Fast solution techniques for high-order Discontinuous Galerkin methods on polygonal/polyhedral grids.

Paola F. Antonietti

MOX-Dipartimento di Matematica, Politecnico di Milano

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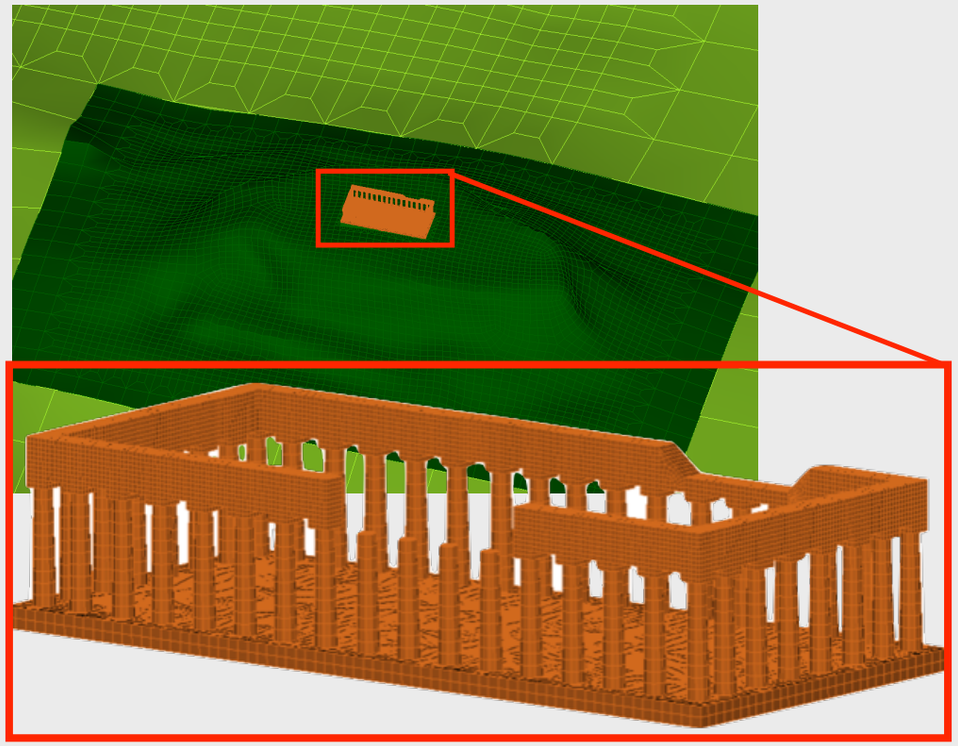
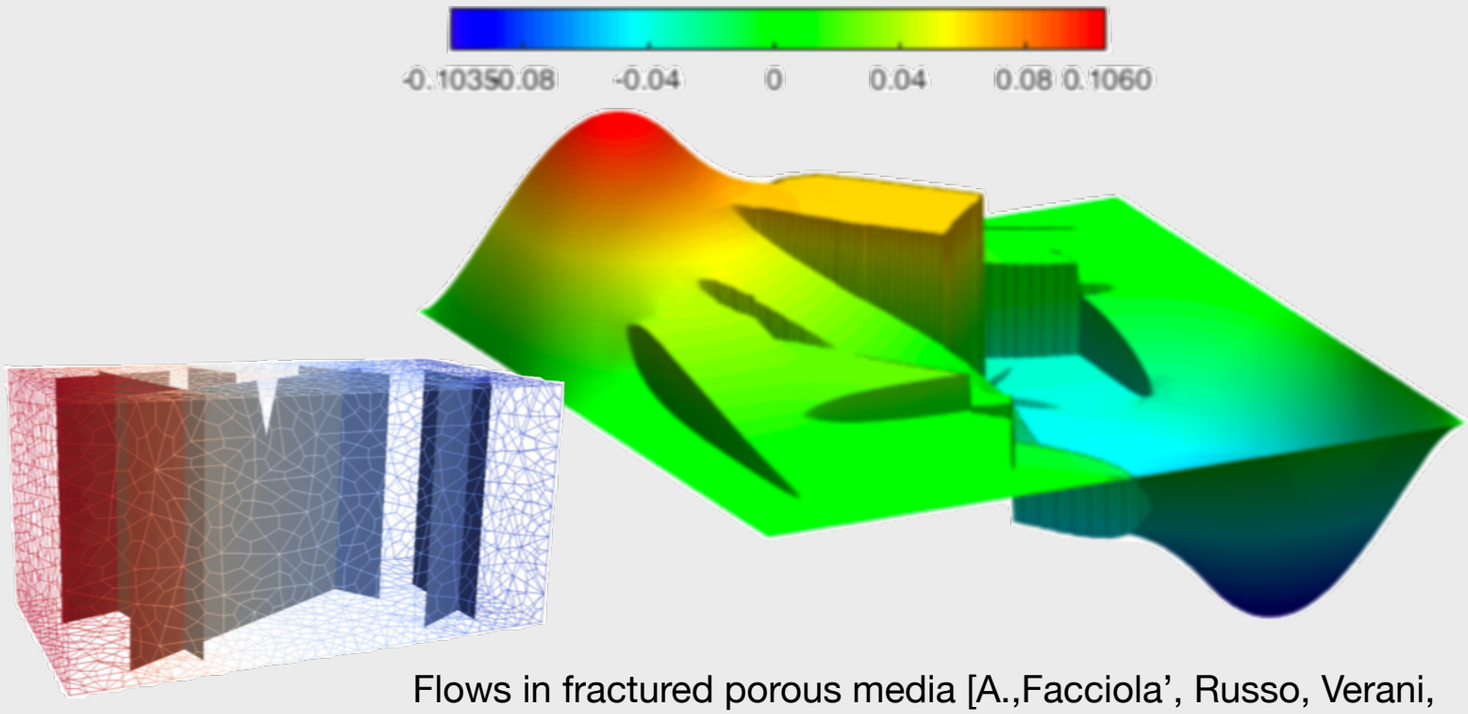
Motivations

For classical FEM approximation based on triangular/tetrahedral or quadrilateral/hexahedral elements the process of mesh generation represents a bottleneck if the computational domain is complicated.



[A., Melas, Mazzieri, Quarteroni, 2019]



Flows in fractured porous media [A.,Facciola', Russo, Verani, 2019], [A., Facciola', Verani, 2019]

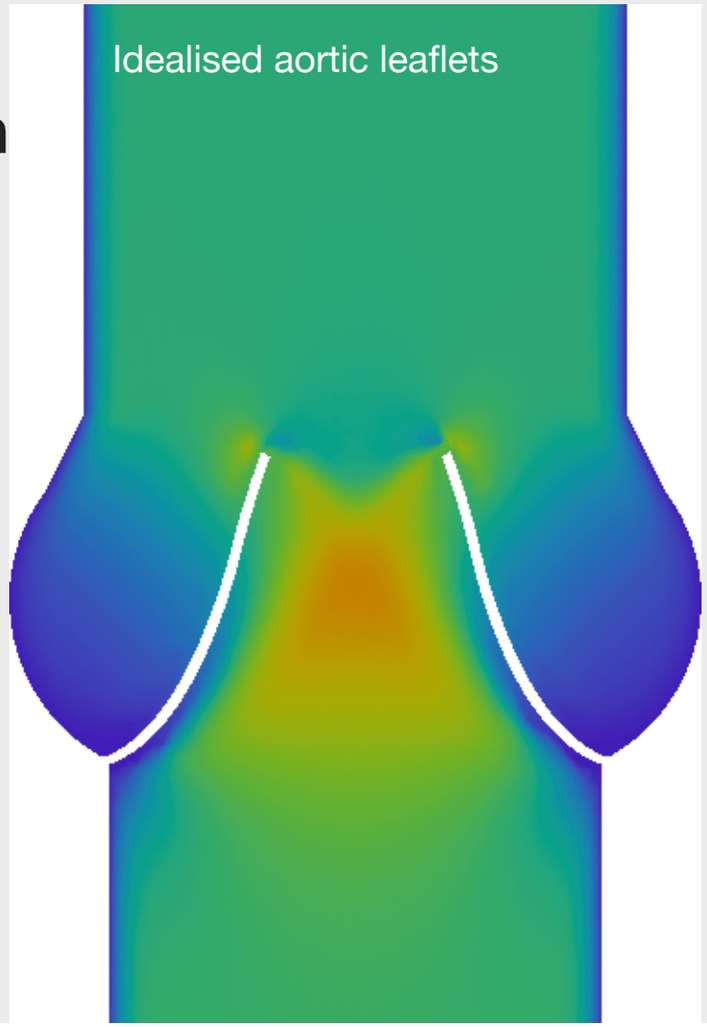
Polygonal/polyhedral shaped elements allow to tame the computational complexity related to the mesh generation process/handling.

