Combinatorial game theory theory deals with *symmetric, finite* games played by two players who take turns making moves. That a game is symmetric means that the legal moves in a given position do not depend on whose turn it is. Chess is thus not symmetric, because each player can only move the white or the black pieces. That a game is finite means that from any position the game will be finished in a finite number of moves.

The game *Nim* is played by two players, with several piles of stones. In a turn, a player is to remove any positive number of stones from any one of the piles, but only from one pile. This means removing from one and up to all the stones in one of the piles. The player to remove the last stone wins.

Any finite symmetric two player game G can be "translated" into Nim, meaning that each position in G is equivalent to a position in Nim, so that a winning move, if one exists, can be computed in Nim and then translated back into G.

Before explaining how this "translation" is done, and why it works, three things should be pointed out:

1. Any position that arises is either a *winning position* or a *losing position* for the player whose turn it is. That is, either that player can guarantee a win, by making the right move, or else she is forced to move to a position from which the opponent can win. Positions where there are no moves left are called *final positions* and we usually (but not always) assume that a final position is a losing position, and thus that a player who can't make a move will lose. A position that is not a losing position is a winning position.

2. It is important to understand that in a winning position there is *some* move (possibly more than one) that will lead to a losing position, whereas in a losing position *every* move will lead to a winning position.

3. In order to use Nim as a model for all finite, symmetric two player games, we need to modify the rules slightly, namely, to allow a player to add any positive number of stones to a pile, instead of removing stones. But, there must be some rule that guarantees that the game is finite. The reason that this doesn't change the winning strategy in Nim, more than trivially, is that if a player in a losing position adds stones, the opponent simply removes the added stones and thus recreates the losing position.

Assume we know all the final positions in a given game (and that final positions are losing positions). Such positions are assigned the *Nim-value* 0 (and are equivalent to Nim piles with 0 stones).

The Nim-value of a position that is *not* a final position is determined recursively in the following way:

Nim-value is als called *Grundy number*

Let P be such a position. List all positions that can arise after one move from P. Assume (by induction) that we know the Nim-values of those "earlier" positions. Let $V(P)$ be the set of Nim-values obtained this way, that is, the set of Nim-values of all positions that can be reached by making one move from the position P.

The Nim-value of P is then defined as the smallest natural number that does *not* belong to $V(P)$.

The number 0 is natural

In a position with many *components* (e.g. many piles where each move can only affect one pile) the Nim-value is computed by taking the *Nim-sum* of the Nim-values of the components, that is, the XOR-sum of their Nim-values. The XOR-sum of two numbers is obtained by writing them in binary and then adding without carrying.

$$\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \end{array}$$

XOR-addition

Example: *Grundy's game* is played with piles of stones and each move consists of selecting a pile and splitting it into two piles of different sizes. We can thus split a pile with 4 stones into piles with 1 and 3 stones, respectively, but not into two 2 stone piles. The final positions are those where each pile has either one or two stones, since these can not be split.

Let's compute the Nim-values for the first few pile sizes in this game:

Number stones	Possible pos. after one move	Values of piles	Nim-values of positions	Value of position
1		0		0
2		0		0
3	1-2	0-0	0	1
4	1-3	0-1	1	0
5	1-4, 2-3	0-0, 0-1	0,1	2
6	1-5, 2-4	0-2, 0-0	2,0	1
7	1-6, 2-5, 3-4	0-1, 0-2, 1-0	1,2,1	0

For the position 7 in this example we see that the values of the piles in the resulting positions 1–6, 2–5 and 3–4 are 0–1, 0–2, and 1–0, respectively. The corresponding Nim-sums are therefore 1, 2, and 1, so the set $M(P)$ becomes $\{1, 2\}$ and the value of the position is 0, which is the smallest natural number that does not belong to $\{1, 2\}$.

Note that the Nim-sum of two piles in the cases above is always very simple, because the Nim-sum of two different powers of 2 is just their ordinary sum. The next two cases look like this:

8	1-7	2-6	3-5		0-0, 0-1, 1-2	0, 1, 3	2
9	1-8	2-7	3-6	4-5	0-2, 0-0, 1-1, 0-2	2, 0, 0, 2	1

Here the position 3–6 has value 0, because both piles (3 and 6) have value 1, and the Nim-sum of a number with itself is always 0 (why?). Another example is $5 \oplus 6 = 3$, as $5 = 4 + 1$ and $6 = 4 + 2$, so $5 \oplus 6 = 1 + 2$, where $\oplus$ is the XOR-sum.