A Course in Bayesian Econometrics

University of Queensland

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Slides for Lecture on

Bayesian State Space Modeling

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1 Introduction

- State space models are a popular way of modeling time series.

- Useful for trend/cycle/irregular/seasonal decompositions

- Also many models in macroeconomics/finance have state space structure (e.g. time varying VARs, stochastic volatility)

- Falls into familiar framework: Gibbs sampling/hierarchical priors

2 A General State Space Model

We will work with the state space model with measurement equation:

$$y_t = X_t\beta + Z_t\alpha_t + \varepsilon_t,$$

and state equation:

$$\alpha_{t+1} = T_t\alpha_t + u_t.$$

- y_t is scalar dependent variable and X_t and Z_t are $1 \times k$ and $1 \times p$ vectors, respectively, containing explanatory variables and/or known constants.
- Extension to multivariate y is straightforward
- α_t is $p \times 1$ vector containing p states.

- ε_t is i.i.d. $N\left(0, h^{-1}\right)$

- u_t is $p \times 1$ vector which is i.i.d. $N\left(0, H^{-1}\right)$

- ε_t and u_s are independent of one another for all s and t.

- T_t is a $p \times p$ matrix of known constants.

- Note: if T_t contains unknown parameters can be handled easily.

- Lots of interesting models structural time series models fall in this framework

- State equation can be interpreted as a hierarchical prior

2.1 Bayesian Computation in the State Space Model

- Conditional on knowing states, model is the Normal regression model:

$$y_t^* = X_t\beta + \varepsilon_t,$$

where $y_t^* = y_t - Z_t\alpha_t$.

- All our results for Normal linear regression model can be used (conditional on states).

- But, conditional on other parameters, can draw states. Thus, Gibbs sampler can be used. Formally, Gibbs sampler involves

$p(\beta|y, \alpha_1, .., \alpha_T, h)$,$p(h|y, \alpha_1, .., \alpha_T, \beta)$,

$p(\alpha_1, .., \alpha_T|y, \beta, h, H)$ and $p(H|y, \alpha_1, .., \alpha_T)$.

2.2 A Prior

- We will derive Gibbs sampler using following prior (although anything possible):

$$p(\beta) = f_N\left(\beta|\underline{\beta}, \underline{V}\right),$$

$$p(h) = f_G\left(h|\underline{s}^{-2}, \underline{\nu}\right),$$

$$p(H) = f_W\left(H|\underline{\nu}_H, \underline{H}\right).$$

- Assume $\alpha_0 = 0$ (there are various ways of treating initial condition, but we will not discuss)

- State equation is a hierarchical prior:

$$p(\alpha_1, .., \alpha_T|H) = p(\alpha_1|H)\, p(\alpha_2|\alpha_1, H) .. p(\alpha_T|\alpha_{T-1}, H),$$

where

$$p(\alpha_{t+1}|\alpha_t, H) = f_N(\alpha_{t+1}|T_t\alpha_t, H)$$

and

$$p(\alpha_1|H) = f_N(\alpha_1|0, H).$$

2.3 Gibbs Sampler

- Gibbs sampler involves drawing from posterior conditionals

- Posterior conditionals for β and h based on usual regression results (with y replaced by y^*, see page 197)

$$\beta|y, h, \alpha_1, .., \alpha_T \sim N\left(\overline{\beta}, \overline{V}\right).$$

and:

$$h|y, \beta, \alpha_1, .., \alpha_T \sim G(\overline{s}^{-2}, \overline{\nu})$$

- What about H? Conditional on $\alpha_1, .., \alpha_T$, state equations are like SUR model (with no explanatory variables).

- Thus (using SUR results from page 140):

$$H|y, \alpha_1, .., \alpha_T \sim W\left(\overline{\nu}_H, \overline{H}\right)$$

where $\overline{\nu}_H$ and $\overline{H}$ are given on page 197

- To complete Gibbs sampler, need $p(\alpha_1, .., \alpha_T | y, \beta, h, H)$ and a means of drawing from it.

- This is multivariate Normal distribution, but hard draw from it since $T-$dimensional (elements can be highly correlated with one another).

- But we can draw on standard state space algorithms (e.g. Carter and Kohn, 1994, and DeJong and Shephard, 1995, Durbin and Koopman, 2002)

- Textbook described DeJong and Shephard's algorithm. I won't repeat details here.

- But note these papers are not explicitly Bayesian, but it turns out that there is an equivalence between standard non-Bayesian methods.

- E.g. Kalman filter produces posterior means of α_t that are required for our Gibbs sampler

- Since writing my textbook, it has come to my attention that the following is the most efficient algorithm for drawing states:

- Durbin, J. and Koopman, S., 2002, A simple and efficient simulation smoother for state space time series analysis, *Biometrika*.

- Thus, a Gibbs sampler can be set up which draws on results for Normal linear regression model, SUR model and a standard algorithm for drawing states in state space models.

2.4 Empirical Illustration: A TVP-AR Model

- A simple example taken from Koop and Potter (2001, Econometrics Journal).

- Economic history data set percentage change in UK industrial production from 1701-1992.

