EC 306: Introductory Econometrics

Class Problem Sheet 3: Deriving Some Results for Time Series Models

1. Assume that a time series variable, Y_t, follows an autoregressive process:

$$Y_t = \rho Y_{t-1} + \varepsilon_t$$

and ε_t satisfies the classical assumptions (i.e. it is i.i.d. with $E(\varepsilon_t) = 0$, $var(\varepsilon_t) = \sigma^2$ and $cov(\varepsilon_t, \varepsilon_s) = 0$ for $s \neq t$). Assume that $\sigma^2 < \infty$. In addition, you can assume that the time series has been running from period $-\infty$ (but we only observe it for time $t = 1, .., T$).

i) Assuming $-1 < \rho < 1$ work out the $var(Y_t)$ and the autocovariance function (i.e. work out $cov(Y_t, Y_{t-s})$ for $s = 1, 2, 3, ...$) and the autocorrelation function (i.e. work out $corr(Y_t, Y_{t-s})$ for $s = 1, 2, 3, ...$ where *corr* means correlation).

ANSWER: In the lectures, I gave a similar derivation when discussing autocorrelated errors. Here let me give the proof in an alternative way. The AR(1) equation can be rewritten in terms of the errors by repeatedly substituting for any terms involving Y on the right hand side of the equation. That is:

$$\begin{aligned}
&Y_t = \rho(\rho Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \rho^2 Y_{t-2} + \rho\varepsilon_{t-1} + \varepsilon_t \\
&Y_t = \rho^2(\rho Y_{t-3} + \varepsilon_{t-2}) + \rho\varepsilon_{t-1} + \varepsilon_t = \rho^3 Y_{t-3} + \rho^2\varepsilon_{t-2} + \rho\varepsilon_{t-1} + \varepsilon_t \\
&... \\
&Y_t = \rho^t Y_0 + \rho^{t-1}\varepsilon_1 + ... + \rho\varepsilon_{t-1} + \varepsilon_t \\
&... \\
&Y_t = \textstyle\sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j}
\end{aligned}$$

The final equation, along with expressions for infinite sums of variables can be used to work out variances and autocovariances. Variances and autocovariances can then be used to calculate the autocorrelation function.

$$\begin{aligned}
var(y_t) &= var\left(\sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j}\right) \\
&= \sum_{j=0}^{\infty} \rho^{2j} var(\varepsilon_{t-j}) \\
&= \sigma^2 \sum_{j=0}^{\infty} \rho^{2j} \\
&= \frac{\sigma^2}{1-\rho^2}
\end{aligned}$$

Where the previous equation used the properties of the variance operator, the fact that the errors are assumed to be uncorrelated with one another and a standard result for infinite sums that, if $0 \leq X < 1$, then $\sum_{j=0}^{\infty} X^j = \frac{1}{1-X}$.

The covariance between any two variables is $E(Y_tY_{t-s}) - E(Y_t)E(Y_{t-s})$*. In this question* $E(Y_t) = E(Y_{t-s}) = 0$ *(since* Y *can be written as the sum of mean zero errors it also has mean zero) and the covariance simplifies to* $E(Y_tY_{t-s})$*.*

$$\begin{aligned} cov\left(Y_t, Y_{t-s}\right) &= E\left(Y_tY_{t-s}\right) \\ &= E\left[\left(\sum_{j=0}^{\infty}\rho^j\varepsilon_{t-j}\right)\left(\sum_{j=0}^{\infty}\rho^j\varepsilon_{t-s-j}\right)\right]. \end{aligned}$$

If we multiply out the product in the previous equation, it is a mess. Note, however, that once you take expected values all terms involving cross-products (e.g. $\varepsilon_t\varepsilon_s$ *for* $t \neq s$*) will drop out (i.e. since we have assumed the errors to be uncorrelated with one another we have* $E\left[\varepsilon_t\varepsilon_s\right] = 0$*) and you are left with only the squared terms. So, for instance, if* $s = 1$*, if we examine the errors that* $\left(\sum_{j=0}^{\infty}\rho^j\varepsilon_{t-j}\right)$ *and* $\left(\sum_{j=0}^{\infty}\rho^j\varepsilon_{t-s-j}\right)$ *have in common (i.e. these are the ones that will end up a squares when we multiply out), we can see*

$$\begin{aligned} cov\left(Y_t, Y_{t-1}\right) &= E\left[\rho\varepsilon_{t-1}^2 + \rho^3\varepsilon_{t-2}^2 + \rho^5\varepsilon_{t-3}^2 + ...\right] \\ &= \rho\sigma^2\left[1 + \rho^2 + \rho^4 + ...\right] \\ &= \frac{\rho\sigma^2}{1-\rho^2}. \end{aligned}$$

In general,

$$cov\left(Y_t, Y_{t-s}\right) = \frac{\rho^s\sigma^2}{1-\rho^2}$$

Using the formula for the correlation,

$$corr\left(Y_t, Y_{t-s}\right) = \frac{\frac{\rho^s\sigma^2}{1-\rho^2}}{\frac{\sigma^2}{1-\rho^2}} = \rho^s.$$

ii) Discuss what happens to your derivations in part i) when $\rho = 1$. What happens when $\rho > 1$?