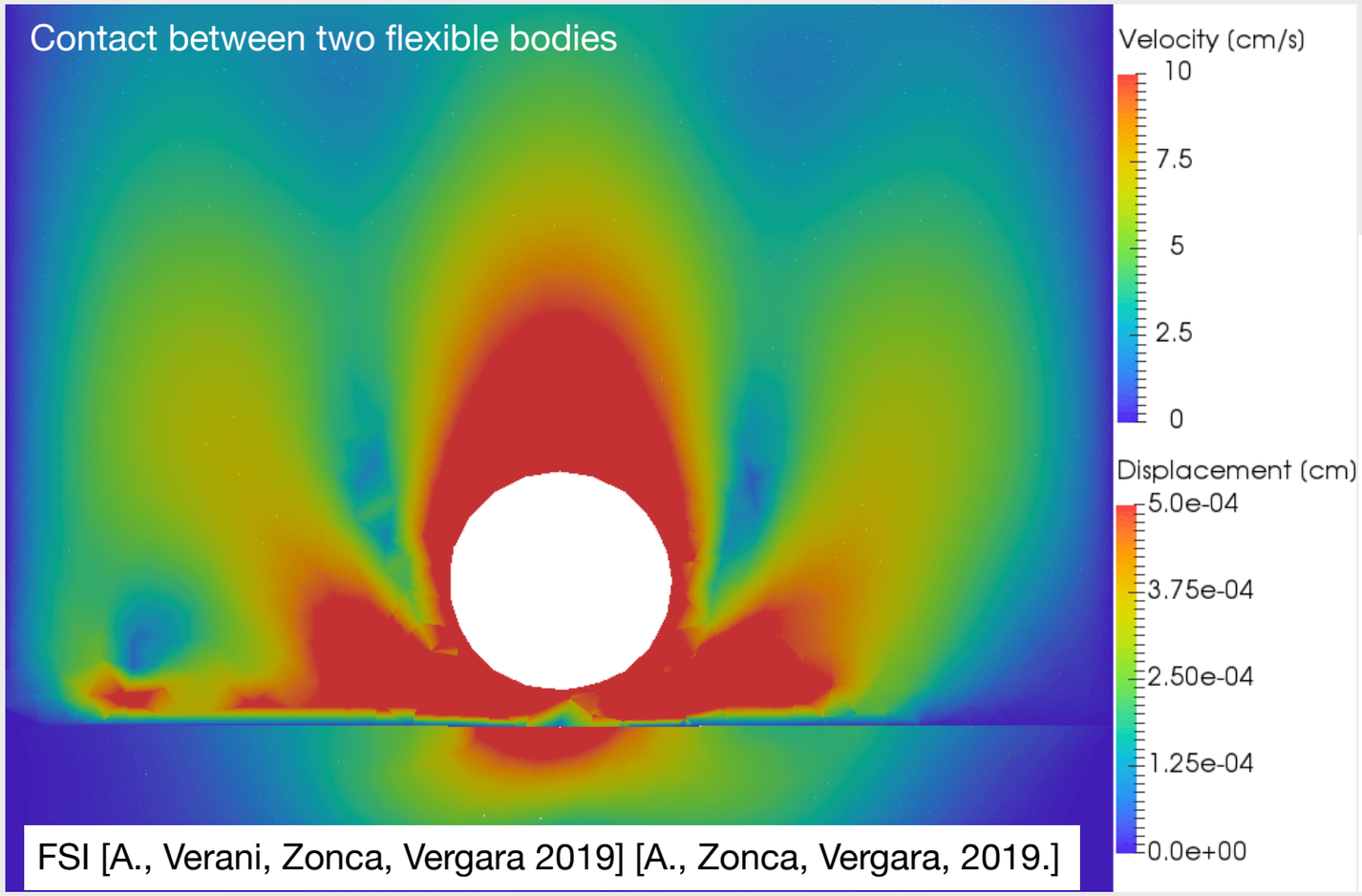
Motivations

Develop discretisation methods for the approximation of PDEs posed on complicated domains:

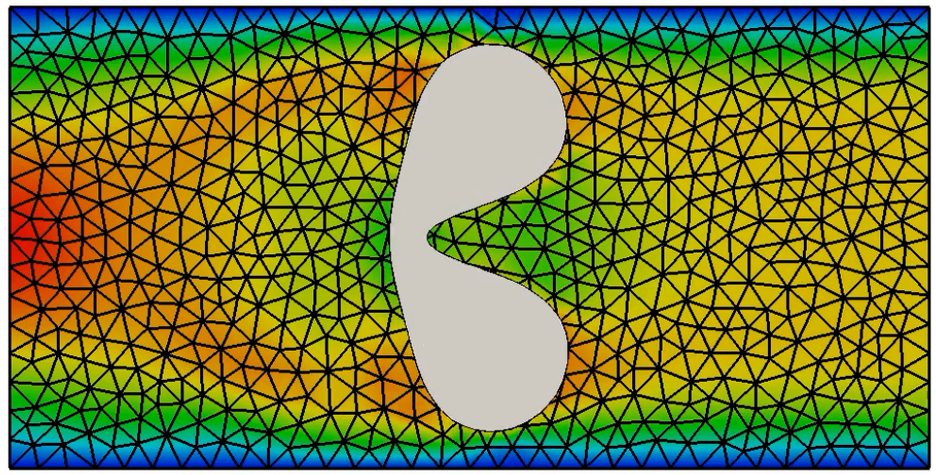
- ☑ high-order accuracy;
- ☑ geometric flexibility (polygonal/polyhedral grids).

- Define and analyse the method;
- Develop and analyse (agglomeration-based) solvers;
- Improve computational efficiency.



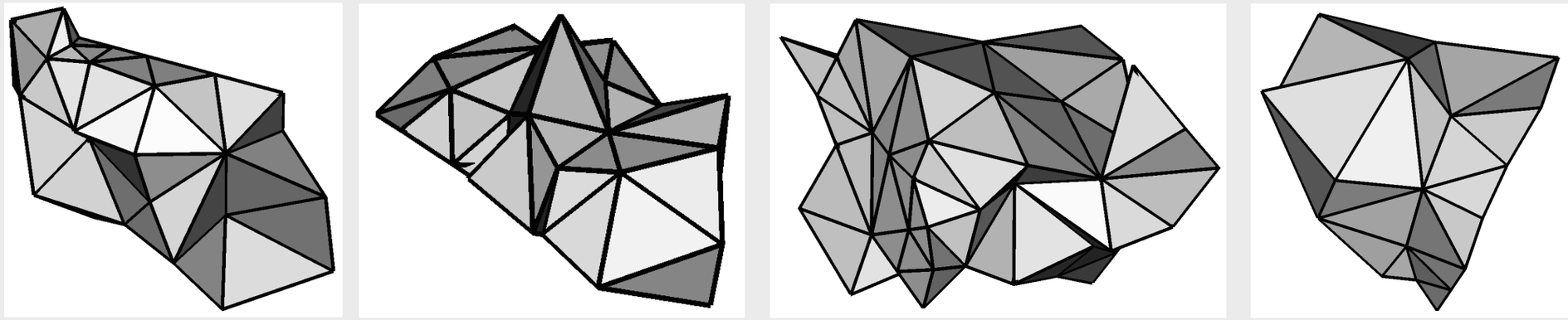


FSI [A., Verani, Zonca, Vergara 2019] [A., Zonca, Vergara, 2019.]



