Construction, transition, demolition?

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The mathematics transition problem

Two important events occurred in October 1995.

- *There is unprecedented concern amongst mathematicians, scientists and engineers in higher education about the mathematical preparedness of new undergraduates.*

[LMS, *Tackling the Mathematics Problem*, 1995]

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  [LMS, *Tackling the Mathematics Problem*, 1995]

- I became a new undergraduate, studying maths.

Three problem areas: [LMS, 1995; cf. Gibson *et al.*, 2005]

- lack of technical fluency;
- decline in "analytical powers";
- failure to appreciate the nature of mathematics.

The mathematics transition problem: explanations

"School teaching or the school system is at fault."

- Popular among university maths teachers.
- It's remarkably hard to pinpoint what is at fault.

"University teaching or the university system is at fault."

- Criticism tends to focus on didactic style in HE.
- Scale and function: university can't work like school.

"The students aren't good enough to be here."

- Not especially constructive...

The mathematics transition problem: explanations

We tend to talk about transition problems as an **anomaly**:

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Alternative: problems exposed by transition:

- are **normal** for most students;
- may be **intrinsic** to how we learn maths.

The "construction" metaphor for learning

Constructivism: learners construct their own knowledge and understanding from their experiences.

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Useful reminder that:

- learning involves a largely hidden process of sense-making (**assimilation** and **accommodation**);
- learners construct an (imperfect) **model** which corresponds in some sense to what we intend.

The "construction" metaphor for learning

Many universities have a building like this...

- Built over a century ago for a very different purpose.
- Walls knocked through; rooms partitioned; floors ripped out; mezzanines installed...
- Occasional doorways that go nowhere.
- Shiny new lab-and-office block tacked onto one corner. The floors don't quite line up.
- Locked cupboard on Floor 6 is full of coprolites.
- Every room is slightly the wrong size and shape.
- Remains of a mocked-up ship's bridge on the roof.
- Wiring diagrams lost in the Great Refiling of 1962.
- Don't go into the cellar.

The "construction" metaphor for learning

Our mathematical knowledge resembles that building.

- **Mixed materials**: formal/informal definitions; examples; images; procedures; heuristics; emotions/motivations...

  [Tall & Vinner, 1981; Schoenfeld, 1985]

- Learning doesn't follow logic (**didactic inversion**) e.g. we learn number before Peano's axioms. [Burn, 2002]
- Can't compare my mental model directly with yours: learning is **underconstrained**.
- Accommodation is **harder** than assimilation: sensible bias in favour of DIY not rebuilding.
- Emotional and cognitive **commitment** to existing model.

NB: this also applies to (communal) development of mathematics. [Lakatos, 1976; Thurston, 1994]

The "construction" metaphor: infrastructure

Some features of infrastructure: [Star & Ruhleder, 1996]

- **embedded** in other structures;
- **transparent** (invisibly supports other tasks);
- **reaches** beyond a single event or practice;
- **taken for granted** by and linked to **conventions** of a community;
- embodies **standards**;
- built on an **existing base**;
- **becomes visible when it goes wrong**.

Compare: *... by the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play the higher faculties of the brain.*

[Whitehead, 1911]

The "construction" metaphor: cruft

During construction and reconstruction, some stuff **inevitably** gets messed up.

- The obvious stuff is probably the easiest to fix
  — gaps, misunderstood terminology etc.
- Much of the rest is infrastructural
  — bodges exposed by change of use (transition).

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Proposal: our main task in first year is to help students **dismantle** and **rebuild** their mathematical knowledge.

Bigger question: how can we help students to build mathematical knowledge that can **adapt** and can mature at **multiple speeds**? [cf. Brand, 1993]

"Mathematics is about implementing procedures"

This is where most of us started mathematics:

- addition, subtraction, multiplication;
- long division ("bus stop sums");
- then basic algebra...

Mathematics as **syntax** comes later.

Mathematical **reasoning** and **algebra** are often seen as separate. [Healy & Hoyles, 2000]

NB: implementing procedures can be very **satisfying**! E.g.:

- reading music;
- Scottish country dancing.

"Algebra is like Lego"

Have you ever said any of the following?

