A Course in Bayesian Econometrics

University of Queensland

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Slides for Lecture on

Bayesian Analysis of Extensions of AR and VAR Models

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1 Introduction

- Autoregressive models can be written as regression models (see lecture 2)

- VAR models closely related to SUR model (see lecture 3)

- Hence, we will briefly deal with AR and VARs.

- Extensions of them are increasingly used with empirical work and we will spend more time on these.

- Basic idea: conditional on a parameter (e.g. probability of transition) or a vector of latent data (e.g. a vector of states) such extensions reduce to regression models.

- Thus, use Gibbs sampler

A Digression: Unit Roots and Cointegration

- We will not have time to discuss issues relating to nonstationary variables.

- Survey paper on cointegration in Palgrave Handbook of Econometrics, Volume 1: Theoretical Econometrics (Koop, Strachan, van Dijk and Villani)

- Non-Bayesians worry since asymptotic distributions of test statistics, etc. become different if unit root present

- Bayesians proceed conditional on data, so are rarely interested in asymptotics.

- Thus, many Bayesians uninterested in unit root issues (e.g. just estimate a VAR, not worry if cointegration is present or not).

- Likelihood function of AR/VAR does not depend on whether variable(s) have unit roots.

- Thus, the Bayesians who do write unit root/cointegration papers typically focus on prior elicitation.

2 Autoregressive Models

- As discussed in lecture on "Forecasting in Dynamic Factor Models Using Bayesian Model Averaging" AR(p) model can be written as a linear regression model.

- All our results for Bayesian analysis of regression model can be used.

- Example based on Geweke (1988, JBES)

- y_t is log real GDP, follows AR(3):

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \beta_3 y_{t-3} + \epsilon_t,$$

where ϵ_t is i.i.d. $N\left(0, h^{-1}\right)$.

- Properties of y_t depend on the roots of the polynomial $1 - \sum_{i=1}^{3} \beta_i z^i$ (denote by r_i for $i = 1, .., p$)

- Features of interest: $C = \{\text{Two of } r_i \text{ are complex}\}$ and $D = \{\min |r_i| < 1\}$

- C and D are regions which are nonlinear functions of $\beta_1, \beta_2, \beta_3$.

- C implies real GDP exhibits an oscillatory response to a shock

- D implies an explosive response to a shock.

- Want $\Pr(\beta \in C|y)$ and $\Pr(\beta \in D|y)$.

2.1 Posterior Analysis of the AR(p) Model

- Noninformative prior:

$$p\left(\beta_0, .., \beta_p, h\right) \propto \frac{1}{h},$$

- Simple treatment of initial conditions: $y = \left(y_{p+1}, .., y_T\right)'$ and treat $y_1, .., y_p$ as fixed initial conditions.

- AR(p) can be written as:

$$y = X\beta + \epsilon,$$

where $\epsilon = \left(\epsilon_{p+1}, .., \epsilon_N\right)'$ and X is the $(T-p)\times(p+1)$ matrix containing an intercept and p lags of the dependent variable.

- Can use derivations from Lecture 2, to say

$$\beta, h|y \sim NG\left(\overline{\beta}, \overline{V}, \overline{s}^{-2}, \overline{\nu}\right)$$

where parameters of distribution are OLS quantities

- Also as in Lecture 2, marginal posterior is multivariate t:

$$\beta|y \sim t\left(\overline{\beta}, \overline{s}^{2}\overline{V}, \overline{\nu}\right)$$

- But how to get $\Pr(\beta \in C|y)$ and $\Pr(\beta \in D|y)$?

- If C and D were defined by linear combinations of β, then no problem ("linear combinations of t's are still t"). But they are nonlinear.

- Monte Carlo integration: randomly draw from $\beta|y$.

- For each draw, calculate solutions to $1-\sum_{i=1}^{p} \beta_i z^i = 0$ and see if C and D are satisfied

- The proportion of draws which satisfy C (or D) will converge to $\Pr(\beta \in C|y)$ (or $\Pr(\beta \in D|y)$)

- Formally, this follows since $\Pr(\beta \in C|y) = E[I(\beta \in C|y)]$ and $\Pr(\beta \in D|y) = E[I(\beta \in D|y)]$, where $I(.)$ is the indicator function.

- US real GDP from 1947Q1 through 2005Q2, $\Pr(\beta \in C|y) = 0.021$ and $\Pr(\beta \in D|y) = 0.132$.

3 Nonlinear Time Series Extensions of AR model

- Many of these have the form "conditional on knowing threshold/state/regime/breakpoint parameter we have a linear regression model"

- E.g. TAR, smooth transition autoregressive, Markov switching models, many structural break models, etc).

- Gibbs sampler (or similar MCMC algorithms) popular

- We will focus on models in the threshold autoregressive (TAR) class

- For TAR analytical results available if we use natural conjugate prior (no Gibbs sampling required)

3.1 Threshold Autoregressive (TAR) Models

- 2 regime TAR:

$$\begin{array}{l} y_t = \beta_{10} + \beta_{11} y_{t-1} + ... + \beta_{1p} y_{t-p} + \epsilon_t \text{ if } y_{t-1} \leq \tau \\ y_t = \beta_{20} + \beta_{21} y_{t-1} + ... + \beta_{2p} y_{t-p} + \epsilon_t \text{ if } y_{t-1} > \tau \end{array},$$

- If threshold τ is known, model can be written as regression model:

$$y = X\beta + \epsilon,$$

- X has t^{th} row given by

$$\left[D_t, D_t y_{t-1}, .., D_t y_{t-p}, (1 - D_t), (1 - D_t) y_{t-1}, .. \right]$$

- D_t is dummy which equals 1 if $y_{t-1} \leq \tau$ and equals 0 if $y_{t-1} > \tau$.