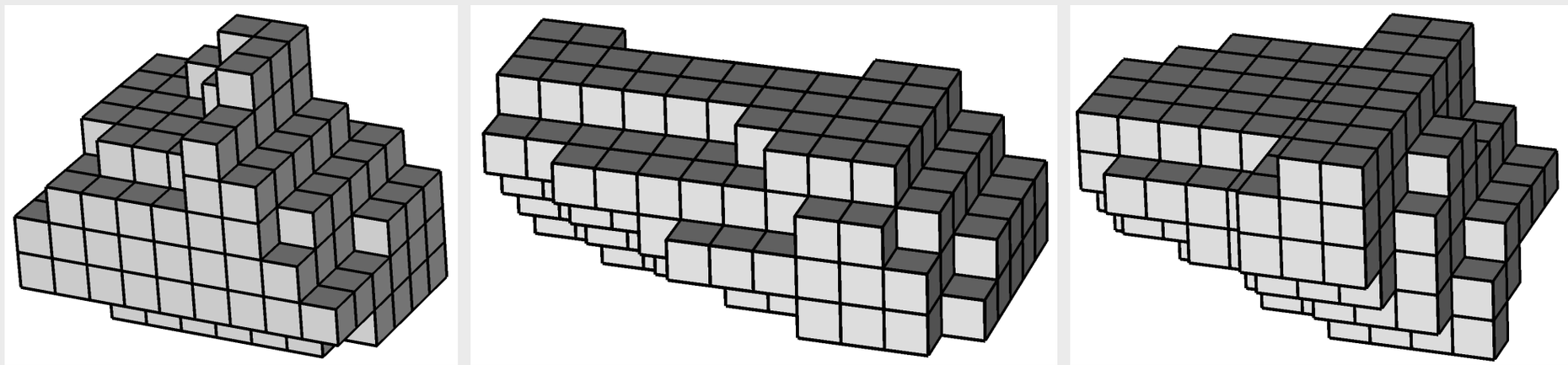
Evolution of an ideal red blood cell experiencing very large displacements

Agglomeration



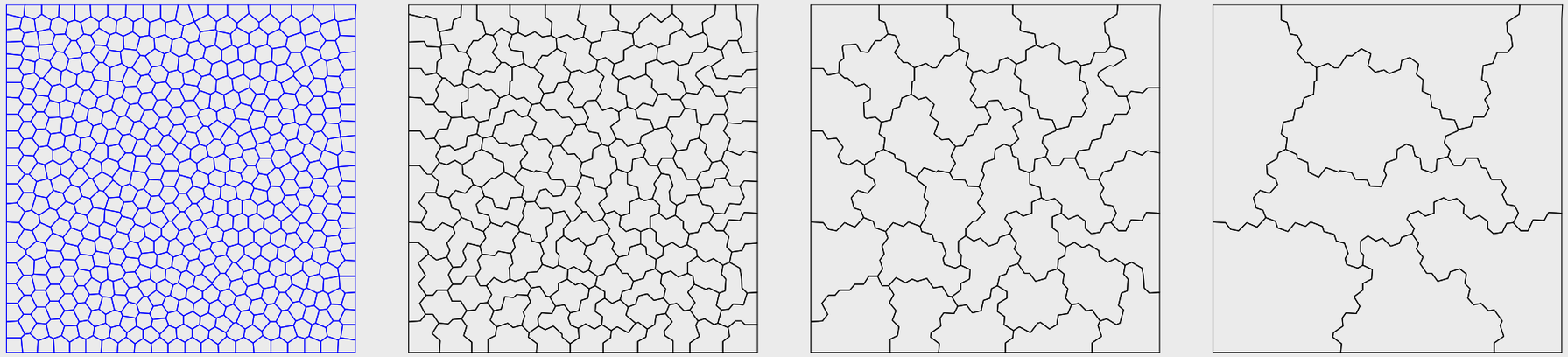
Agglomeration of a tetrahedral Mesh

Courtesy of P. Houston



Agglomeration of a voxel Mesh

Courtesy of P. Houston



Agglomeration of polygonal mesh

Mathematical challenges

- Elements' of arbitrarily shape
- Many faces per elements (grow as the number of agglomeration steps)
- "Small" faces compared with the diameter of the corresponding element

Outline

- DGFEMs on Polytopic Meshes
- Agglomeration-based multigrid solvers
- A Massively Parallel Additive Schwarz preconditioner
- Summary and Outlook

Model problem and its hp-DGFEM approximation

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $f \in L_2(\Omega)$: find u such that

$$-\nabla \cdot (\rho \nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $0 < \rho_0 \leq \rho$.

We consider a tessellation of the domain into polygonal/polyhedral elements

We set

$$V_h = \{u \in L_2(\Omega) : u|_\kappa \in \mathcal{P}_{p_\kappa}(\kappa)\ \forall \kappa \in \mathcal{T}_h\},$$

where $\mathcal{P}_p(\kappa)$ denotes the set of polynomials of degree at most $p \geq 1$ over κ.

hp–DGFEM: find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v_h) = \int_\Omega f v_h \mathrm{d}\mathbf{x}$$

for all $v_h \in V_h$.

DGFEMs on polytopic grids

Find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v_h) = \int_\Omega f v_h \, d\mathbf{x}$$

for all $v_h \in V_h$, where

$$\begin{aligned} \mathcal{A}_h(u, v) \quad = \quad & \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \rho \nabla u \cdot \nabla v \, d\mathbf{x} + \sum_{F \in \mathcal{F}_h} \int_F \sigma \, [\![u]\!] \cdot [\![v]\!] \, ds \\ & - \sum_{F \in \mathcal{F}_h} \int_F \left(\{\!\{ \rho \nabla_h v \}\!\}_\omega \cdot [\![u]\!] + \{\!\{ \rho \nabla_h u \}\!\}_\omega \cdot [\![v]\!] \right) ds \end{aligned}$$

$\{\!\{\cdot\}\!\}_\omega$: ρ–Weighted Average Operator $\quad$ $[\![\cdot]\!]$: Jump Operator

For the analysis see, e.g., [Cangiani, Dong, Georgoulis, Houston, 2017] and the references therein.

The algebraic problem

Find $u \in \mathbb{R}^n$ such that $Au = F$

➡ A is a large sparse, s.p.d. and ill-conditioned:

$$\kappa(A) = \mathcal{O}(p^4 h^{-2})$$

A. & Houston 2011

- The solver should be scalable with respect to both *h*- and *p*-, as well as the diffusion coefficient.

- Exploit agglomeration-based strategies as coarse level corrections with multilevel solvers.

