23.1 We have
\[ \Delta \nabla y_m = \Delta (y_m - y_{m-1}) = y_{m+1} - y_m - (y_m - y_{m-1}) = y_{m+1} - 2y_m + y_{m-1} \]
and
\[ \nabla \Delta y_m = \nabla (y_{m+1} - y_m) = y_{m+1} - y_m - (y_m - y_{m-1}) = y_{m+1} - 2y_m + y_{m-1}. \]
So \( \Delta \nabla = \nabla \Delta. \)

Also,
\[ (\Delta - \nabla)(y_m) = (y_{m+1} - y_m) - (y_m - y_{m-1}) = y_{m+1} - 2y_m + y_{m-1}. \]
So \( \Delta \nabla = \Delta - \nabla. \)

Also,
\[ \delta^2 y_m = \delta(\delta y_m) = \delta(y_{m+\frac{1}{2}} - y_{m-\frac{1}{2}}) = y_{m+1} - y_m - (y_m - y_{m-1}) = y_{m+1} - 2y_m + y_{m-1}. \]
So \( \delta^2 = \Delta \nabla. \)

Also,
\[ \mu \delta y_m = \mu(\delta y_m) = \mu(y_{m+\frac{1}{2}} - y_{m-\frac{1}{2}}) = \frac{1}{2}(y_{m+1} - y_m + (y_m - y_{m-1})) = \frac{1}{2}(y_m - y_{m-1}). \]
So \( \mu \delta = \Delta_0. \)

Similarly,
\[ \delta \mu y_m = \delta(\mu y_m) = \delta(\frac{1}{2}(y_{m+\frac{1}{2}} + y_{m-\frac{1}{2}})) = \frac{1}{2}(y_{m+1} + y_m - (y_m + y_{m-1})) = \frac{1}{2}(y_m - y_{m-1}). \]
So \( \delta \mu = \Delta_0. \)

Also,
\[ \Delta^2 y_m = \Delta(\Delta y_m) = \Delta(y_{m+1} - y_{m-1}) = y_{m+2} - y_{m+1} - (y_{m+1} - y_m) = y_{m+2} - 2y_{m+1} + y_m. \]
and
\[ \delta^2 E y_m = \delta^2 (E y_m) = \delta^2 (y_{m+1}) = y_{m+2} - 2y_{m+1} + y_m. \]
So \( \Delta^2 = \delta^2 E \).

Further,
\[ E \delta^2 y_m = E(\delta^2 y_m) = E(y_{m+1} - 2y_m + y_{m-1}) = y_{m+2} - 2y_{m+1} + y_m. \]
So \( \Delta^2 = E \delta^2 \).

**23.3** First row of (23.9):
\[ U_{1}^{i+1} = (1 - 2\nu)U_{1}^{i} + \nu U_{2}^{i} + p_{1}^{i} = (1 - 2\nu)U_{1}^{i} + \nu U_{2}^{i} + \nu a(ik). \]
Generally,
\[ U_{j}^{i+1} = (1 - 2\nu)U_{j}^{i} + \nu U_{j+1}^{i} + U_{j-1}^{i}. \]

Last row of (23.9):
\[ U_{N_{x}-1}^{i+1} = (1 - 2\nu)U_{N_{x}-1}^{i} + \nu U_{N_{x}-2}^{i} + p_{N_{x}-1}^{i} = (1 - 2\nu)U_{N_{x}-1}^{i} + \nu U_{N_{x}-2}^{i} + \nu b(ik). \]
Hence, the formulation is correct.

**23.5** Following the FTCS analysis, we have for BTCS
\[ R_{j}^{i} = \left( \frac{\partial u}{\partial t} - \frac{4}{k} \frac{\partial^2 u}{\partial t^2} + O(k^2) \right) - \left( \frac{\partial^2 u}{\partial t^2} - \frac{1}{12} h^2 \frac{\partial^4 u}{\partial x^4} + O(h^4) \right). \]
Since \( u \) satisfies the PDE (23.2), we have
\[ R_{j}^{i} = -\frac{1}{k} \frac{\partial^2 u}{\partial t^2} - \frac{1}{12} h^2 \frac{\partial^4 u}{\partial x^4} + O(k^2) + O(h^4). \]

**23.7** Expanding the equation in the exercise gives
\[ U_{j}^{i+1} - \frac{1}{2} \nu [U_{j+1}^{i+1} - 2U_{j}^{i+1} + U_{j-1}^{i+1}] = U_{j}^{i} + \frac{1}{2} \nu [U_{j+1}^{i} - 2U_{j}^{i} + U_{j-1}^{i}], \]
which rearranges to
\[ (1 + \nu)U_{j}^{i+1} = \frac{1}{2} \nu U_{j+1}^{i+1} + \frac{1}{2} \nu U_{j-1}^{i+1} + (1 - \nu)U_{j}^{i} + \frac{1}{2} \nu U_{j+1}^{i} + \frac{1}{2} \nu U_{j-1}^{i}. \]
Multiplying by 2 gives (23.18).

**23.9** General row of (23.19) gives
\[ (1 + \nu)U_{j}^{i+1} - \frac{1}{2} \nu U_{j+1}^{i+1} - \frac{1}{2} \nu U_{j-1}^{i+1} = (1 - \nu)U_{j}^{i} + \frac{1}{2} \nu U_{j+1}^{i} + \frac{1}{2} \nu U_{j-1}^{i}, \]
which agrees with (23.18). The vector \( r^{i} \) has \( r_{1}^{i} = \frac{1}{2} \nu [U_{0}^{i} + U_{0}^{i+1}] \) and \( r_{N_{x}-1}^{i} = \frac{1}{2} \nu [U_{N_{x}}^{i} + U_{N_{x}}^{i+1}] \), as required.

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23.11 General row of equation in exercise is

\[ \frac{1}{2} U_{j}^{i+1} + \frac{1}{2} \left\{ (1 + 2\nu) U_{j}^{i+1} - \nu U_{j-1}^{i+1} - \nu U_{j+1}^{i+1} \right\} = \frac{1}{2} U_{j}^{i} + \frac{1}{2} \left\{ (1 - 2\nu) U_{j}^{i} + \nu U_{j-1}^{i} + \nu U_{j+1}^{i} \right\}. \]

This rearranges to

\[ (1 + \nu) U_{j}^{i+1} = \frac{1}{2} \nu U_{j+1}^{i+1} + \frac{1}{2} \nu U_{j-1}^{i+1} + (1 - \nu) U_{j}^{i} + \frac{1}{2} \nu U_{j-1}^{i} + \frac{1}{2} \nu U_{j+1}^{i}, \]

which is equivalent to (23.18). Also, \( \frac{1}{2} (p^i + q^i)_1 = r_1^i \) and \( \frac{1}{2} (p^i + q^i)_{N_x-1} = r^i_{N_x-1} \), as required.