- Can use all our old results for regression model.

- E.g. if prior is $NG\left(\underline{\beta}, \underline{Q}, \underline{s}^{-2}, \underline{\nu}\right)$ then posterior is $NG\left(\overline{\beta}, \overline{Q}, \overline{s}^{-2}, \overline{\nu}\right)$ (see Lecture 2 for exact definitions of arguments)

- But what if τ is an unknown parameter?

- We need the posterior: $p\left(\beta, h, \tau | y\right)$.

- Could do Gibbs sampling, but an even simpler strategy is possible.

- Rules of conditional probability imply:

$$p(\beta, h, \tau | y) = p(\beta, h | \tau, y) \, p(\tau | y) \, .$$

- $p(\beta, h | \tau, y)$ use results for regression model.
- But what about $p(\tau | y)$?
- Remember that the marginal likelihood (see Lecture 1, used in calculating posterior model probability, etc.) is $p(y)$
- For given τ, the marginal likelihood can be calculated for regression model (Lecture 2):

$$p(y|\tau) = c \left(\frac{|\overline{V}|}{|\underline{V}|} \right)^{\frac{1}{2}} \left(\overline{\nu s}^2 \right)^{-\frac{\overline{\nu}}{2}}$$

- Details of formula unimportant (see Lecture 2 for details). Key thing is that we can calculate $p(y|\tau)$ for every value of τ.

- Bayes' theorem implies:

$$p(\tau|y) \propto p(y|\tau)\, p(\tau),$$

- so combine $p(y|\tau)$ with a prior for τ to get $p(\tau|y)$.

- Any prior for τ can be used.

- A common choice is a restricted noninformative one.

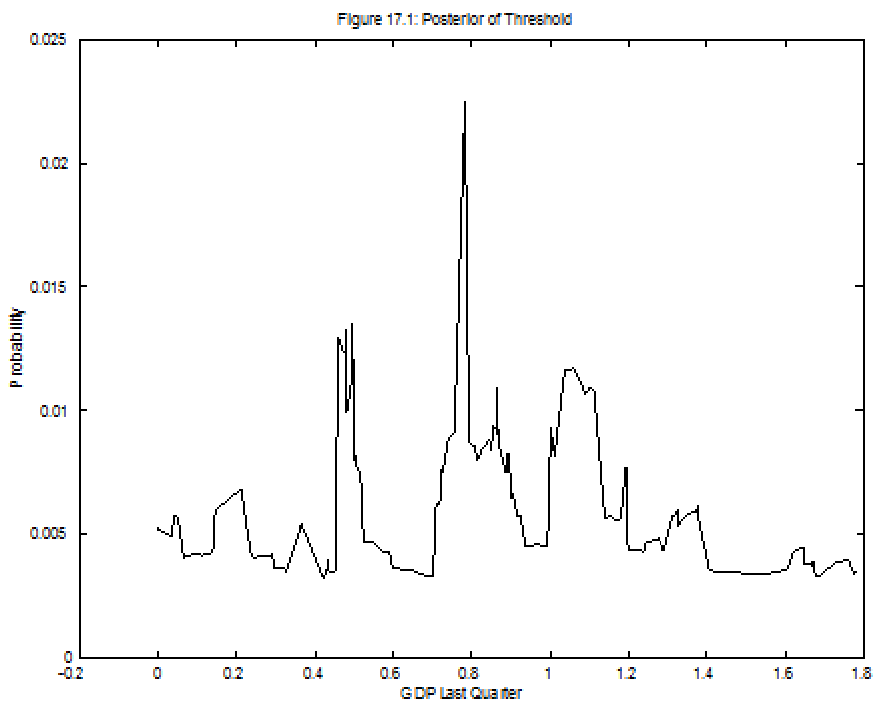
- E.g. Every value for τ which implies each regime contains a minimum number of observations (e.g. 15% of the observations).

- Note: even though y_{t-1} is a continuous variable, τ will be a discrete random variable since there are a finite number of ways of dividing a given data set into two regimes.

- Hence, if $\tau \in \{\tau_1, .., \tau_{T^*}\}$ denotes possible threshold values, then

$$p(\beta, h|y) = \sum_{i=1}^{T^*} p(\beta, h|\tau = \tau_i, y)\, p(\tau = \tau_i|y).$$

3.1.1 Example: US GDP Growth



Posterior Results for TAR Model with Unknown Threshold		
Parameter	Mean	Standard Dev.
β_{10}	0.542	0.119
β_{11}	0.269	0.139
β_{12}	0.373	0.101
β_{20}	0.249	0.242
β_{21}	0.384	0.127
β_{22}	0.196	0.100
σ^2	0.896	0.083

3.2 Extensions of the Basic TAR Model

- Many extensions of the TAR can be done in this basic framework

- Basic TAR has last quarter's GDP growth triggering regime switch.

- But it might be another (exogenous or lagged) variable, z, that is the *threshold trigger*.

