
$\frac{1}{128}$	7	6	6	6	6	6	5	4	4	4	3

$P_h < 1$ $\qquad$ $P_h \geq 1$

Further Remarks

- Separate approximation and smoothing matrices:
  - semiperiodic analysis for smoothing matrix norm is representative of Dirichlet problem behaviour for all values of P_h,
  - semiperiodic analysis for approximation matrix norm is representative of Dirichlet problem behaviour for $P_h \geq 1$: for $P_h < 1$, one ‘bad’ eigenvalue again causes trouble.

- Replacing the Dirichlet condition by a Neumann condition on the outflow boundary leads to similar computational results.

Conclusions

- Linear algebra can be used to give a useful insight into convergence of two-grid iteration.
- We have obtained bounds on the multigrid convergence factor for a problem with semiperiodic boundary conditions.
- Boundary effects associated with a Dirichlet condition on the inflow boundary appear to be transient.
- Semiperiodic analysis gives an accurate description of MG behaviour for the full Dirichlet problem.