Chapter Eighteen. American options

Outline Solutions to odd-numbered exercises from the book:
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18.1 Let \( C^{Am}(S, t) \) denote the American call value. Arguing as in section 18.3, we find that

\[
C^{Am}(S, t) \geq \Lambda(S(t)), \quad \text{for all } 0 \leq t \leq T, \ S \geq 0, \tag{1}
\]

with \( \Lambda(S(t)) = \max(S(t) - E, 0) \). Then

\[
\frac{\partial C^{Am}}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C^{Am}}{\partial S^2} + rS \frac{\partial C^{Am}}{\partial S} - rC^{Am} \leq 0. \tag{2}
\]

Then for each \( S, t \) one of (1) or (2) is at equality. \( \tag{3} \)

Then

\[
C^{Am}(S, T) = \Lambda(S(T)), \quad \text{for all } S \geq 0, \tag{4}
\]

\[
C^{Am}(S, t) \to 0, \quad \text{as } S \to 0, \quad \text{for all } 0 \leq t \leq T. \tag{5}
\]

and

\[
C^{Am}(S, t) \approx S, \quad \text{as } S \to \infty, \quad \text{for all } 0 \leq t \leq T. \tag{6}
\]

Let’s see whether we can solve these conditions with \( C^{Am}(S, t) \equiv C(S, t) \), where \( C(S, t) \) is the European option value. We know from Section 10.4 that \( C(S, t) \) satisfies the Black–Scholes PDE, so (2) holds with equality. Hence, (3) is also true. We also know that \( C(S, t) \) satisfies (4), (5) and (6)—see Exercise 8.3. Now the time-zero result (2.4) that we proved in Chapter 2 may be generalized to a time \( t \) result \( C(S, t) \geq \max(S - E e^{-r(T-t)}, 0) \), which implies (1).

Since the European value, \( C(S, t) \), satisfies all the conditions needed for the American value, we must have \( C^{Am}(S, t) \equiv C(S, t) \).

18.3 At any time \( t \), it is clearly optimal not to exercise when \( S(T) > E \) (since exercising would give zero payoff.) Hence all points \( (S, t) \) with \( S > E \) must be above the exercise boundary, for all \( 0 \leq t \leq T \). At expiry it is clearly optimal to exercise if \( S(T) < E \). Hence all points \( (S, T) \) with \( S < E \) must
be below the exercise boundary. It follows that \((E, T)\) must be the location of the exercise boundary at expiry; that is, \(S^*(T) = E\).

To show that \(S^*(t)\) is a non-decreasing function of \(t\), it is equivalent to show the following.

For a fixed asset price \(\hat{S}\) and two times \(t_1\) and \(t_2\), with \(0 \leq t_1 < t_2 \leq T\), if it is optimal to exercise at \((\hat{S}, t_1)\) then it is optimal to exercise at \((\hat{S}, t_2)\). (In other words, if \((\hat{S}, t_1)\) is in the exercise region then \((\hat{S}, t_2)\) is in the exercise region.)

Suppose that it is optimal to exercise at \((\hat{S}, t_1)\). We can prove that it is therefore optimal to exercise at \((\hat{S}, t_2)\) via the following sequence of arguments.

1. We have \(P^{Am}(\hat{S}, t_1) \geq P^{Am}(\hat{S}, t_2)\); that is, the American put at time \(t_1\) with asset price \(\hat{S}\) is worth more than the American put at time \(t_2\) with asset price \(\hat{S}\). (This is because the former case gives all the opportunities of the latter, plus some extra opportunities.)

2. We have \(P^{Am}(\hat{S}, t_1) = E - \hat{S}\); that is, the value of the American put at time \(t_1\) and asset price \(\hat{S}\) is that given by exercising.

3. We have \(P^{Am}(\hat{S}, t_2) \leq E - \hat{S}\). This follows from points 1 and 2.

4. If the asset price is \(\hat{S}\) at time \(t_2\), then we can obtain the value \(E - \hat{S}\) by exercising at that time. From point 3 it must therefore be optimal to exercise in this circumstance.

18.5 Strategy 1 is OK, as it uses a stopping time. (Decision to exercise at time \(t^*\) uses only information about \(S(t)\) between \(0 \leq t \leq t^*\). Actually it only uses \(S(t^*)\).)

Strategy 2 is not OK, as it does not use a stopping time. (Decision to exercise at time \(t^*\) requires knowledge of \(S(t)\) at times beyond \(t^*\).)

Strategy 3 is OK, as it uses a stopping time. (Decision to exercise at time \(t^*\) uses only information about \(S(t)\) between \(0 \leq t \leq t^*\).)