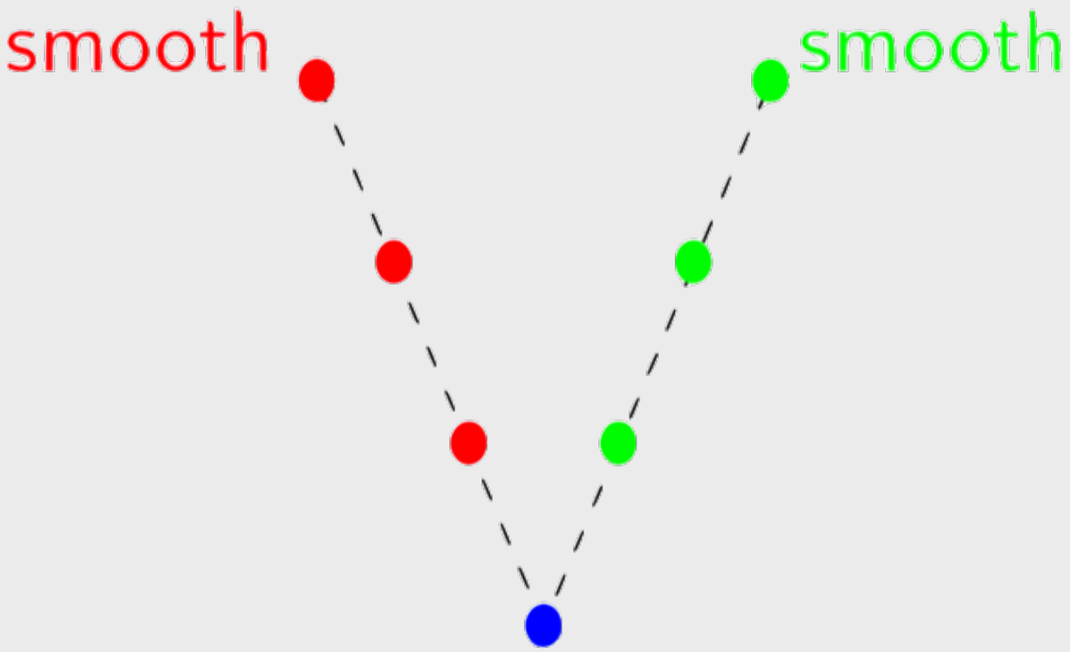
Agglomeration-based multigrid solvers

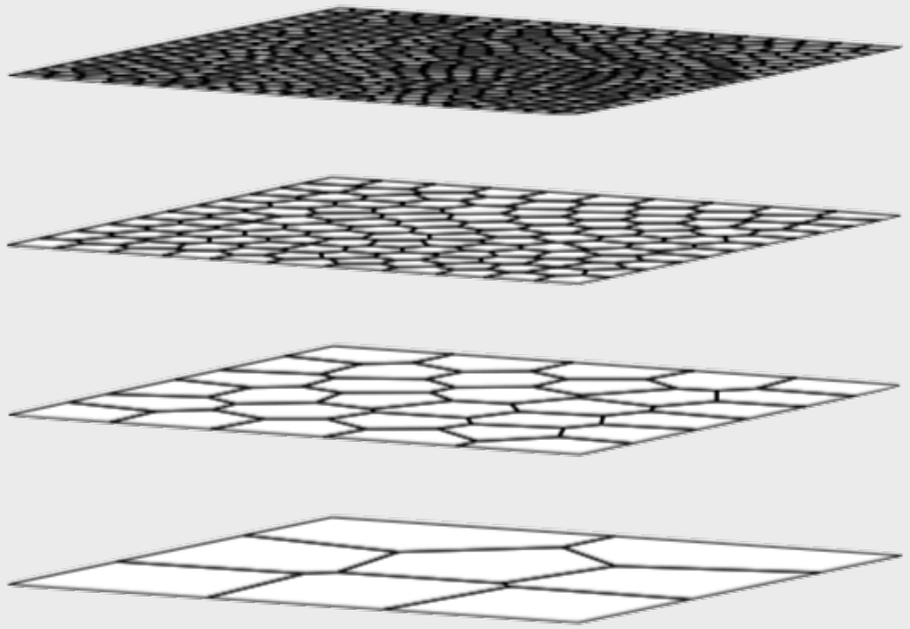
Multigrid methods: the idea

Find $u \in \mathbb{R}^n$ such that $Au = F$

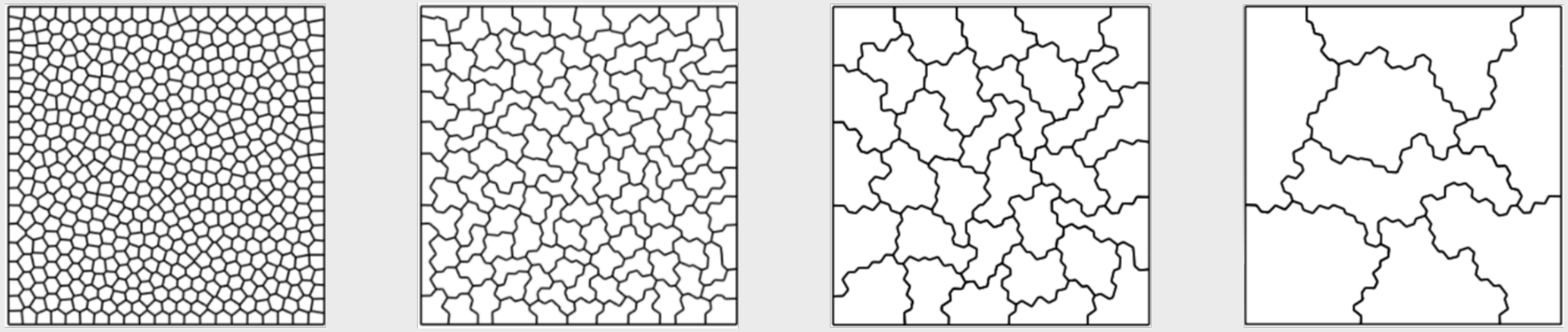
Construct a sequence $u^{(k)}, k = 1,2,3,$ such that $u^{(k)} \longrightarrow u,$ exploiting a set of coarser (and cheaper) subproblems

$$u^{(k)} \longrightarrow u^{(k+1)}$$





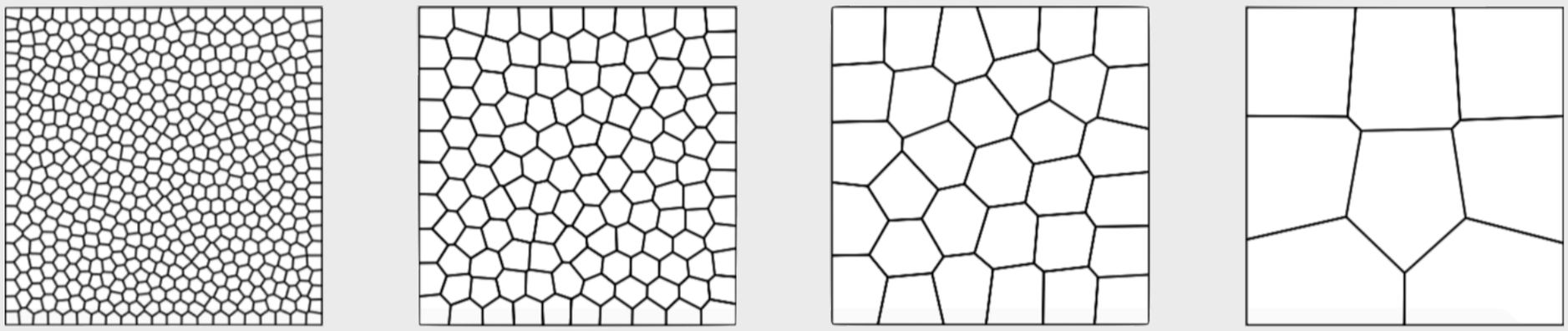
Agglomeration-based Multigrid (MG) solvers



Pure agglomeration (nested grids)

Fine grid

Coarse grid

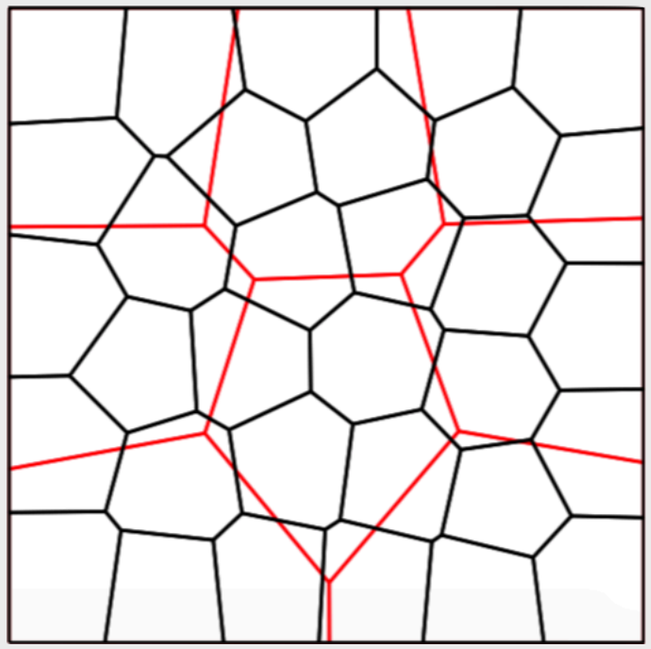


Agglomeration with edge/face coarsening (non-nested grids)

Ingredients for MG methods

1. $\left\{\mathcal{T}_j\right\}_{j=1,\ldots,J}$: a sequence of grids (with diameters $h_j \geq h_{j-1}$)
2. $\left\{V_j\right\}_{j=1,\ldots,J}$: a sequence of DG spaces defined over $\mathcal{T}_j$
3. $\mathcal{A}_j : V_j \times V_j \longrightarrow \mathbb{R}$ DG bilinear forms on the level $j = 1,\ldots,J$

Find $u \in \mathbb{R}^n$ such that $Au = F$ ⟶ By convention the finest level correspond to the level J, i.e. $\mathcal{T}_J \equiv \mathcal{T}_h, V_J \equiv V_h$,



Two levels non-nested $\mathcal{T}_j$ fine and $\mathcal{T}_{j-1}$ coarse grids.