- It may take longer than one period to induce the regime switch. Thus, introduce d: *delay parameter*

$$\begin{array}{c} y_t = \beta_{10} + \beta_{11} y_{t-1} + ... + \beta_{1p} y_{t-p} + \epsilon_t \text{ if } z_{t-d} \leq \tau \\ y_t = \beta_{20} + \beta_{21} y_{t-1} + ... + \beta_{2p} y_{t-p} + \epsilon_t \text{ if } z_{t-d} > \tau \end{array},$$

- Bayesian inference basically the same as for basic TAR, but now have two unknown parameters: τ and d and so interest centres on:

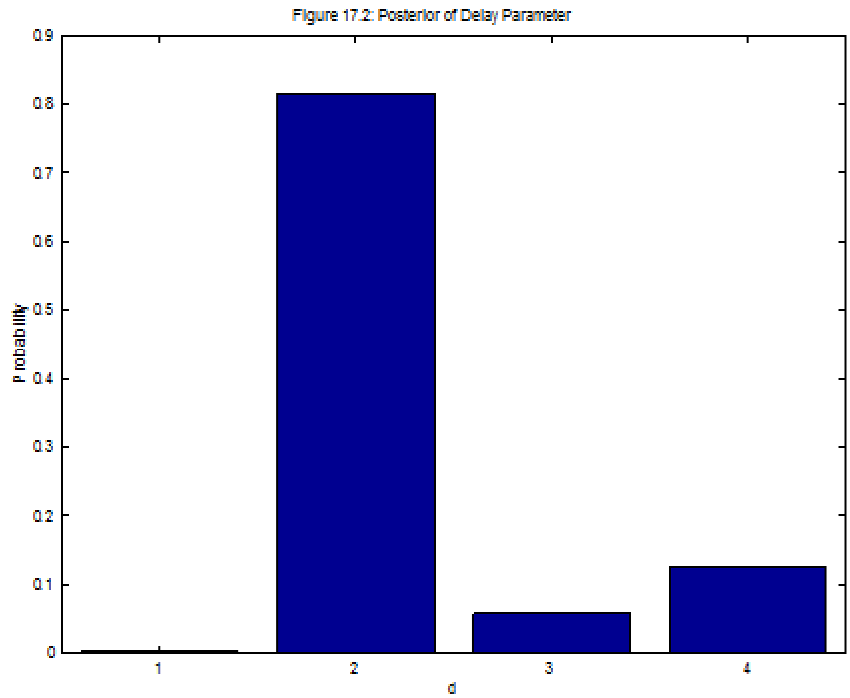
$$p(\beta, h, \tau, d|y) = p(\beta, h|\tau, d, y)\, p(\tau, d|y)\,.$$

- Same idea as for basic TAR: evaluate marginal likelihood for every value for τ and d and this can be used to produce $p(\tau, d|y)$.

- Extension to more than 2 regimes (with τ_1 and τ_2 being two thresholds) same idea (evaluate marginal likelihood for every value for τ_1, τ_2 and d and this can be used to produce $p(\tau_1, \tau_2, d|y)$

- Note: BMA can be used to average over different choices for z (which may be empirically important).

- Allowing for error variance to differ across regimes may be important (more about this later)



- Example: US real GDP growth data, are provided in Table 17.4.

- Figure plots the posterior of d.

- Strong support for $d = 2$ [same as in Potter (1995, JAE)].

4 Vector Autoregressive (VAR) Models

- There is a large literature on Bayesian VARs (e.g. work of Sims)

- Lots of different priors tried (priors = shrinkage which seems to help forecasting)

- E.g. Kadiyala and Karlsson (1997, JAE) a good source for different priors.

- Just as AR could be put in form of regression model, VAR can be put in form of multivariate regression model (e.g. like a SUR)

- There is a natural conjugate prior for the VAR model which yields analytical results (comparable to analytical results for AR/TAR done above).

- Bayesian Econometric Methods, Exercise 17.6 does natural conjugate case.

- I will discuss VAR using an independent Normal-Wishart prior (same as we used for SUR model).

- Do this partly to provide a variety of priors, but also because natural conjugate has some restrictive properties in the VAR case.

- VAR(p) model is:

$$y_t = a_0 + \sum_{j=1}^{p} A_j y_{t-j} + \varepsilon_t$$

where y_t for $t = 1, .., T$ is an $M \times 1$ vector containing observations on M variables

- a_0 is an $p \times 1$ vector of intercepts and A_j is an $M \times M$ matrix of coefficients.

- ε_t are independent $N\left(0, H^{-1}\right)$

- But note (similar to our SUR derivations), we can write the VAR as

$$y_t = Z_t \alpha + \varepsilon_t$$

where Z_t is an $M \times m$ matrix of data on explanatory variables (i.e. lags of all dependent variables and an intercept and other deterministic terms) arranged in the same manner as we did for SUR model.

- α is now long vector containing all the VAR coefficients in every equation

- Put in this form, we can do derivations exactly like for SUR model.

