Hand in solutions to problems 1, 2 and 3 by 4PM on 26 February. Follow instructions carefully!

Write your tutorial room in the top right corner of the cover sheet, collaborators at the bottom.

1. A regular deck of 52 playing cards has 13 ranks in 4 suits. The ranks of Jack, Queen, King and Ace of each suit are *top cards*. Suppose you are randomly dealt seven cards. What is the probability of getting

   (a) Three top cards and four non-top cards?

   (b) Three top cards in the same suit and any four cards in another suit (but all four in one suit)?

   (c) Three top cards *not all* in the same suit, and any four non-top cards?

   (d) Seven non-top cards all in the same suit? (E.g: $2, 4, 5, 6, 7, 8, 10$, all in spades)

   (e) Two pairs of top cards of different ranks (e.g. Q-Q, A-A) and any three cards not containing a pair, whose ranks are different from those of both pairs?

   As always, write *coherent text* that explains carefully how you got your answers. At the end of your solution to this problem, write a list with the answers from each of the five parts, *one answer per line*. Give both an exact answer as a fraction with integers (including binomial coefficients) and a *rounded* approximation to at least 2 decimal places after the inital string of 0s. For example, an answer could look like this:

   Very important: Follow the instructions! See footnote

   $$\text{(c):} \qquad \frac{\binom{13}{3} \cdot \binom{4}{3}}{\binom{52}{6}} \approx 0.000056$$

2. On the course website, input your nine digit registration number to get the numbers P, W and L for this problem. A coin has probability P of coming up heads. Suppose you play a game where you flip this coin until you have either gotten 3 heads or 3 tails. You win if you get 3 heads before getting 3 tails.

   (a) What is your probability of winning? *8pts*

   (b) Suppose you get W pounds if you win, but otherwise lose L pounds, What is your expected net outcome? *2pts*

   You have to do this "from scratch"; you cannot use formulas from others (unless you first prove them), only things we have gone through in the course so far.

   Do NOT use the binomial formula!

   For each part give the answer as a correctly rounded approximation with 7 digits after the decimal point[1]. Also explain your reasoning carefully, in addition to showing your calculations. At the end of your solution of this problem, *write your Reg. number, your P, W and L, thus*: 201712345, $P = 0.432, W = 543, L = 456$, *and list each of your answers, one per line.*

   Important instructions

3. Two players take turns making moves in the following game: There are M matches on a table (get M on website as in Problem 2). In a move, a player can remove 1, 2, or 3 matches. The last person to take a match wins. If both play optimally, who wins, the first or second player?
   Write your number M at the beginning of your solution.

   Important instructions: Write a clear and concise solution, with a convincing argument. Solutions will be marked partly on the clarity of the text, in addition to their correctness. Thus, an essentially correct solution is not enough to get full marks, if it is not efficiently explained. See the info on Combinatorial Games overleaf.

   Describe final, winning and losing positions

[1] www.wolframalpha.com has powerful calculators. But you are responsible for the accuracy of your calculations.

A BONUS PROBLEM

4. **(No collaboration allowed, no use of external resources)**

   Redo problem 2a with the following modification: You need 4 heads to win, but lose at 4 tails. At each stage, if you have obtained at least as many tails as heads so far, your probability of getting heads on the next flip is $(1 - P)$, but otherwise the probability is P.
   For example, if you have gotten 2 tails and 1 heads, or 2 of each, your probability is $(1 - P)$, but if you have 1 tails and 2 heads the probability is P.

   Can you explain the outcome and solve this for any n instead of 4?

ANALYSING SYMMETRIC COMBINATORIAL GAMES

A *finite symmetric combinatorial game* is a game, guaranteed to end in a finite number of moves, played by two players who take turns making moves, where in any position the allowed moves do not depend on whose turn it is (which rules out chess and Tic-tac-toe, for example).

To find a strategy for winning such a game, you need to do a few things:

• Identify a set of *final positions*, in which no more moves can be made.

• Describe a set of *losing positions* ($\mathcal{L}$), which must contain all the final positions, and a set of *winning positions* ($\mathcal{W}$). The sets $\mathcal{L}$ and $\mathcal{W}$ must have the following properties:

1. Every position in the game belongs to either $\mathcal{L}$ or $\mathcal{W}$, but not both.
2. From any non-final position in $\mathcal{L}$, *every* move leads to a position in $\mathcal{W}$.
3. From any position in $\mathcal{W}$, there is *some* move that leads to a position in $\mathcal{L}$.

Once you have described the sets $\mathcal{L}$ and $\mathcal{W}$, and shown that they have the properties in points 1-3 above, to complete a strategy for the game you need to explain how to find, given any position in $\mathcal{W}$, a move to a position in $\mathcal{L}$.

MORE PROBLEMS

5. Consider a chessboard with two of the diagonally opposite corners removed (for example $a1$ and $h8$). Is it possible to cover the board with domino pieces (whose size is exactly two board squares)?
6. On a chessboard, suppose you have a piece that moves just one square at a time, up, down, left or right. Two people play by taking turns moving the piece, and are not allowed to put it on a square it has been on before. The player who makes the last move wins. If the piece starts out on the bottom left square, who wins, first or second player, and how?
7. Suppose that when a child is born, the probability is 1/2 that it is a girl, 1/2 that it is a boy. Do men or women have more sisters on average?