- "Rearrange the equation."
- "Take the $2x$ to the other side."
- "Hang on, I think we've lost a minus sign."
- "Where's the π gone?"

Have you ever written something like this?

$$
\begin{aligned}
(x-1)^2 + 2 &= 2x^2 - 3 \\
x^2 - 2x + 1 + 2 &= 2x^2 - 3 \\
-x^2 - 2x + 6 &= 0
\end{aligned}
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When calculus is assimilated into Lego algebra

How do we read this?

$$\frac{d}{dx}\ x^2 - 3x + 2 \Rightarrow 2x - 3$$

Expert: "The derivative of $x^2 - 3x + 2$ implies $2x - 3$"? ☹

Student: "Differentiate $x^2 - 3x + 2$ and we get $2x - 3$." ☺

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- E expects **statements** and logical **connections**.
- S reads $\implies$ as "we get" (assimilated from algebra) and $\dfrac{\mathrm{d}}{\mathrm{d}x}$ as an imperative (assimilated from procedures).

For the student, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is an **instruction**.

Quadratic equations

Instruction: "Solve the equation $x^2 + 3x + 2 = 0$."

We might write something like this:

$$
\begin{aligned}
& x^2 + 3x + 2 = 0 \\
\Leftrightarrow\ & (x+2)(x+1) = 0 \\
\Leftrightarrow\ & x+2 = 0 \quad \text{or} \quad x+1 = 0 \\
\Leftrightarrow\ & x = -2 \quad \text{or} \quad x = -1.
\end{aligned}
$$

Quadratic equations

Instruction: "Solve the equation $x^2 + 3x + 2 = 0$."

A student might write it (or read it) like this:

$$x^2 + 3x + 2 = 0$$

$$\Rightarrow (x+2)(x+1) = 0$$

$$\Rightarrow x = -2, \quad x = -1$$

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- The "algebra" is present and correct.
- The logical structure is (at best) implicit.
- "Factorise $x^2 + 3x + 2$" would get the same response.

Quadratic equations

Instruction: "Solve the equation $x^2 - x - 1 = 0$."

$$x^2 - x - 1 = 0$$

$$a = 1 \qquad b = -1 \qquad c = -1$$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4.1.(-1)}}{2.1}$$

$$= \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

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- Failure to factorise activates a different procedure.
- The $\pm$ sign conceals the logical "or".
- "Factorise $x^2 - x - 1$" would baffle many students.

Quadratic inequalities

Instruction: "Solve the inequality $x^2 + 3x + 2 \geq 0$."

$$x^2 + 3x + 2 \geqslant 0$$

$$(x+2)(x+1) \geqslant 0$$

$$x \geqslant -2 \quad x \geqslant -1$$

Quadratic inequalities

Instruction: "Solve the inequality $x^2 + 3x + 2 \geq 0$."

$$x^2 + 3x + 2 \geq 0$$

$$a = 1 \quad b = 3 \quad c = 2$$

$$x \geq \frac{-3 \pm \sqrt{9 - 4 \times 1 \times 2}}{2.1}$$

$$\geq \frac{-3 \pm 1}{2} \qquad \geq -2, -1$$

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- The change from equation to inequality exposes the underlying lack of syntax.

Quadratic inequalities

Instruction: "Solve the inequality $x^2 + 3x + 2 \geq 0$."

We now teach them an efficient method:

$$x^2 + 3x + 2 \geqslant 0$$

$$(x+2)(x+1) \geqslant 0$$

x	-3	-2	$-\frac{3}{2}$	-1	0
$x+2$	$-$	0	$+$	$+$	$+$
$x+1$	$-$	$-$	$-$	0	$+$
	$+$	0	$-$	0	$+$

$$x \leqslant -2 \quad \text{or} \quad x \geqslant -1$$

- A new procedure conceals the lack of syntax.

A thought experiment

Imagine a native English speaker who knows no Chinese locked in a room full of boxes of Chinese symbols, together with a book of instructions for manipulating the symbols.

People outside the room send in other Chinese symbols which, unknown to the man in the room, are questions in Chinese. By following the instructions, he is able to pass out Chinese symbols which are correct answers to the questions.

