Modelling Volatility: ARCH and GARCH Models

January 24, 2012
Regressions (and extensions of regression considered in this course) seek to explain the *mean* of $Y_t$.

That is, the regression model says:

$$E(Y_i) = \beta X_i$$

But in some cases we want a model for the *variance* of $Y_t$.

This usually (but not always) occurs in finance.

Variance (volatility) of the price of an asset relates to its riskiness.

ARCH and GARCH models which are the most popular ways of modelling volatility.

Reading: Gujarati, Chapter 14 and Koop, pages 197-205.
• ARCH and GARCH are time series topics
• Notation: $Y_t$ for $t = 1, \ldots, T$ are our observations (e.g. on a stock return for each of $T$ days)
• Univariate time series econometric methods were discussed in 3rd year course.
• You may wish to review this material (although I will try and briefly explain relevant background below).
• Review of basic univariate time series: my textbook (Koop, Introduction to Econometrics) pages 173-196.
• Note: material on autoregressive (AR) model is of particular use
• The material in this set of slides is taken from my textbook (pages 197-205) and Gujarati chapter 15.
Recall the random walk (with drift) model:

\[ Y_t = \alpha + Y_{t-1} + \varepsilon_t \]

or

\[ \Delta Y_t = \alpha + \varepsilon_t \]

where \( \Delta Y_t \) is the first difference or change in \( Y_t \).

Note: if \( Y_t \) is the log of a variable (e.g. \( Y_t = \log(P_t) \) where \( P_t \) is the price of an asset) then \( \Delta Y_t \) is the percentage change in the variable.

E.g. \( \Delta Y_t \) could be stock return (exclusive of dividends)
- Financial assets like stock prices often found to follow random walk with drift.
- Stock prices, on average, increase by $\alpha$ per period, but are otherwise unpredictable.
- Stock returns (exclusive of dividends) are on average $\alpha$ but are otherwise unpredictable.
- Hard to predict stock returns (if it were easy to do, many econometricians would be rich).
- However, it is easier to predict variance (volatility) of stock returns.
- Why is this interesting?
- Volatilities appear in many places in finance relating to risk: portfolio management, option pricing, pricing of financial derivatives, VIX, Black-Scholes, etc.
- This is not a finance course, so I will just say: having estimates of the volatilities of asset prices is very useful for finance people.
Assume asset price does follow a random walk.

To eliminate worrying about the drift, work with

\[ \Delta y_t = \Delta Y_t - \overline{\Delta Y} \]

where \( \overline{\Delta Y} = \frac{\sum \Delta Y_t}{T} \)

Taking deviations from the mean implies that there is no intercept in the model.

So assume the model is:

\[ \Delta y_t = \varepsilon_t \]
Before getting to ARCH and GARCH, let me introduce some ideas using simpler methods.

Remember some basic statistics:

If you have a random sample, $X_1, \ldots, X_N$ then an estimate of the variance is:

\[
\sum_{i=1}^{N} \frac{(X_i - \bar{X})^2}{N}
\]

This result assumes all of $X_i$ have same variance.

What if each $X_i$ has a different variance? Cannot average them all together.

Think of $X_i$ as a random sample of size $N = 1$ and assume its mean is zero and you get $X_i^2$ as an estimate of the variance.

Applying these ideas to $\Delta y_t$:

$\Delta y_t^2$ is an estimate of volatility of the asset price at time $t$. 
Figures on next 3 slides illustrate common pattern with asset prices.

First figure plots of weekly observations of the (logged) stock price of certain company.

Second figure plots $\Delta Y$, the percentage change in $Y$.

Third figure plots $\Delta y_t^2$ - an estimate of volatility

Note *clustering in volatility*: there are times stock price is much more volatile than other times

Often happens with financial assets
Figure 6.7: Time Series Plot of Log of Stock Price

Week

0 50 100 150 200 250

3.15
3.2
3.25
3.3
3.35
3.4
3.45

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Figure 6.8: Time Series Plot of Change in Stock Price

Week

0 50 100 150 200 250
-1.5
-1
-0.5
0
0.5
1
1.5
2
Figure 6.9: Time Series Plot of Volatility
In 3rd year course, we used the AR($p$), model for $Y_t$: 

$$Y_t = \alpha + \rho_1 Y_{t-1} + \ldots + \rho_p Y_{t-p} + \varepsilon_t$$ 

for $t = p + 1, \ldots, T$

Properties of this model:

- $Y_t$ depends on past values of itself
- $Y_t$ depends on lags of itself
Properties of AR(1) model:

- If $Y$ is high at time $t-1$ it will also tend to be high at time $t$ (if $\rho > 0$)
- Clustering in time of high values of $Y$
- Assume $|\rho| < 1$ (this is stationarity restrictions, if $\rho = 1$ we have unit root)

- $\text{corr}(Y_t, Y_{t-1}) = \rho$
- $\text{corr}(Y_t, Y_{t-s}) = \rho^s$

We will use AR models (or similar) but not for $Y_t$, instead for volatilities
**Clustering in volatility** is often present in financial time series data.

Why not use an AR(1) model for volatility?