- Prior $p(\alpha, H) = p(\alpha) p(H)$

- Prior for VAR coefficients

$$p(\alpha) = f_N(\alpha | \underline{\alpha}, \underline{V})$$

- Noninformative prior has $\underline{V}^{-1} = 0$

- Wishart prior for the error precision matrix:

$$H \sim W\left(\underline{\nu}_H, \underline{H}\right)$$

- Noninformative prior has $\underline{\nu}_H = 0$ and $\underline{H}^{-1} = 0$

- As we did with SUR, Gibbs sampler can be set up using $p\left(\alpha|y, H\right)$ and $p\left(H|y, \alpha\right).$

- Can show

$$\alpha|y, H \sim N\left(\overline{\alpha}, \overline{V}\right),$$

where formula for $\overline{\alpha}, \overline{V}$ are on page 140 of textbook.

- And the posterior for H conditional on α is Wishart:

$$H|y, \alpha \sim W\left(\overline{\nu}, \overline{H}\right)$$

where formula for $\overline{\nu}$ and $\overline{H}$ are on pages 140-141 of textbook.

- So can set up a Gibbs sampler involving drawing from Normal and Wishart distributions.

5 Nonlinear VARs

- Many key recent papers use Bayesian methods in nonlinear extensions of VARs

- E.g. Sims and Zha (2006, AER), Primiceri (2005, ReStud) Cogley and Sargent (2001, 2005a, 2005b, various journals), Bernanke, Boivin and Eliasz (2005, QJE)

- To illustrate this literature and as a way of linking many threads together (VARs, state space models, stochastic volatility, etc.) I will present an empirical application based on Primiceri (2005)

- Cogley and Sargent use a similar framework in their work.

5.1 Primiceri's TVP-VAR

- Macroeconomic background: evolution of monetary policy

- Extensions of VARs using a few macro variables used.

- Primiceri: US quarterly data on inflation and unemployment rates ("non-policy block") and interest rates ("policy block")

- I will not get into macro issues much.

- Suffice it to note: Primiceri has identification restrictions to define "monetary policy shock", calculates various impulse responses using this (and many other things).

5.1.1 Primiceri's Model

- Remember: we wrote the VAR as

$$y_t = Z_t \alpha + \varepsilon_t$$

- Z_t is an $M \times m$ matrix of data on lags of all dependent variables and an intercept.

- ε_t are independent $N(0, \Omega)$

- Note: to be consistent with Primiceri's notation, error covariance matrix is Ω

- Primiceri extends this in two important ways.

- α becomes α_t (VAR coefficients can change over time)

- Multivariate stochastic volatility: The error covariance matrix evolves over time.

- Note: Primiceri allows error variances and covariances to evolve over time in a very general way (many other papers more restrictive).

- There is much interest in volatility issues in empirical macro today ("Great Moderation" of business cycle)

5.1.2 Evolution of VAR coefficients

- A standard state space model:

$$y_t = Z_t \alpha_t + \varepsilon_t$$

and

$$\alpha_{t+1} = \alpha_t + \eta_t,$$

where α_t an $m \times 1$ vector of states and η_t are independent $N(0, Q)$.

- Random walk evolution of VAR coefficients

- In Lecture 5 we discussed Bayesian inference (Gibbs sampling) with scalar version of this model (extension to M dependent variables straightforward)

- I have been using Durbin and Koopman (2002) algorithm for this (but many possible).

5.1.3 Multivariate Stochastic Volatility

- Now let Ω become Ω_t. Many ways to do this.

- Important issue: want error covariances to evolve over time (many specifications do not allow for this).

- Primiceri (2005) uses a triangular reduction Ω_t, such that:

$$A_t \Omega_t A_t' = \mathbf{\Sigma}_t \mathbf{\Sigma}_t'$$

or

$$\Omega_t = A_t^{-1} \mathbf{\Sigma}_t \mathbf{\Sigma}_t' \left(A_t^{-1} \right)'$$

- $\mathbf{\Sigma}_t$ is a diagonal matrix with diagonal elements $\sigma_{j,t}$ (loosely speaking error variances)

- A_t is lower triangular matrix with ones on diagonal (loosely speaking correlations between errors)

- For $\mathbf{\Sigma}_t$ – stochastic volatility.

- to be precise $h_{i,t} = \ln\left(\sigma_{i,t}\right)$, $h_t = \left(h_{1,t}, .., h_{p,t}\right)'$ then:

$$h_{t+1} = h_t + u_t,$$

where u_t is $N\left(0, W\right)$

- Gibbs sampling: draws of $h = \left(h_1', .., h_T'\right)'$ (conditional on α and the parameters of the model) use algorithm of Kim, Shephard and Chib (1998)

- What about A_t?

- Stack into vector as $a_t = \left(a_{21,t}, a_{31,t}, a_{32,t}, .., a_{p(p-1),t}\right)'$ and use

$$a_t = a_{t-1} + \zeta_t,$$

where ζ_t is $N(0, C)$

- But now we have another state space model

- Durbin and Koopman (2002) algorithm can be used to draw the states (conditional on other model parameters).

- Bottom line: Gibbs sampling algorithm can be set up which draws on off-the-shelf algorithms

5.1.4 Extension I (with coauthors) am Working On

- Primiceri's model says "all model parameters change every time period" (gradual evolution of coefficients)

- But other structural break models have a small number of more substantive breaks.

- Why not nest these two options?

- Dynamic mixture models (Gerlach, Carter and Kohn, 2000, JASA or Giordani and Kohn, 2008, JBES).

