Chapter Twelve. Risk neutrality

Outline Solutions to odd-numbered exercises from the book:

*An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation*,
by Desmond J. Higham, Cambridge University Press, 2004
ISBN 0521 83884 3 (hardback)
ISBN 0521 54757 1 (paperback)

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12.1 We have

\[ W(S, t) = e^{-r(T-t)} \int_0^\infty \frac{\Lambda(x)}{x\sigma\sqrt{2\pi}} \frac{1}{\sqrt{T-t}} e^{- \frac{\{\log(x) - \log(S) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)\}^2}{2\sigma^2(T-t)}} \, dx \]

We will use the shorthand \{\cdot\} to denote \{\log(x) - \log(s) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)\}.

First, note that

\[ \frac{\partial}{\partial t} \left[ \frac{\{\cdot\}^2}{2\sigma^2(T-t)} \right] = \frac{2\sigma^2(T-t)2\{\cdot\}(r - \frac{1}{2}\sigma^2) + \{\cdot\}^22\sigma^2}{(2\sigma^2(T-t))^2} \]

\[ = \frac{(r - \frac{1}{2}\sigma^2)\{\cdot\}}{\sigma^2(T-t)} + \frac{\{\cdot\}^2}{2\sigma^2(T-t)^2} \]

Also,

\[ \frac{\partial}{\partial S} [\{\cdot\}^2] = -2\frac{\{\cdot\}}{S} \]

Hence

\[ \frac{\partial W}{\partial t} = rW + \frac{1}{2(T-t)}W - \frac{(r - \frac{1}{2}\sigma^2)\{\cdot\}W}{\sigma^2(T-t)} - \frac{\{\cdot\}^2W}{2\sigma^2(T-t)^2} \]

\[ \frac{\partial W}{\partial S} = \frac{2\{\cdot\}W}{S2\sigma^2(T-t)} = \frac{\{\cdot\}W}{S\sigma^2(T-t)} \]

\[ \frac{\partial^2 W}{\partial S^2} = \frac{\{\cdot\}^2}{S^2\sigma^4(T-t)^2} - \frac{\{\cdot\}W}{S^2\sigma^2(T-t)} - \frac{W}{S^2\sigma^2(T-t)^2} \]

So

\[ \frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + rS \frac{\partial W}{\partial S} - rW = \]

\[ \left[ r + \frac{1}{2(T-t)} - \frac{r\{\cdot\}}{\sigma^2(T-t)} + \frac{\{\cdot\}}{2(T-t)} - \frac{\{\cdot\}^2}{2\sigma^2(T-t)^2} + \frac{1}{2} \frac{\{\cdot\}^2}{\sigma^2(T-t)^2} \right] \]
Using the density function, (6.10), the probability that a European call option will be exercised is

\[
P(S(T) \geq E) = \int_E^\infty \exp\left(\frac{-\log(x/S) - (\mu - \sigma^2/2)T}{2\sigma^2 T}\right) \frac{1}{x\sigma\sqrt{2\pi}} \, dx.
\]

We obviously need to change variable. The trick is to spot that the lower limit of integration should be \(-d_2\). Hence, let

\[
y = \frac{\log(x/S) - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}.
\]

When \(x = E\), we have

\[
y = \frac{\log(E/S) - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}} = -d_2,
\]

where we used the risk-neutrality condition, \(\mu = r\). Also,

\[
\frac{dy}{dx} = \frac{1}{\sigma\sqrt{T}} \frac{1}{x}.
\]

The integral above then simplifies to

\[
\frac{1}{\sqrt{2\pi}} \int_{-d_2}^\infty e^{-\frac{1}{2}y^2} \, dy = 1 - N(-d_2) = N(d_2).
\]

To replicate the option, the portfolio must have value at time \(T\) given by \(\Lambda_{\text{up}}\) when \(S(T) = S_{\text{up}}\), and \(\Lambda_{\text{down}}\) when \(S(T) = S_{\text{down}}\). This gives two the equations

\[
AS_{\text{up}} + Ce^{rT} = \Lambda_{\text{up}}, \quad (1)
\]

\[
AS_{\text{down}} + Ce^{rT} = \Lambda_{\text{down}}. \quad (2)
\]

Subtracting (2) from (1) gives (12.5). Substituting this into (2) then gives (12.6). The price of the portfolio at time \(t = 0\) is therefore \(AS_S + C\); which, using (12.5)–(12.6), can be rearranged to (12.7). If the option is valued above this level, then at \(t = 0\) an arbitrageur could sell the option and buy the portfolio. This gives an instant, riskless, profit at \(t = 0\), because whichever of the two asset prices, \(S_{\text{up}}\) or \(S_{\text{down}}\), prevailed at expiry, the arbitrageur can pay off the option holder using the funds in the portfolio. The arbitrageur
has locked into a guaranteed, instantaneous, riskless profit, which violates the no arbitrage principle. We conclude that the time-zero option value cannot exceed (12.7).

An analogous argument using the words below, buy and sell shows that the time-zero option value cannot be less than (12.7), and hence (12.7) is the fair price.

Now, if \( q < 0 \) then \( S_0 e^{rT} < S_{\text{down}} \). This means that both of the possible asset values at expiry correspond to better performance than cash in the bank. Thus, there is an arbitrage opportunity: using loan from the bank to buy the asset (and selling the asset at expiry to replay the loan) guarantees a profit with no outlay. Hence \( q < 0 \) cannot hold. Similarly, if \( q > 1 \) then \( S_0 e^{rT} > S_{\text{up}} \). This means that both of the possible asset values at expiry correspond to worse performance than cash in the bank. Thus, there is an arbitrage opportunity: short selling the asset and investing the proceeds in the bank (and buying the asset with that cash at expiry to cover the short sale) guarantees a profit with no outlay. Hence \( q > 1 \) cannot hold. Thus, by the no arbitrage principle we have \( 0 < q < 1 \).

Using the definition of \( q \), we may rearrange (12.7) into the form

\[
e^{-rT} [(1 - q)\Lambda_{\text{down}} + q\Lambda_{\text{up}}],
\]

which is precisely the discounted expected payoff for an asset taking the values

\[
S(T) = S_{\text{up}} > S_0, \quad \text{with probability } q
\]
\[
S(T) = S_{\text{down}} < S_0, \quad \text{with probability } 1 - q.
\]

Although this question is based on an artificially simple scenario, a number of features ring bells from the Black–Scholes analysis.

1. The probability \( p \) does not affect the option value, just as the drift parameter \( \mu \) does not appear in the Black–Scholes PDE. (So, two investors who agree on the two possible asset values \( S_{\text{up}} \) and \( S_{\text{down}} \), but have wildly different views about the probability \( p \), will agree on the option value. An analogous statement for the Back–Scholes case appeared in Chapter 11.)

2. With a little imagination, the expression (12.5) for the asset holding can be likened to the delta value \( \partial C / \partial S \) that arose in the Black–Scholes hedging argument.

3. The time-zero option value is not simply the discounted expected payoff for the asset model, \( p\Lambda_{\text{up}} + (1 - p)\Lambda_{\text{down}} \). However, it is the discounted expected payoff for a different asset model that does not involve \( p \). This chapter showed that an analogous statement is true in the Black–Scholes case.