Hand in solutions to nos. 4, 5 and 6 at 4PM on 5 February. Follow the instructions for each.

Write names of your collaborators at the bottom of the cover sheet

Write your tutorial room in the top right corner of cover sheet

The probability of an event in a given situation is the number of outcomes we consider to be occurrences of that event, divided by the number of all outcomes possible (assuming that all cases are equally likely). Note that this is always a number p such that $0 \leq p \leq 1$.

When flipping a coin there are two possible outcomes, heads or tails. There is only one outcome of the type "heads", so, with a fair coin, the probability of getting heads is $1/2$.

In an ordinary deck of 52 cards there are 13 spades. So the probability of a randomly chosen card being a spade is $13/52 = 1/4 = 0.25$.

If two ordinary dice are thrown, one after the other, there are $6 \cdot 6 = 36$ possible outcomes. There are three ways to get the sum 10, namely 4+6, 5+5, and 6+4. So the probability that they sum to 10 is $3/36 = 1/12 \simeq 0.0833$.

Problems 1, 2, and 3 below should be rather easy warmups. The answers to them are as follows:

1: $1/4$, $1/2$, $1/2$

2: $1/6$, $1/6$, $2/3$, $1/3$

3: $2/5$, $3/5$, 1, $3/5 \cdot 2/4$, $2/5 \cdot 1/4$, $1 - (3/5 \cdot 2/4 + 2/5 \cdot 1/4)$, $2/5 \cdot 3/4$

Before you tackle the harder poblems make sure you *understand* the answers above.

1. A fair coin is flipped twice. What is the probability that
   (a) it comes up heads both times?
   (b) the two outcomes are different?
   (c) that it comes up heads on the second flip?
2. An ordinary die is thrown. What is the probability that
   (a) the outcome is 3?
   (b) the outcome is 6?
   (c) the outcome is a number smaller than 5?
   (d) the outcome is a number at least as large as 5?
3. A box contains 2 white balls and 3 black ones.
   (a) One ball is picked at random. What is the probability that it is white? That it is black? What is the sum of the two answers? Why?
   (b) Two balls are picked at random, one after the other, without putting the first one back. What is the probability that
      i. both are black?
      ii. both are white?
      iii. one is white, the other black?
      iv. the first one is white, the second one black?

4. **Hand-in:** Two ordinary dice are cast, one after the other. What is the probability of getting

Each problem is worth 10 points

   (a) A 4 on both?
   (b) A 4 on the first, a 5 on the second?
   (c) A 4 on one, a 5 on the other?
   (d) A sum of 6?
   (e) A sum of 9 or more?

   Give your solutions both in exact form (as fractions reduced to lowest terms) and as a rounded decimal number with three decimal places. Also, you must *explain* in a few words *how* you got each answer. (This applies to *all* problems, *always*.)

   Read all instructions carefully. You will not get full credit unless you follow them in detail.

   At the end of your solution, *after* you explain in text how you got each answer, list your answers to each part, one on each line. For example, if your answer to part (a) is 14/36, which is 7/18 in lowest terms, and equal to 0.389 . . ., you should write

   (a) $\frac{7}{18} \approx 0.389$

5. **Hand-in:** Two people play the following game: On the table we have twenty cards, numbered $1, 2, 3, \ldots, 20$. In a turn, a player can remove any set of odd numbered cards or any set of even numbered cards (but not both odd and even cards on the same turn), and must remove at least one card. For example, you can remove $2, 6, 10, 12, 18$ in a move, or $7, 11$, but not $4, 7, 10$. The player who removes the last card wins.

   (a) Who wins, the first or second player, and how? 7pts
   (b) What if we start with cards numbered $1, 2, 3, \ldots, 23$? 3pts

   Important instructions: Write a clear and concise solution, with a convincing argument. Solutions will be marked on the clarity of the text, in addition to their correctness. Thus, an essentially correct solution is not enough to get full marks, if it is not efficiently explained. See the info on Combinatorial Games overleaf.

6. **Hand-in:** Input your nine digit registration number at a website you can find from the course webpage[1] to obtain the numbers a, b, c. In a bag we have three coins. Their respective probabilities of coming up heads, are $a/11$, $b/11$ and $c/11$.

This number begins with 2017 if that's the year you started at Strathclyde

   (1) If you randomly pick one of these coins and flip it, what is the probability that it will come up heads?
   (2) If you randomly pick two of these coins and flip both, what is the probability that at least one of them will come up tails? Draw a tree to represent the possible outcomes and use this to compute your answer.

   After your explanations for how you got your solution, write, at the very end of your solution, your registration number, and the numbers a, b, c. For example, if your number is 201312345 you would get $a = 5$, $b = 9$, $c = 2$, and you should write something like this (assuming these were your answers to the questions):

Important instructions

   201312345, $a = 5$, $b = 9$, $c = 2$,

   1: $735/1234 \approx 0.596$ 2: $157/1234 \approx 0.127$

More problems, including a bonus problem, on following pages

[1] https://personal.cis.strath.ac.uk/einar.steingrimsson/cs110/

A BONUS PROBLEM

7. **(No collaboration allowed; only your own thinking in isolation)** Two companies, X and Y, both employ two types of people, A-types and B-types. Both X and Y have only two kinds of salaries, low and high. In each of X and Y a higher proportion of A-types than B-types is on high salaries. Does it follow that when X and Y are taken together, there is a higher proportion of A-types than B-types on high salaries?
If the answer is yes, explain why that must be so. If the answer is no, give an example demonstrating that.