- Work with Primiceri's model except for the following modifications:

$$\alpha_{t+1} = \alpha_t + K_{1t}\eta_t,$$

$$h_{t+1} = h_t + K_{2t} u_t$$

$$a_t = a_{t-1} + K_{3t} \zeta_t.$$

- K_{1t}, K_{2t} and K_{3t} are 1/0 variables indicating whether a break has/has not taken place

- Hierarchical prior

$$p\left(K_{jt}\right) = p_j$$

for $j = 1, 2, 3$ where p_j is unknown parameter (probability of break)

- Can use Primiceri's with some extra blocks added to the Gibbs sampler

- Gerlach, Carter and Kohn (2000) is a very efficient algorithm for drawing K_{jt} (conditional on other model parameters).

- Bottom line: I have set up a Gibbs sampler which combines existing algorithms to carry out posterior inference in this extension of Primiceri's model.

- Much recent Bayesian empirical macro adapts this kind of strategy. Gibbs sampling naturally divides problems into blocks, each block taken from existing literature.

5.1.5 Preliminary Empirical Results

- US data on inflation and unemployment rates and interest rates 1947Q1-2006Q3

- With my extension, I am getting results suggesting Primiceri's specification is a good one.

- E.g. $E(p_1|Data) = 0.92$, $E(p_2|Data) = 0.97$ and $E(p_3|Data) = 0.62$.

- Breaks occur in most periods (like in a TVP model).

- Figures 1 through 6 illustrate the kind of things Primiceri presents in his paper (of interest to macro policy)

- Figures 1 through 3 are volatilities

- Figures 4 through 6 are correlations between the errors in the VAR

- These are posterior means (i.e. estimates), but could calculate posterior standard deviations or anything else (e.g. prediction)