The program enables the man to pass the Turing Test for understanding Chinese but he does not understand a word of Chinese.

[after Searle, 1999]

Ways of unlearning?

To unlearn / reorganise our understanding we need:

- sufficient **cognitive** pressure to change;
- but not too much **affective** pressure

This is not easy...

Ways of unlearning?

The cognitive pressure is (relatively) easy to supply.

- Questions that **violate expectations** or defy short-cuts:
  - problems with extraneous/inadequate information;
  - "messy numbers" as final answers. [Ruesser, 1988]
- Practice **reading and writing** mathematical arguments:
  - self-explanation and group/paired explanation;
  - proof-appraisal ("good, bad, or ugly?") exercises;
  - pair-and-swap writing exercises.

  [e.g. Zerr & Zerr 2011; Inglis & Alcock, 2012; Hodds *et al.*, 2014]

  NB: has to be reinforced in regular marking etc.
- Don't always teach the most **efficient** methods.

“That’s not fair!”

A useful idea: **didactic contracts**: [e.g. Pepin, 2014]

- implicitly “negotiated” between students and teachers;
- shape expectations and override explicit instructions.

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A useful idea: **didactic contracts**: [e.g. Pepin, 2014]

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- shape expectations and override explicit instructions.

Students reach us with a variety of contracts:

- inherited from school or college;
- formed during early weeks of university.

Many of these contracts are malfunctional.

Didactic contracts: malfunction

One contract begins with "I was good at maths at school."

- Maths is about carrying out **procedures** efficiently.
- Motivation depends on regular **affirmation** of success.
- Teaching is about providing **templates** for assessments.

[cf. Alcock, 2013; Pepin, 2014]

A common consequence:
The teacher ends up choosing a situation that enables the student's system of knowledge to furnish the desired response.

[Brousseau & Warfield, 1981/1999]

Worse still: "pseudo-" behaviours: [Vinner, 1997]

- navigating by vague association (**pseudo-conceptual**);
- applying templates unthinkingly (**pseudo-analytical**).

Didactic contracts: breach of contract

What happens when students' and lecturers' expectations don't coincide? All too often:

- **anxiety**, avoidance, and self-handicapping;
- stubborn persistence with **ineffective strategies**;
- a sense of **betrayal**; rebellion and revenge.

[Rhodewalt & Vohs 2005; Rodd 2009]

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[Rhodewalt & Vohs 2005; Rodd 2009]

Some symptoms (complaints from student feedback):

- "The lecturer makes maths seem academic."
- "He asked questions that we hadn't seen before."
- "Time was spent on examples not needed in the exam."
- "Proof was interesting but it was more like literature than mathematics."

Ways forward?

In a perfect world...

- Less emphasis on refreshing; more on **rebuilding**.
- Allow **time** for accommodation.
- Consider the culture of mathematics implied by tasks:
  - technical **fluency** is important but not all-important;
  - make space for **legitimate peripheral participation**.

[Solomon, 2007]

- Emphasise **low-stakes but highly-challenging** activities rather than high-stakes but less challenging activities.
- Advertise the **provisional** nature of knowledge — needs to be open for maintenance and replacement!

[Adams, 1992]

Summary

Mathematical knowledge is inevitably **self-constructed**, **complex**, continually **changing**, and **ramshackle**.

Changing our hidden (infrastructural) ideas is:

- **essential** for continued learning;
- **difficult**, time-consuming, and sometimes painful.

As teachers we need to:

- look for the signs of **defective infrastructure**;
- provide both **cognitive pressure** and **social/affective room** to change;
- **plan** for the misconceptions that our students' next teacher will want them to correct!

A closing thought

It is some comfort in the confusions & puzzles one makes, that they are always exceedingly amusing to me, after they are cleared away. And this is at least some compensation for the plague of them before.

With many thanks,
Yours most truly

A. A. Lovelace

[Letter to A. de Morgan, 11 July 1841]

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No, I wasn't making it up

The "Deviascope" on the roof of the Royal Technical College, Glasgow, 1913.

[University of Strathclyde Archives]

No, I wasn't making it up

The coprolite cupboard on Level 6 of the Royal College Building.