That is, use $\Delta y_t^2$ as the dependent variable:

$$\Delta y_t^2 = \alpha + \rho \Delta y_{t-1}^2 + \epsilon_t$$

Volatility in a period depends on volatility in the previous period.

If $\rho > 0$ (as it often is in financial applications), then if volatility was unusually high last period (e.g. $\Delta y_{t-1}^2$ was very large), also tends to be unusually high this period.

If volatility unusually low last period (e.g. $\Delta y_{t-1}^2$ was near zero) then this period’s volatility will also tend to be low.

Standard OLS econometric techniques can be used to estimate this model (provided $|\rho| < 1$)
Example: Volatility in Stock Prices (continued)

- Use data in Figure 6.9 and estimate AR(p) model using $\Delta y_t^2$ as dependent variable.
- Remember: to select lag length can use sequential testing procedure.
- E.g. estimate AR(4) and if coefficient on 4th lag insignificant, move the AR(3) model, etc.
- This strategy yields the AR(1) model for $\Delta y_t^2$.

<table>
<thead>
<tr>
<th>Table 6.6: AR(1) Model for $\Delta y_t^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
</tr>
<tr>
<td>Intercept</td>
</tr>
<tr>
<td>$\Delta y_{t-1}$</td>
</tr>
</tbody>
</table>

- Last week’s volatility has strong explanatory power for this week’s volatility.
- Clustering in volatility is found.
ARCH models (including extensions of them) are the most popular models for financial volatility.

To allow for generality and conform with how econometrics packages work context of regression model:

\[ Y_t = \alpha + \beta_1 X_{1t} + \ldots + \beta_k X_{kt} + \varepsilon_t \]

Note if \( X_{1t} = Y_{t-1} \) then this is an AR model.

If no explanatory variables at all (i.e. \( \alpha = \beta_1 = \ldots = \beta_k = 0 \)) then modelling volatility in dependent variable

Common for dependent variable is a stock return

Common to include at least an intercept
ARCH model relates to the variance (or volatility) of the error, $\epsilon_t$.

Use the following notation:

$$\sigma_t^2 = \text{var} (\epsilon_t)$$

$\sigma_t^2$ is volatility at time $t$.

Error variances are not constant: thus we have heteroskedasticity which accounts for the “H” in ARCH.
The ARCH model with $p$ lags is denoted by ARCH($p$).

Today’s volatility is an average of past errors squared:

$$\sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2 + \ldots + \gamma_p \varepsilon_{t-p}^2$$

$\gamma_0, \gamma_1, \ldots, \gamma_p$ are coefficients that can be estimated in Gretl.

We will not discuss estimation of ARCH models: usually maximum likelihood is used (but other estimators exist).
If no explanatory variables and dependent variable is the de-meaned stock return, $\Delta y_t$, then $\epsilon_t = \Delta y_{t-1}$

In this case the ARCH model becomes:

$$\sigma_t^2 = \gamma_0 + \gamma_1 \Delta y_{t-1}^2 + \ldots + \gamma_p \Delta y_{t-p}^2$$

and volatility depends on recent values of $\Delta y_t^2$ – the metric for volatility we were using in the preceding section.

ARCH model is closely related to AR

ARCH models have similar properties to AR models – except that these properties relate to the volatility of the series.
With ARCH models do not need to subtract the mean off of stock returns as we did previously.

By simply including an intercept in regression model we are allowing for a random walk with drift.

Use logged stock price data and take first difference to create $\Delta Y$.

Estimate ARCH(1) model with $\Delta Y$ as dependent variable and an intercept in the regression equation I get Table 6.7 (next slide).

The upper part of Table 6.7 refers to the coefficients in the regression equation (here only an intercept).

The lower part of the table refers to the ARCH equation.

I have specified ARCH(1) model, so have an intercept (with coefficient labeled $\gamma_0$ in the ARCH equation) and one lag of the errors squared (labeled $\gamma_1$ in the ARCH equation).
Table 6.7: ARCH(1) Model using Stock Return Data

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient Estimate</th>
<th>P-value</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression Equation with $\Delta Y$ as Dependent Variable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>0.105</td>
<td>0.000</td>
<td>[0.081, 0.129]</td>
</tr>
<tr>
<td>ARCH Equation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>0.024</td>
<td>0.000</td>
<td>[0.016, 0.032]</td>
</tr>
<tr>
<td>$\Delta \varepsilon_{t-1}^2$</td>
<td>0.660</td>
<td>0.000</td>
<td>[0.302, 1.018]</td>
</tr>
</tbody>
</table>
Numbers labeled “Coefficient Estimate” are estimates of the coefficients.

Numbers labeled “P-value” are P-values for testing whether corresponding coefficient equals zero.

Here all P-values are all less than 0.05 so all coefficients are significant at the 5% level.

Estimate of $\gamma_1$ is 0.660, indicating that volatility this week depends strongly on the errors squared last week.

Persistence in volatility of similar degree to what we found before.

Remember we previously found the AR(1) coefficient for the variable $\Delta y_t^2$ to be 0.737.
Lag length selection in ARCH models can be done in the same manner as with any time series model.