By handing in a solution to a bonus problem you are declaring that you have not collaborated with anybody on it, and not used anything except your own thinking to solve it.

	A	B
Hi	25	50
Lo	40	90

ANALYSING SYMMETRIC COMBINATORIAL GAMES

A *finite symmetric combinatorial game* is a game, guaranteed to end in a finite number of moves, played by two players who take turns making moves, where in any position the allowed moves do not depend on whose turn it is (which rules out chess and Tic-tac-toe, for example).

To find a strategy for winning such a game, you need to do a few things:

• Identify a set of *final positions*, in which no more moves can be made.

• Describe a set of *losing positions* ($\mathcal{L}$), which must contain all the final positions, and a set of *winning positions* ($\mathcal{W}$). The sets $\mathcal{L}$ and $\mathcal{W}$ must have the following properties:

1. Every position in the game belongs to either $\mathcal{L}$ or $\mathcal{W}$, but not both.
2. From any non-final position in $\mathcal{L}$, *every* move leads to a position in $\mathcal{W}$.
3. From any position in $\mathcal{W}$, there is *some* move that leads to a position in $\mathcal{L}$.

Once you have described the sets $\mathcal{L}$ and $\mathcal{W}$, and shown that they have the properties in points 1-3 above, to complete a strategy for the game you need to explain how to find, given any position in $\mathcal{W}$, a move to a position in $\mathcal{L}$.

MORE PROBLEMS

8. A fair coin is flipped three times. What is the probability that we get
   (a) all heads?
   (b) heads, heads, tails (in this order)?
   (c) two heads and one tails (in any order)?
   (d) an even number of tails? (zero is an even number)
   (e) at least two heads?

   Draw a tree to represent the possible outcomes and use this to compute each answer. List your answers at the end as in Problem 4, but first *explain* each one briefly.

9. In a bag, you have three coins. One is fair, one has two tails, and the third one has probability 2/3 of giving a head. If you randomly pick a coin from the bag and flip it, what's the probability you will get a head?

10. Poker is played with a regular deck of 52 cards: 13 ranks in 4 suits. A poker hand consists of 5 cards. What is the probability of being dealt

    (a) four of a kind?

    (b) three of a kind? (three cards of the same rank, plus two cards which are not of this rank nor the same as each other)

    (c) a flush? (five cards of the same suit, but with ranks not in sequence)

    (d) does not contain two cards of the same rank?

    (e) contains exactly two cards of the same rank? (in other words, the cards on your hand have exactly four different ranks)

If you play against a perfect player, knowing probability won't make you rich. But, it can make you less poor.

11. You are standing ten feet away from a bottle perched on a fence post. You have a stone and a shoe that you can throw at it. Your probability of hitting with the stone is 1/3 and your probability of hitting with the shoe is 2/5.

    (a) How likely is it that you will hit the bottle if you get to throw both things?

    (b) Does is matter whether you first throw the stone or the shoe?

Your hitting is so poor because you're shoeless out in the fields and you've drunk the contents of the bottle

12. Look at the expansion of $(a + b)^3$. What does this say about the probabilities of having three girls, two girls, etc. in a family with three children?

13. If we expand $(a + b)^3$ *without* letting a and b commute, how are the terms in this expansion related to the order in which children are born in a family with three children?

14. Is there anything special about 3 in the previous two exercises, or does this generalize to $(a + b)^n$ for any n?

15. If a sibling group with five children is chosen randomly, what is the probability that there will be an odd number of boys? How could you prove that without doing any computations?

16. In opinion polls one often tries to get ca. 1000 respondents. If the same number of voters in the UK were for and against leaving the EU, what is the probability that that would be the exact result with 1000 respondents in a perfectly executed opinion poll? What is the probability that the result would be less than one percentage point away from 50%? Two percentage points?

17. Suppose I offer to play the following game with you a hundred times: I roll 4 dice. If I get at least one 6 you pay me £100. Otherwise I pay you £100.

    (a) If you play, are you more likely to win or lose money?

    (b) How much do you expect to win/lose in a hundred games?

    (c) Suppose we change the rules so that if I lose I pay you £200, but you still pay me only £100 if I win. How many rolls should I demand to get in each turn in order not to lose money?

18. A number is a *palindrome* if it is the same when read forward and backward. For example, 414 and 85633658 are palindromes. What is the probability that a randomly chosen decimal number with $2n$ digits is a palindrome? What about $2n + 1$ digits? Note that a number cannot start with the digit 0. (Make sure your answer is correct in the cases $n = 0$ and $n = 1$)

19. Here is a relatively easy problem that several professors of mathematics made fools of themselves trying to answer: There are three doors, with a car behind one, and goats behind the other two. You get whatever is behind the door you decide to open. You pick a door (without opening it). I open one of the other two doors, and there is a goat behind it. I now ask you whether you want to stick to the door you first picked, or switch to the other closed door. Assuming you prefer a car to a goat, should you switch?

20. A hard problem (with a simple elegant solution, if you find it (it can also be solved by brute force, but that will be quite brutal)): A movie theatre has a hundred numbered seats and tickets are sold to specific seats. Tonight it is sold out. The first person to arrive has lost her ticket, but we are nice and let her go in. She doesn't remember her seat number and picks a random seat. Each subsequent guest takes his or her assigned seat if it is available, but otherwise a random free seat. What is the probability that the last guest to enter finds his or her seat free?

21. Assume that when a child is born it is as likely to be a girl as a boy. Do men or women, on average, have more sisters?