Figure 1: Volatility (st dev) in Inflation Equation

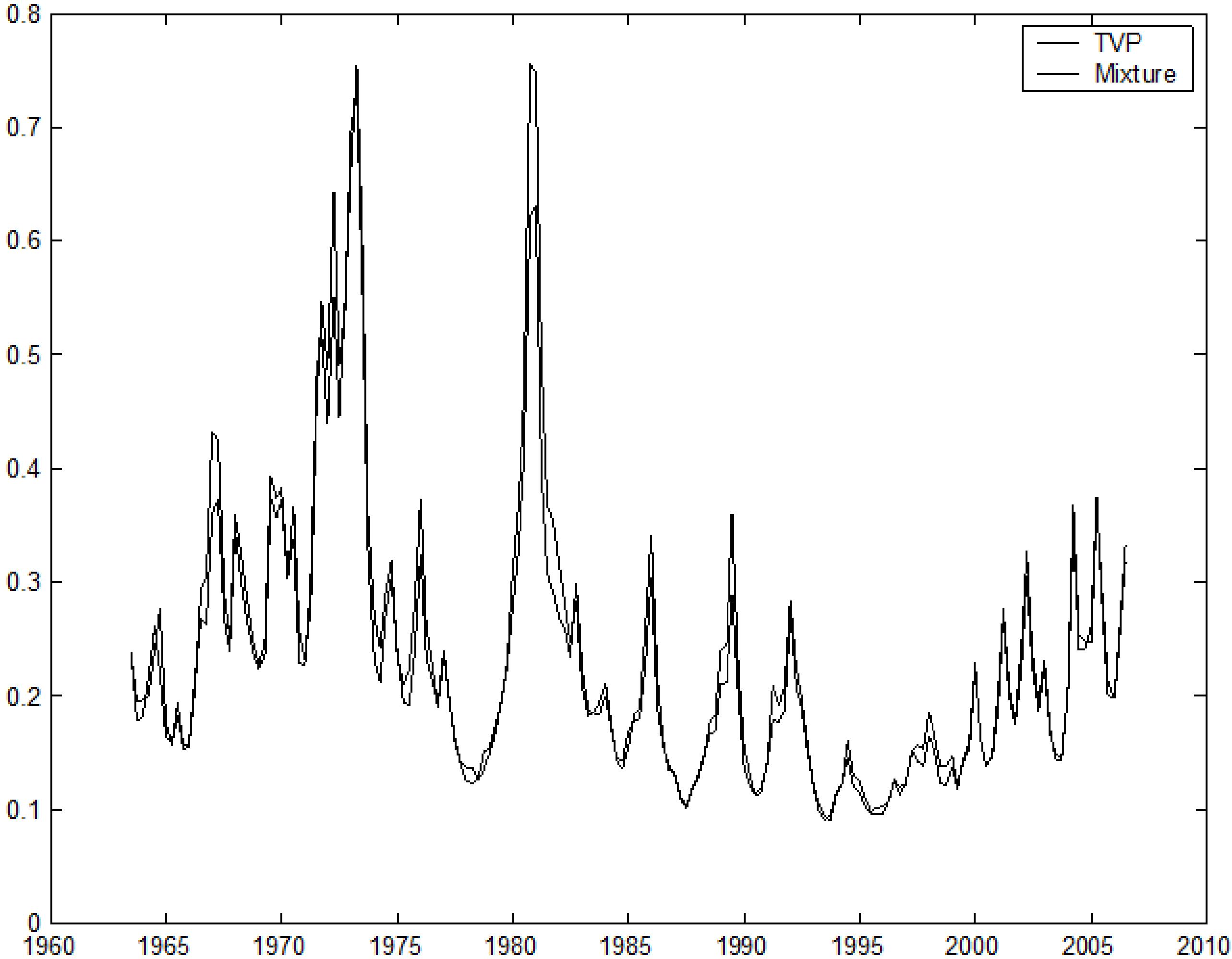


Figure 2: Volatility (st dev) in Unemployment Equation

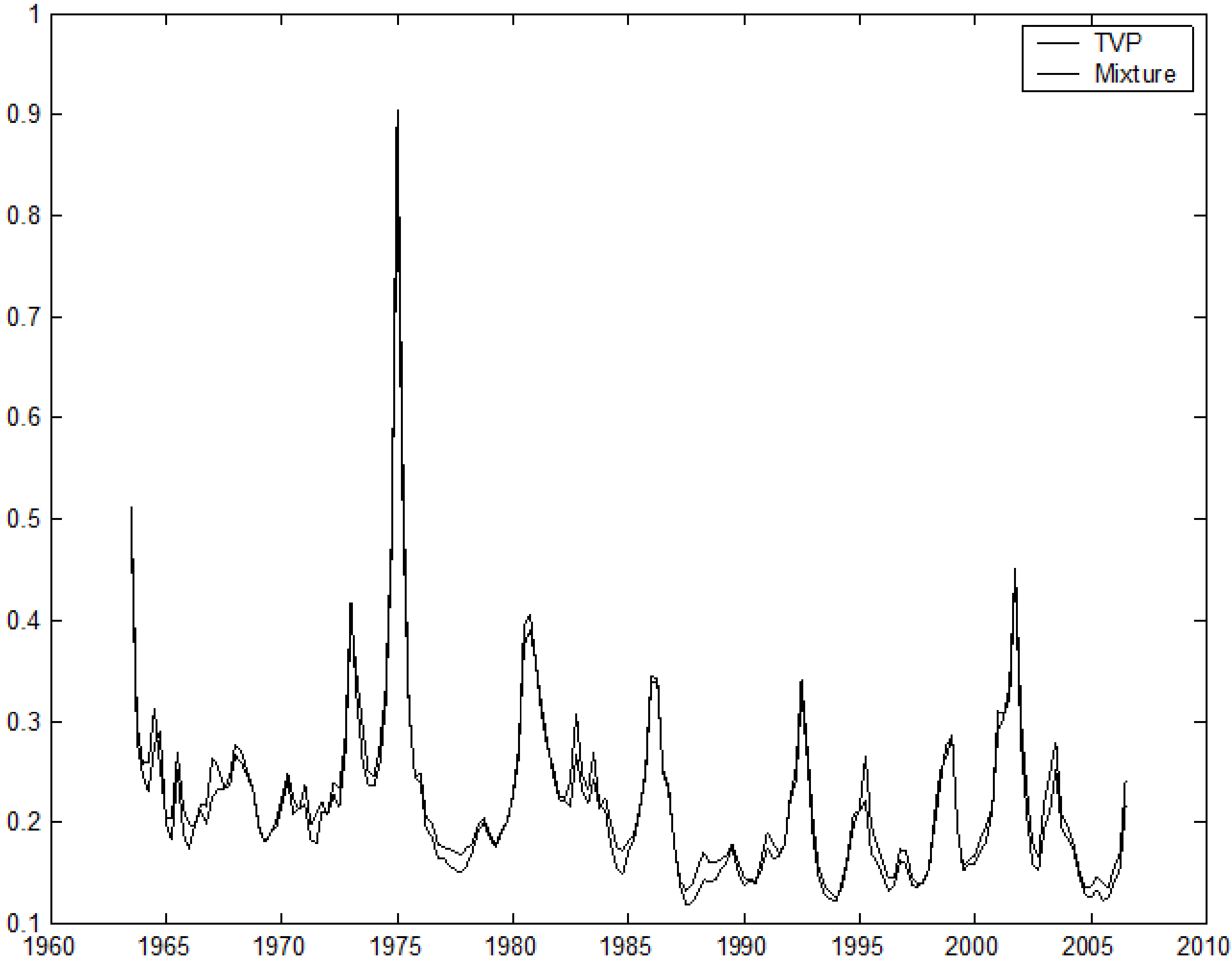