Can use an information criterion to select a model.

Or look at P-values for whether coefficients equal zero (and, if they do seem to be zero, the accompanying variables can be dropped).

E.g. if we estimate an ARCH(2) model table on next slide.

Results similar to ARCH(1) model.

But coefficient on $\Delta \varepsilon_{t-2}^2$ is not significant, since P-value is greater than 0.05.

Thus, ARCH(1) model is adequate and the second lag added by ARCH(2) model does not add significant explanatory power.
### Table 6.8: ARCH(2) Model using Stock Return Data

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient Estimate</th>
<th>P-value</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
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<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>0.109</td>
<td>0.000</td>
<td>[0.087, 0.131]</td>
</tr>
</tbody>
</table>

| ARCH Equation | | | |
| Intercept | 0.025 | 0.000 | [0.016, 0.033] |
| $\Delta \varepsilon_t^{2}$ | 0.717 | 0.000 | [0.328, 1.107] |
| $\Delta \varepsilon_{t-2}^{2}$ | $-0.043$ | 0.487 | $[-0.165, 0.079]$ |
Many extensions of ARCH model are used by financial econometricians.

Generalized ARCH (or GARCH) is most popular and we will discuss it below.

Gretl has a menu of “GARCH variants” with seven variants of the with acronyms like TARCH, EGARCH, NARCH, etc.

Will not discuss these in this course, but if you are interested in financial volatility, you can learn more about these models.

Here focus on GARCH
ARCH had volatility depending on lagged errors squared

GARCH adds to this lags of volatility itself

Properties of GARCH similar to ARCH

But GARCH is much more flexible, much more capable of matching a wide variety of patterns of financial volatility

Financial time series often have “fat tails”: more extreme outcomes in the tails of the distribution than a Normal distribution would allow for

Important property of GARCH: allows for fat tails

GARCH model with \((p, q)\) lags is denoted by GARCH\((p, q)\) and has a volatility equation of:

\[
\sigma^2_t = \gamma_0 + \gamma_1 \varepsilon^2_{t-1} + \ldots + \gamma_p \varepsilon^2_{t-p} + \lambda_1 \sigma^2_{t-1} + \ldots + \lambda_p \sigma^2_{t-p}
\]

Maximum likelihood estimation can be done by Gretl (and other packages)
Table contains GARCH(1,1) results, we obtain the results in the following table.

Looks like ARCH(1) was good enough for this data set since coefficient on $\sigma^2_{t-1}$ is insignificant.

<table>
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</tr>
<tr>
<td>ARCH Equation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>0.026</td>
<td>0.000</td>
<td>[0.015, 0.038]</td>
</tr>
<tr>
<td>$\Delta \varepsilon^2_{t-1}$</td>
<td>0.714</td>
<td>0.000</td>
<td>[0.327, 1.101]</td>
</tr>
<tr>
<td>$\sigma^2_{t-1}$</td>
<td>$-0.063$</td>
<td>0.457</td>
<td>[$-0.231, 0.104$]</td>
</tr>
</tbody>
</table>
So far our most general model is a regression with GARCH errors.

To summarize, this means:

\[ Y_t = \alpha + \beta_1 X_{1t} + \ldots + \beta_k X_{kt} + \varepsilon_t \]

\( \varepsilon_t \) is \( N(0, \sigma^2_t) \)

and

\[ \sigma^2_t = \gamma_0 + \gamma_1 \varepsilon^2_{t-1} + \ldots + \gamma_p \varepsilon^2_{t-p} + \lambda_1 \sigma^2_{t-1} + \ldots + \lambda_p \sigma^2_{t-p} \]

But it is useful to introduce one more extension.
What if the volatility directly influences the dependent variable?

E.g. volatility in stock markets causes people to worry about risk which impacts on stock returns

Volatility is an explanatory variable in the regression:

\[ Y_t = \alpha + \beta_1 X_{1t} + .. + \beta_k X_{kt} + \lambda \sigma_t + \varepsilon_t \]

\( \varepsilon_t \) is \( N(0, \sigma_t^2) \)

and

\[ \sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2 + .. + \gamma_p \varepsilon_{t-p}^2 + \lambda_1 \sigma_{t-1}^2 + .. + \lambda_p \sigma_{t-p}^2 \]

Exactly like GARCH model except we have added \( \sigma_t \) as explanatory variable in the regression.

This is the GARCH-M model
- GARCH-M model can be estimated using maximum likelihood in Gretl
- These models will be investigated in the computer tutorial
With time series regression models errors are often heteroskedastic in a particular way: clustering in volatility.

This is especially true with financial assets like stock prices.

Obtaining estimates of volatilities is important in many areas of finance (e.g. option pricing, portfolio management).

If $Y_t$ is the log of a stock price, then $\Delta y_t^2$ can be used as a measure of volatility and used in AR models or regressions.

However, ARCH and GARCH are better ways of modelling volatilities.

ARCH and GARCH share similar intuition with AR models but intuition holds for the volatilities.

Estimation of ARCH and GARCH can be done in econometrics software packages such as Gretl.

Many extensions of ARCH and GARCH are also popular of which I covered GARCH-M.