The previous proofs used a formula for the infinite sum of a series which only held if $\rho < 1$*.If* $\rho \geq 1$*, then the previous derivations are not valid. To provide some intuition about what happens in this case, notice that in our derivation for the variance (before we used the formula for the infinite sum) we had* $var\left(Y_t\right) = \sigma^2\sum_{j=0}^{\infty}\rho^{2j}$ *For* $\rho \geq 1$ *this variance will be infinite. Alternatively, above we have written* $Y_t = \sum_{j=0}^{\infty}\rho^j\varepsilon_{t-j}$*. If* $\rho = 1$*, we have* $Y_t = \sum_{j=0}^{\infty}\varepsilon_{t-j}$ *so even errors in the infinitely distant past are effecting* Y_t*.* If $\rho > 1$ *(this is called an explosive case), then past errors receive much more weight than current ones. For instance, in* $Y_t = \sum_{j=0}^{\infty}\rho^j\varepsilon_{t-j}$*, consider the coefficient on the error a million periods ago. It is* $\rho^{1,000,000}$ *which, if* $\rho > 1$ *is enormous. Informally, speaking* *these problems (e.g. variances going to infinity, infinite weight on errors from the infinitely distant past, etc.) mean that the standard statistical methods (e.g. for deriving hypothesis tests/confidence intervals) do not work when $\rho \geq 1$. As we shall see, the so-called unit root case of $\rho = 1$ which is quite important in empirical work.*

iii) The formal definition of a stationary time series is as follows: Y_t is stationary if a) $E(Y_t) = \mu$ for all t (i.e. the time series has a constant mean which is that same at all times), b) $var(Y_t) < \infty$ and c) $cov(Y_t, Y_{t-s}) = \gamma_s$ (i.e. the correlation between two values of the series s periods only depends on s and not on t). In light of your answers to parts i) and ii), under what condition is Y_t stationary?

ANSWER: Examining the results from part i) you can see that if $-1 < \rho < 1$, then the conditions for stationarity are satisfied (but they are not satisfied for other values of ρ).

2. Consider the autoregressive distributed lag (ADL) model:

$$Y_t = \alpha + \phi_1 Y_{t-1} + .. + \phi_p Y_{t-p} + \beta_0 X_t + \beta_1 X_{t-1} + ... + \beta_q X_{t-q} + \varepsilon_t.$$

Show that the ADL can be rewritten as:

$$\Delta Y_t = \alpha + \rho Y_{t-1} + \gamma_1 \Delta Y_{t-1} + .. + \gamma_{p-1} \Delta Y_{t-p+1} + \theta X_t + \omega_1 \Delta X_t + ... + \omega_q \Delta X_{t-q+1} + \varepsilon_t.$$

ANSWER: There are many ways of answering this question. Perhaps the simplest is just to begin with the second equation and use the definition of the differencing operator. That is, the second equation can be written as:

$$\begin{aligned} Y_t - Y_{t-1} &= \alpha + \rho Y_{t-1} + \gamma_1 (Y_{t-1} - Y_{t-2}) + .. + \gamma_{p-1} (Y_{t-p+1} - Y_{t-p}) + \\ & \quad \theta X_t + \omega_1 (X_t - X_{t-1}) + ... + \omega_q (X_{t-q+1} - X_{t-q}) + \varepsilon_t. \end{aligned}$$

If we then isolate Y_t on the left hand side of the equation and collect terms on the level of each variable we obtain:

$$\begin{aligned} Y_t &= \alpha + (1 + \rho + \gamma_1) Y_{t-1} + (\gamma_2 - \gamma_1) Y_{t-2} + .. + (\gamma_{p-1} - \gamma_{p-2}) Y_{t-p+1} - \gamma_{p-1} Y_{t-p} + \\ & \quad (\theta - \omega_1) X_t + X_{t-1} (\omega_2 - \omega_1) + ... - \omega_q X_{t-q} + \varepsilon_t. \end{aligned}$$

If we define $\phi_1 = (1 + \rho + \gamma_1), \phi_2 = (\gamma_2 - \gamma_1)$, etc. then we can see that this last equation is identical to the one given at the beginning of the question. Thus the two forms of the ADL define the same model.