Figure 3: Volatility (st dev) in Interest Rate Equation

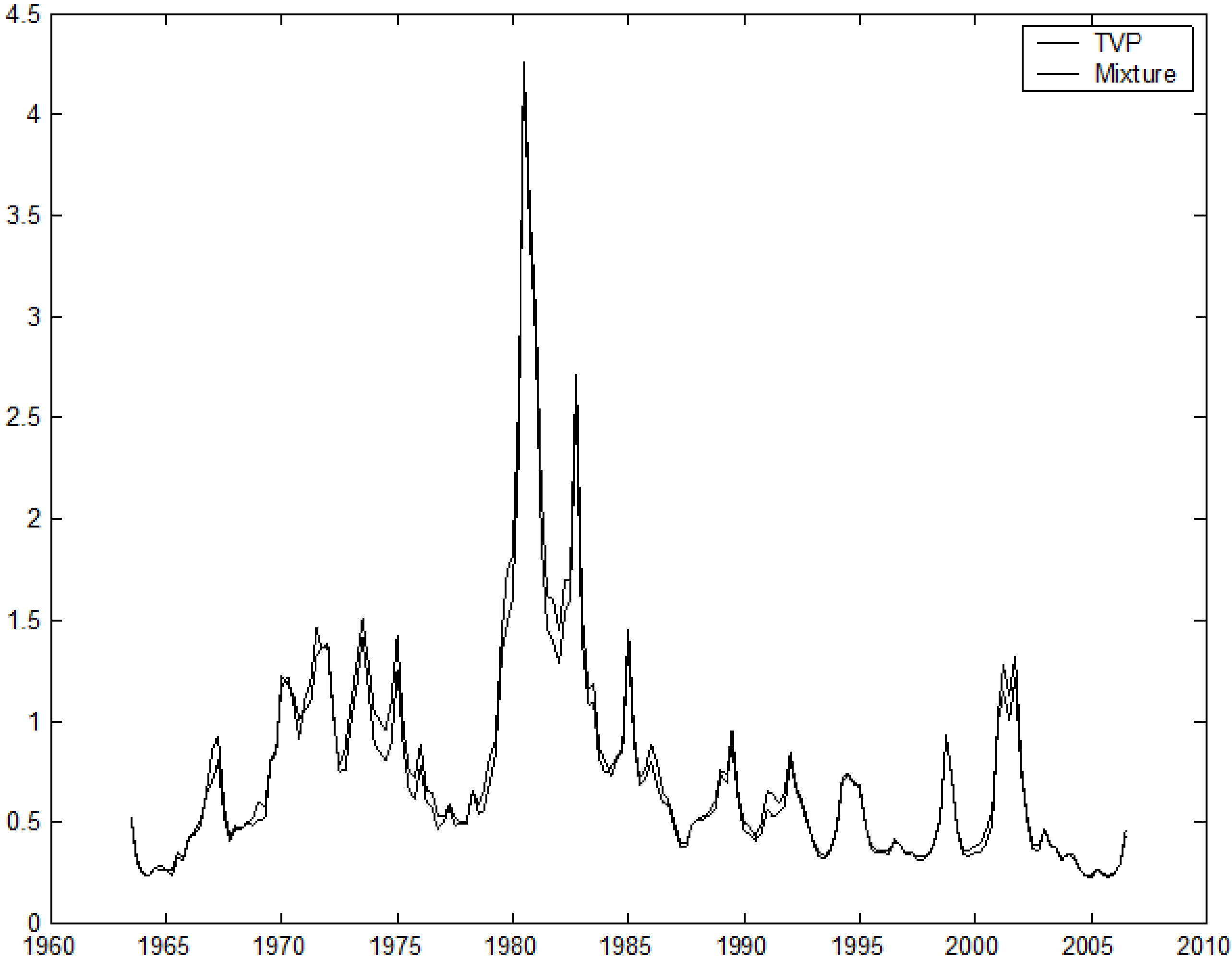


Figure 4: Correlation between Variance in Inflation and unemployment Equation

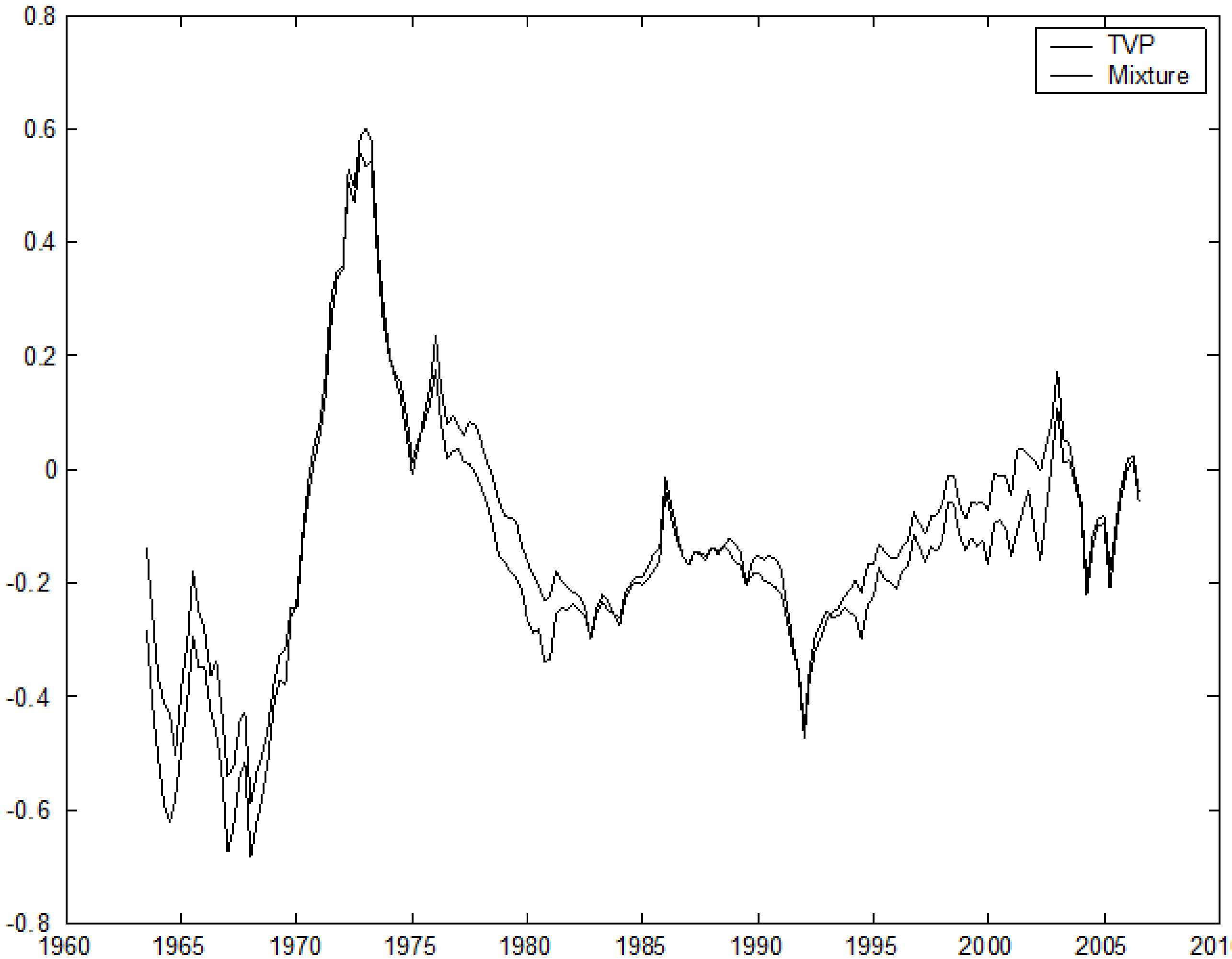