V -cycle MG iteration

One iteration of the V-cycle MG algorithm is given by

$$u^{(k)} \longrightarrow u^{(k+1)} := \mathrm{MG}(J, f, u^{(k)}, m_1, m_2)$$

Function MG(j, g, z, m_1, m_2)

$z^{(0)} = z$

if j=1 **then**

$\mathrm{MG}(1, g, z_0, m_1, m_2) = A_1^{-1} f.$

else

Pre-smoothing:

for $i = 1, \ldots, m_1$ **do**

$z^{(i)} = z^{(i-1)} + B_j^{-1}(f - A_j z^{(i-1)});$

end for

Recursive coarse grid correction:

$r_{j-1} = I_j^{j-1}(f - A_j z^{(m_1)});$

$e_{j-1} = \mathrm{MG}(j-1, r_{j-1}, 0, m_1, m_2);$

$z^{(m_1+1)} = z^{(m_1)} + I_{j-1}^{j} e_{j-1};$

Post-smoothing:

for $i = m_1 + 2, \ldots, m_1 + m_2 + 1$ **do**

$z^{(i)} = z^{(i-1)} + B_j^{-1}(f - A_j z^{(i-1)});$

end for

$\mathrm{MG}(j, g, z_0, m_1, m_2) = z^{(m_1+m_2+1)}.$

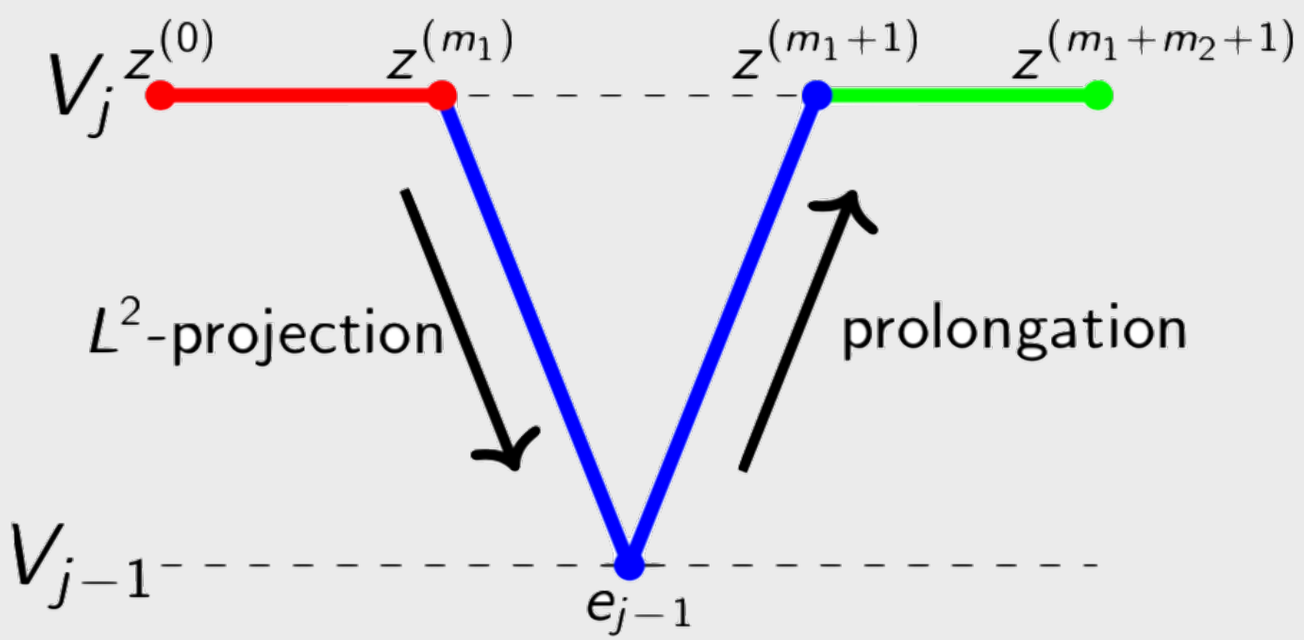
end if

Iterative V-cycle for $Au = f$

Given an initial guess $u^{(0)}$

until convergence do

$$u^{(k+1)} = \mathrm{MG}(J, f, u^{(k)}, m_1, m_2)$$



Convergence analysis

Theorem

Let the smoother B be the Richardson operator, then $u^{(k)} \longrightarrow u$ uniformly wrt h, p, and the number of levels provided that $m_1 = m_1(p)$ and $m_2 = m_2(p)$ are chosen sufficiently large.

[A., Pennesi, 18]

Stability estimate:

$$\|(\mathrm{Id}_j - I_{j-1}^{j} P_j^{j-1}) u_h\|_{1,j} \lesssim \|u_h\|_{1,j} \quad \forall u_h \in V_j, \quad j \geq 2;$$

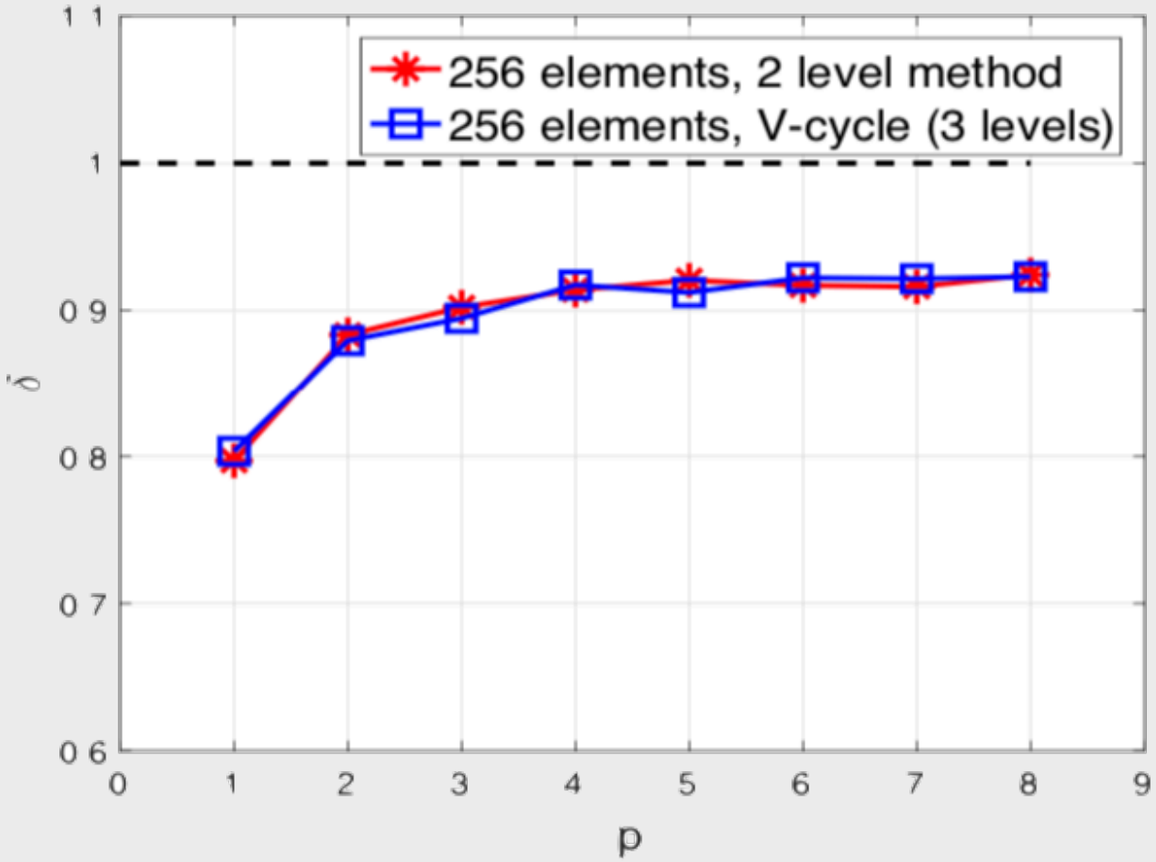
Regularity-approximation property:

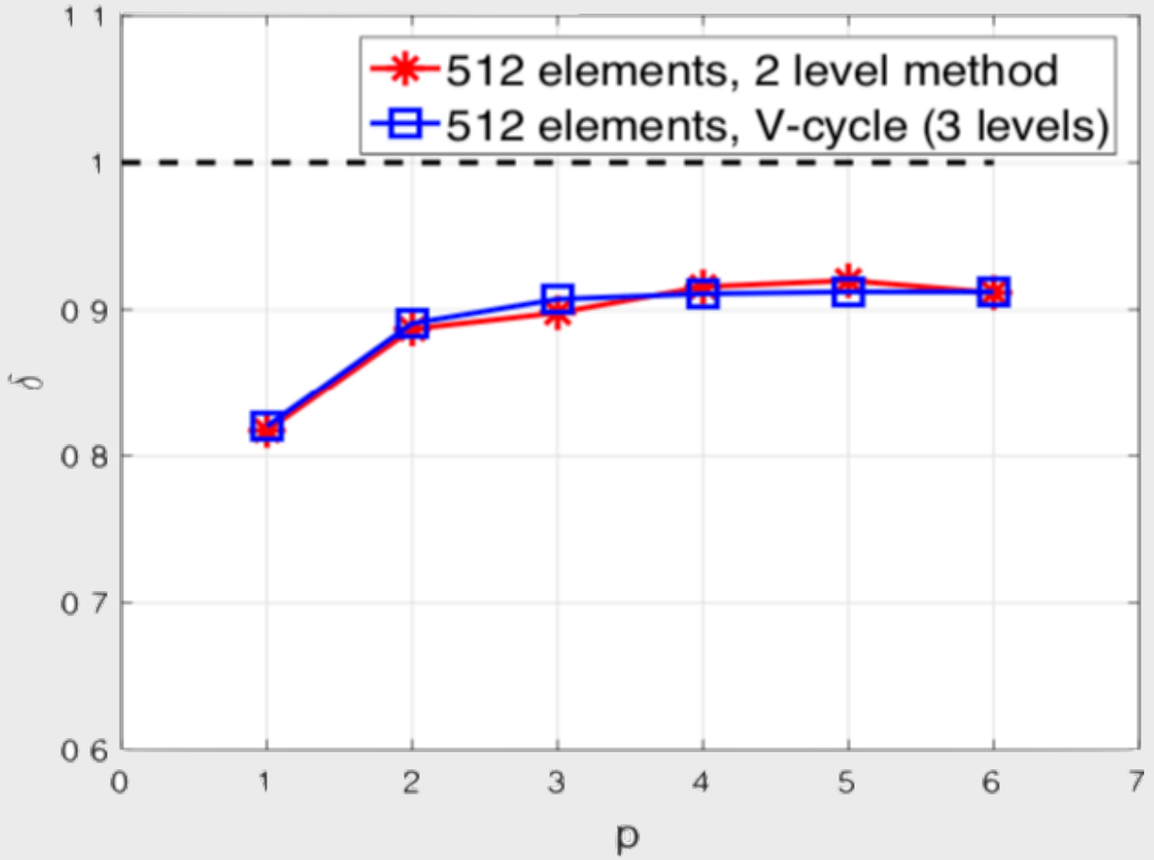
$$\left|\mathcal{A}_j((\mathrm{Id}_j - I_{j-1}^{j} P_j^{j-1}) u_h, u_h)\right| \lesssim \frac{\|u_h\|_{2,j}^2}{\Lambda_j} \quad \forall u_h \in V_j, \quad j \geq 2;$$

Smoothing property: $\mathcal{R} = \left(\mathrm{Id}_j - G_j^* G_j\right) A_j^{-1}$, $G_j = \mathrm{Id}_j - B_j^{-1} A_j$,

$$\frac{\|u_h\|_{L^2(\Omega)}}{\Lambda_j} \lesssim \left(\mathcal{R} u_h, u_h\right)_{L^2(\Omega)} \quad \forall u_h \in V_j, \quad j \geq 2.$$

Numerical results





Estimate of the contraction factor (δ) wrt p. Richardson smoother. Nested grids.

Analogous result for

- scalability wrt to h
- scalability wrt the number of levels

A Massively Parallel Additive Schwarz smoother

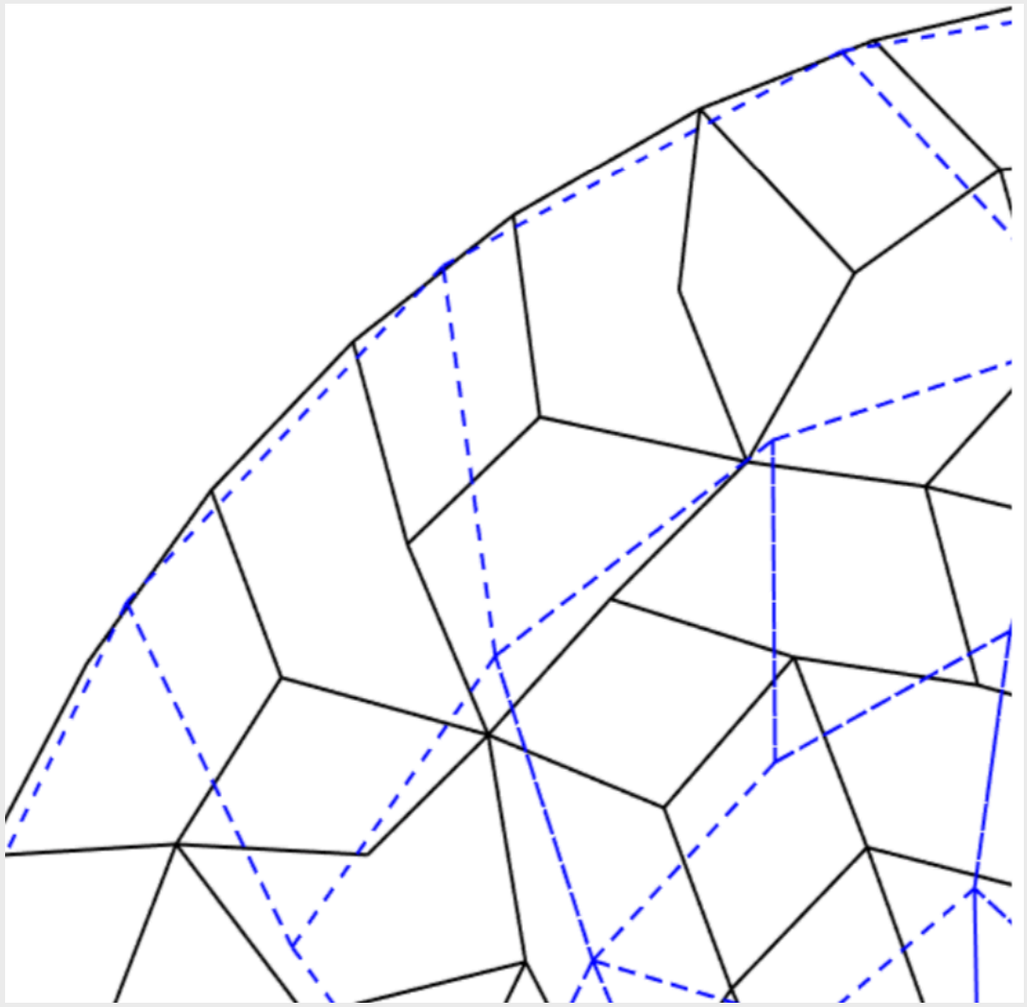
Replace the Richardson smoother with the Preconditioned Gradient Method where the preconditioner is based on the Massively Parallel Additive Schwarz (MPAS) preconditioner (see later).