- AR(p) model with time varying parameters:

$$y_t = \alpha_{0t} + \alpha_{1t} y_{t-1} + ... + \alpha_{pt} y_{t-p} + \varepsilon_t,$$

  where for $i = 0, .., p$

$$\alpha_{it+1} = \alpha_{i,t} + u_{it}.$$

- ε_t is i.i.d. $N\left(0, h^{-1}\right)$ and u_{it} is i.i.d. $N\left(0, \lambda_i h^{-1}\right)$ where ε_t, u_{is} and u_{jr} are independent of one another

- Special case of our state space model if we exclude X_t and define:

$$\alpha_t = \begin{pmatrix} \alpha_{0t} \\ \alpha_{1t} \\ . \\ . \\ \alpha_{pt} \end{pmatrix},$$

$$u_t = \begin{pmatrix} u_{0t} \\ u_{1t} \\ . \\ . \\ u_{pt} \end{pmatrix},$$

$$Z_t = \begin{pmatrix} 1 & y_{t-1} & . & . & y_{t-p} \end{pmatrix}$$

and set $T_t = I_{p+1}$ and

$$H^{-1} = h^{-1} \begin{bmatrix} \lambda_0 & 0 & 0 & . & 0 \\ 0 & \lambda_1 & . & . & . \\ . & 0 & . & . & . \\ . & . & . & . & 0 \\ 0 & . & . & 0 & \lambda_p \end{bmatrix}.$$

- Posterior results obtained by running Gibbs sampler.

- As an example of empirical results.

Figure 8.3a: Posterior Density for λ_0

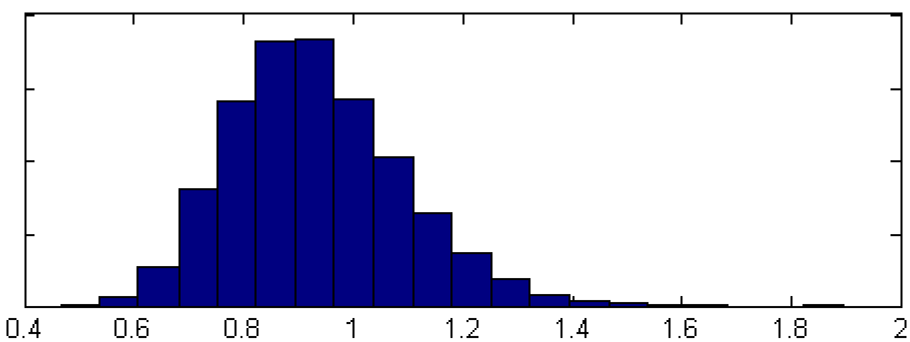
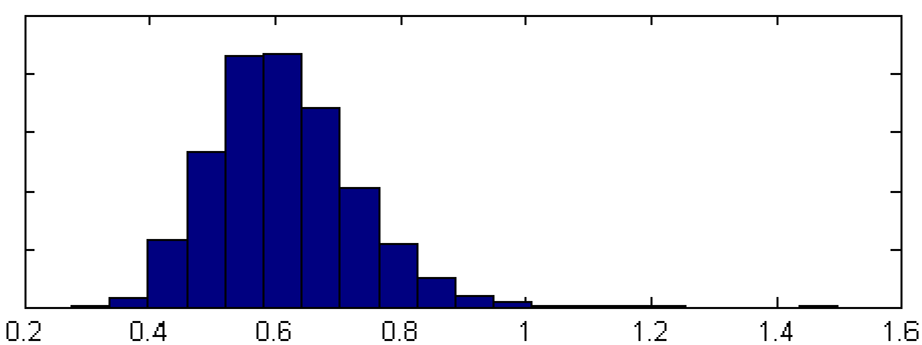


Figure 8.3b: Posterior Density for λ_1



3 Extensions

- Note that blocking nature of Gibbs sampler means we can handle many extensions by just adding one more block (i.e. one more conditional posterior distribution) to the Gibbs sampler

- For instance, adding *Student-t* errors to the state space model can be done by combining Gibbs sampler Normal state space model with Gibbs sampler we talked about before for regression model with Student-t errors.

- Many other nonlinear/non-normal extensions of state space models

- Of particular importance: stochastic volatility model:

$$y_t = \exp\left(\frac{\alpha_t}{2}\right)\varepsilon_t,$$

where ε_t is i.i.d. $N(0,1)$

$$\alpha_{t+1} = \mu + \phi\alpha_t + u_t$$

where u_t is i.id. $N\left(0,\sigma_u^2\right)$.

- Rewrite (nonlinear) measurement equation as:

$$\log\left(y_t^2\right) = \alpha_t + \log\left(\varepsilon_t^2\right)$$

- The only thing which stops direct use of our previous Gibbs sampler is that $\log\left(\varepsilon_t^2\right)$ is not Normal

- But as noted in Shephard (1993, Biometrika), Carter and Kohn (1997, JRSS,B) and Kim, Shephard and Chib (ReStud, 1998), $\log\left(\varepsilon_t^2\right)$ can be approximated extremely well by a mixture of Normal distributions.

- Hence, add block to Gibbs sampler drawing on results from mixture of Normals literature (remember Student-t errors is an example of mixture, but more general mixtures possible).