20.1 From Section 6.5 we know that if \(a_M \sim N(a, b^2/M)\) then
\[
\left[ a_M - \frac{1.96b}{\sqrt{M}}, \ a_M + \frac{1.96b}{\sqrt{M}} \right]
\]
is a 95% confidence interval for \(a\). We have
\[
a = \left( r - \frac{\sigma^2}{2} \right) \Delta t \Rightarrow \left( r - \frac{\sigma^2}{2} \right) \frac{t^*}{M}
\]
and
\[
b^2 = \sigma^2 \frac{\Delta t}{M}; \text{ hence } b^2 = \sigma^2 \Delta t.
\]
So our 95% confidence interval is
\[
\left[ a_M - \frac{1.96\sigma\sqrt{t^*}}{M}, \ a_M + \frac{1.96\sigma\sqrt{t^*}}{M} \right].
\]
The length of the confidence interval is \(2 \times \frac{1.96\sigma\sqrt{t^*}}{M}\).

The quantity we are estimating, \(a\), is also proportional to \(1/M\). Hence, in this case the amount of uncertainty in the result is of the same order as the exact result itself.

20.3 We have \(E(Y) = E(\alpha + \beta Z) = \alpha + \beta E(Z) = \alpha\).

Hence,
\[
Y - E(Y) = \beta Z
\]
and
\[
\mathbb{V} \mathbb{A} \mathbb{R}(\mathbb{E}(Y) - E(Y))^2 = \mathbb{V} \mathbb{A} \mathbb{R}((\beta Z)^2) = \beta^4 \mathbb{V} \mathbb{A} \mathbb{R}(Z^2)
\]
Now
\[
\mathbb{V} \mathbb{A} \mathbb{R}(Z^2) = E(Z^4) - (E(Z^2))^2 = 3 - 1^2 = 2.
\]
Hence, \(\mathbb{V} \mathbb{A} \mathbb{R}((Y - E(Y))^2) = 2\beta^4\).

Since \(\hat{U}_i \sim \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma Z\), the result above applies with \(\beta = \sigma\) to give
\[
\mathbb{V} \mathbb{A} \mathbb{R}((\hat{U}_i - E(\hat{U}_i))^2) = 2\sigma^4.
\]
20.5 Let $\hat{U}_i = U_i / \sqrt{\Delta t}$.

Let $z = \sigma^2$. We want to maximise $G(z) := \prod_{i=1}^{M} \frac{1}{\sqrt{2\pi z}} \exp \left( -\frac{\hat{U}_{n+1-i}^2}{2z} \right)$ for $z \geq 0$. Equiv. to maximising

$$\log G(z) = -\frac{1}{2} \left[ M \log z + \sum_{i=1}^{M} \frac{\hat{U}_{n+1-i}^2}{z} \right] + \text{constant}$$

Equiv. to minimising $F(z) := M \log z + \sum_{i=1}^{M} \frac{\hat{U}_{n+1-i}^2}{z}$.

Well $\frac{dF}{dz} = \frac{M}{z} - \sum_{i=1}^{M} \frac{\hat{U}_{n+1-i}^2}{z^2}$, so $\frac{dF}{dz} = 0$ when $z = \frac{1}{M} \sum_{i=1}^{M} \hat{U}_{n+1-i}^2$.

Since $F(z) \to \infty$ when $z \to 0$ and when $z \to \infty$, this is the global minimum required.

20.7 Going back as far as $t_0 = t_{n+1} - (n + 1)\Delta t$, we have

$$\Delta t \sigma_{n+1}^2 = \omega \Delta t \sigma_n^2 + (1 - \omega)U_{n+1}^2$$

$$= \omega [\omega \Delta t \sigma_{n-1}^2 + (1 - \omega)U_n^2] + (1 - \omega)U_{n+1}^2$$

$$= \omega [\omega [\Delta t \sigma_{n-2}^2 + (1 - \omega)U_{n-1}^2] + (1 - \omega)U_n^2] + (1 - \omega)U_{n+1}^2$$

$$\vdots$$

$$= \omega^{n+1} \Delta t \sigma_0^2 + \sum_{i=0}^{n} (1 - \omega)\omega^i U_{n+1-i}^2$$

First term on RHS is exponentially small. Second term has geometrically declining weights. (But note that $M$ is not fixed.)