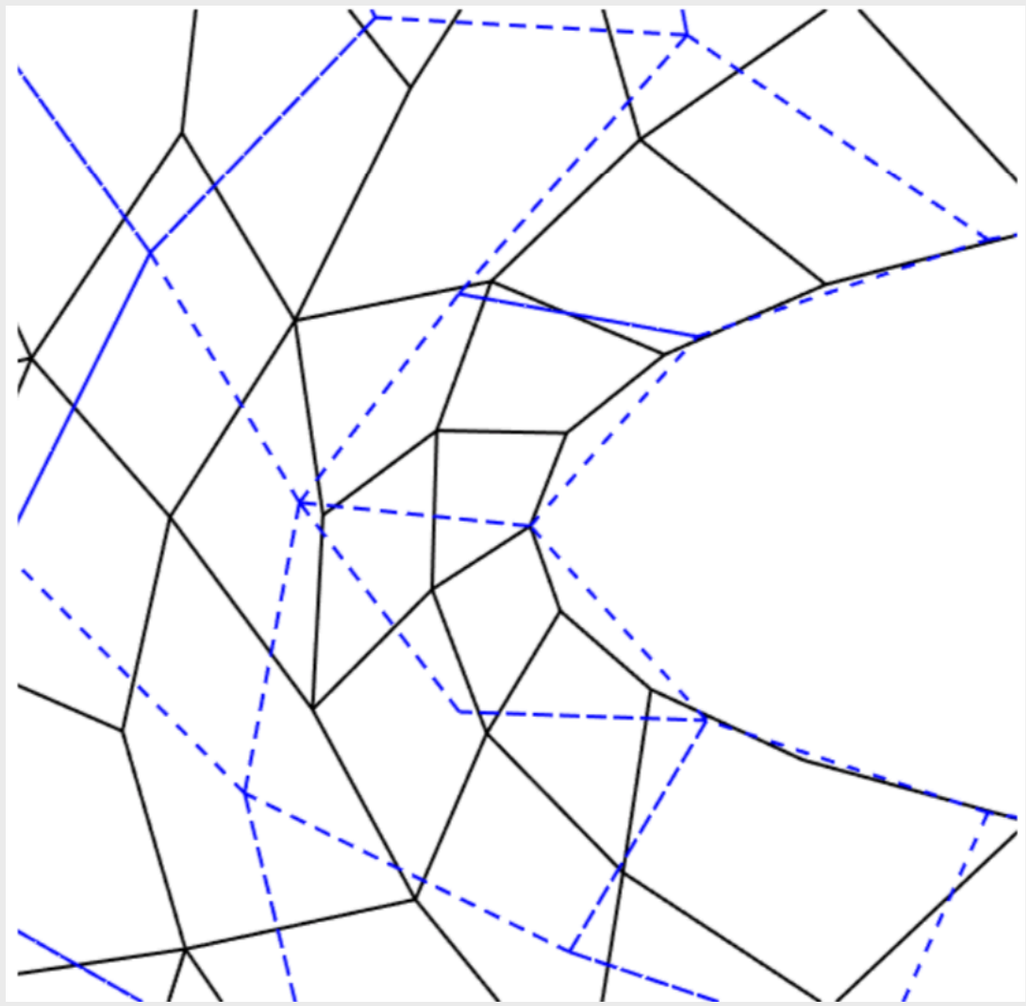
$p=2$	$\lvert\mathcal{T}_h\rvert = 1024$			$\lvert\mathcal{T}_h\rvert = 2048$			$\lvert\mathcal{T}_h\rvert = 4096$		
	2 lev.	3 lev.	4 lev.	2 lev.	3 lev.	4 lev.	2 lev.	3 lev.	4 lev.
$m=3$	15	15	15	14	14	14	12	12	12
$m=5$	8	8	8	7	7	7	7	7	7
$m=8$	4	4	4	4	4	4	4	4	4

$p=4$	$\lvert\mathcal{T}_h\rvert = 1024$			$\lvert\mathcal{T}_h\rvert = 2048$			$\lvert\mathcal{T}_h\rvert = 4096$		
	2 lev.	3 lev.	4 lev.	2 lev.	3 lev.	4 lev.	2 lev.	3 lev.	4 lev.
$m=3$	22	23	23	24	26	26	24	25	25
$m=5$	12	12	12	11	11	11	12	12	12
$m=8$	6	6	6	6	6	6	6	6	6

Iteration counts of the V -cycle MG solver with the MPAS smoother as a function of h and the number of smoothing steps $m := m_1 = m_2$. Nested grids.

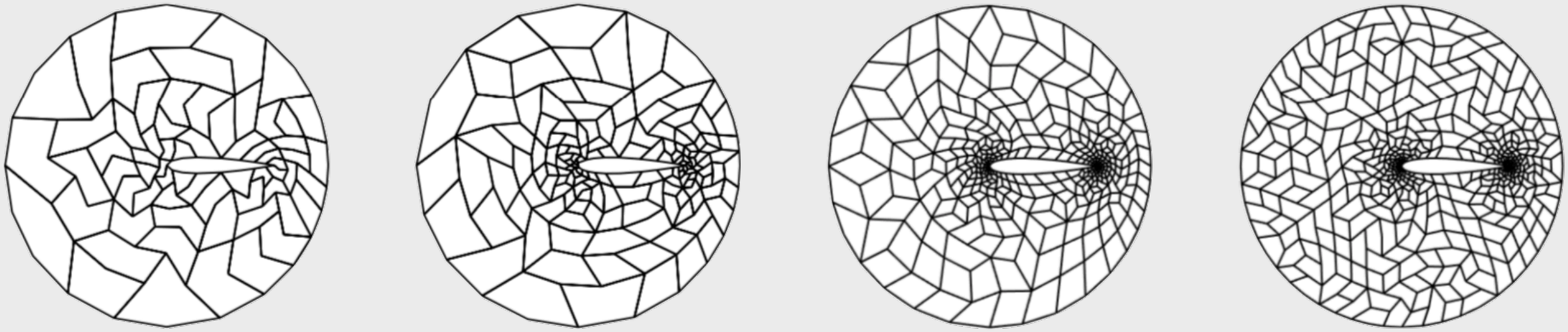
Application: domains with curved boundary





Non-nested fine $\mathscr{T}_h$ (solid line) and coarse $\mathscr{T}_H$ (dashed line) grids for a domain with a curved boundary.

Airfoil profile



$p = 1$	$\lvert\mathcal{T}_h\rvert = 292$			$\lvert\mathcal{T}_h\rvert = 676$			$\lvert\mathcal{T}_h\rvert = 2709$		
	2 lev	3 lev	4 lev	2 lev	3 lev	4 lev	2 lev	3 lev	4 lev
$m = 3$	12	11	11	15	14	14	15	15	15
$m = 5$	6	6	6	8	8	8	8	8	8
$m = 8$	4	4	4	4	4	4	4	4	4

$p = 3$	$\lvert\mathcal{T}_h\rvert = 292$			$\lvert\mathcal{T}_h\rvert = 676$			$\lvert\mathcal{T}_h\rvert = 2709$		
	2 lev	3 lev	4 lev	2 lev	3 lev	4 lev	2 lev	3 lev	4 lev
$m = 3$	40	55	43	48	44	49	41	36	38
$m = 5$	14	16	14	17	17	18	16	16	16
$m = 8$	7	7	7	9	9	9	8	8	8

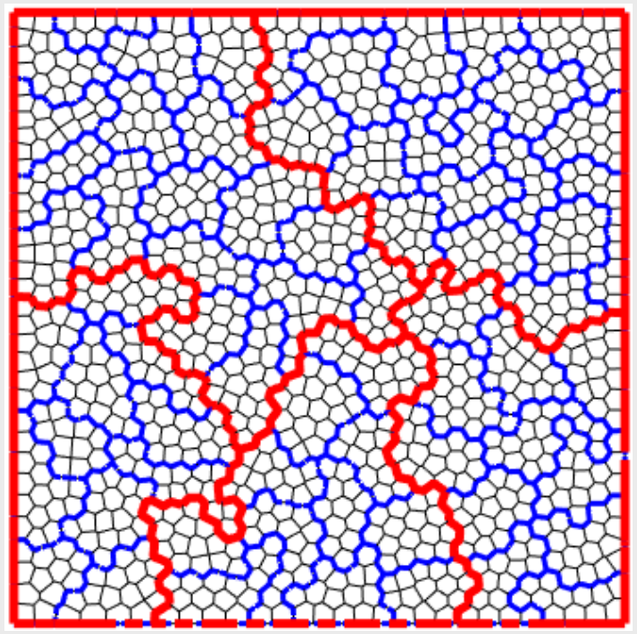
Iteration counts of the V-cycle MG solvers with the MPAS smoother as a function of $m := m_1 = m_2$ and the number of smoothing steps.