Figure 5: Correlation between Variance in Inflation and interest rate Equation

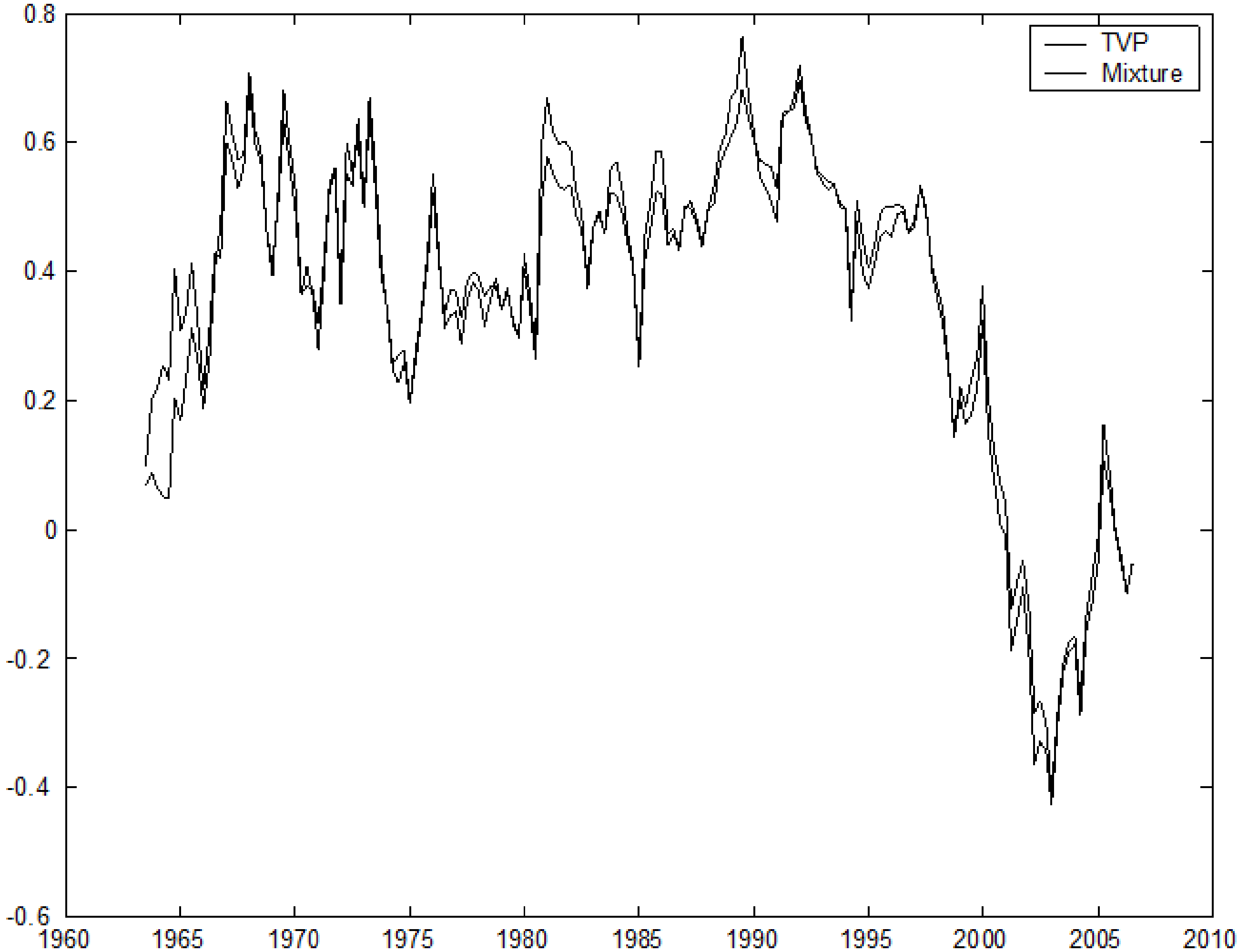


Figure 6: Correlation between Variance in Unemployment and interest rate Equation