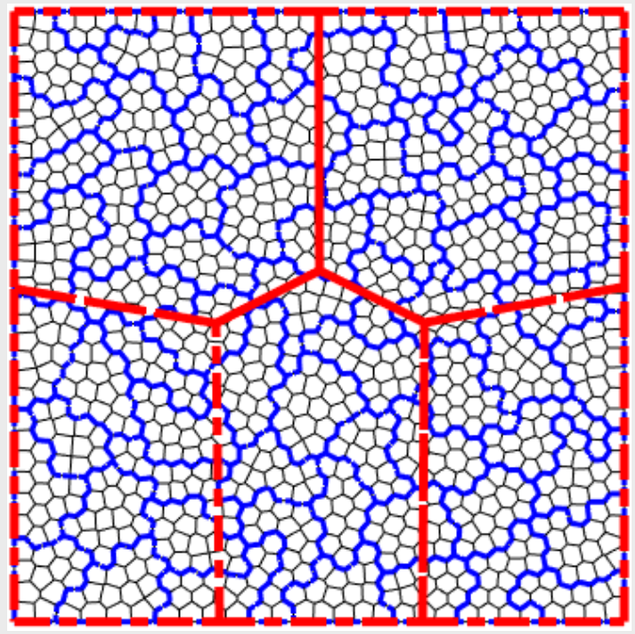
A Massively Parallel Additive Schwarz preconditioner

Partitions

Coarse partion

Subdomains



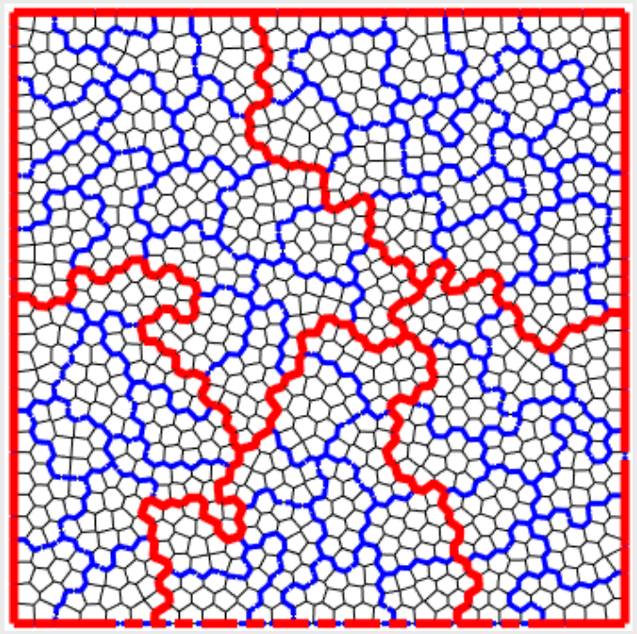


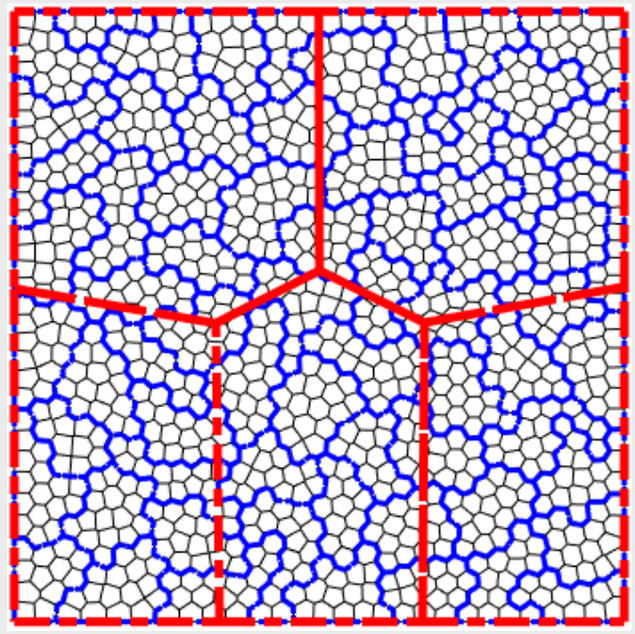
- $\mathcal{T}_{\mathbb{H}} := \{\Omega_1, \ldots, \Omega_{N_{\mathbb{H}}}\}$ of size $\mathbb{H} := \max_{1 \le i \le N_{\mathbb{H}}} \{\mathrm{diam}(\Omega_i)\}$.
- $\mathcal{T}_H := \{\mathcal{D}_1, \ldots, \mathcal{D}_{N_H}\}$ of size $H := \max_{1 \le j \le N_H} \{\mathrm{diam}(\mathcal{D}_j)\}$.

Fine grid $\mathcal{T}_h$ is the fine grid;
Sub-domains $\mathcal{T}_{\mathbb{H}}$ obtained by agglomeration of elements of $\mathcal{T}_h$;
Coarse grid $\mathcal{T}_H$ obtained by:

1. agglomeration of elements of $\mathcal{T}_{\mathbb{H}}$ (nested);
2. agglomeration + edge-coarsening (non-nested).

Partitions (cont'd)





Coarse partion

Subdomains

- $\mathcal{T}_{\mathbb{H}} := \{\Omega_1, \ldots, \Omega_{N_{\mathbb{H}}}\}$ of size $\mathbb{H} := \max_{1 \le i \le N_{\mathbb{H}}} \{\text{diam}(\Omega_i)\}$.
- $\mathcal{T}_H := \{\mathcal{D}_1, \ldots, \mathcal{D}_{N_H}\}$ of size $H := \max_{1 \le j \le N_H} \{\text{diam}(\mathcal{D}_j)\}$.

Nested case

$$\mathcal{T}_h = \mathcal{T}_{\mathbb{H}} \Rightarrow \text{Massively parallel setting}$$

Dryja & Krzyzanowski, 2016

Preconditioner's ingredients

Local Solvers

Prolongation (injection) operator $R_i^\top : V_i \to V_h$, where

$$\begin{aligned} V_i &:= \{v_i \in L^2(\Omega_i) : \ v_i|_\kappa \in \mathcal{P}_p(\kappa) \ \forall \kappa \in \mathcal{T}_h, \ \kappa \subseteq \Omega_i\} \equiv V_h|_{\Omega_i}, \\ \mathcal{A}_i(u_i, v_i) &:= \mathcal{A}_h(R_i^\top u_i, R_i^\top v_i) \quad \forall u_i, v_i \in V_i, \qquad [A_i = R_i A R_i^\top] \end{aligned}$$

for $i = 1, \ldots, N_{\mathbb{H}}$.

Coarse Solver

Prolongation (L^2-)operator $R_0^\top : V_0 \to V_h$, where

$$\begin{aligned} V_0 \equiv V_H &:= \{v_H \in L^2(\Omega) : v_H|_{\mathcal{D}_j} \in \mathcal{P}_q(\mathcal{D}_j), \ j = 1, \ldots, N_H\}, \\ \mathcal{A}_0(u_0, v_0) &:= \mathcal{A}_h(R_0^\top u_0, R_0^\top v_0) \quad \forall u_0, v_0 \in V_0, \quad [A_0 = R_0 A R_0^\top] \end{aligned}$$

for $1 \leq q \leq p$.

The massively-parallel additive Schwarz operator

Schwarz Operator

- Projection operators: $P_i := R_i^\top \widetilde{P}_i : V_h \to V_h,\ i = 0, 1, \ldots, N_{\mathbb{H}}$, where

$$\widetilde{P}_i : V_h \to V_i,\ \mathcal{A}_i(\widetilde{P}_i v_h, w_i) := \mathcal{A}_h(v_h, R_i^\top w_i) \quad \forall\, w_i \in V_i,\ i = 1, \ldots, N_{\mathbb{H}},$$
$$\widetilde{P}_0 : V_h \to V_0,\ \mathcal{A}_0(\widetilde{P}_0 v_h, w_0) := \mathcal{A}_h(v_h, R_0^\top w_0) \quad \forall\, w_0 \in V_0.$$

- Additive Schwarz Operator:

$$P_{ad} := \sum_{i=0}^{N_{\mathbb{H}}} P_i.$$

Algebraic Representation

$$P_{ad} = \sum_{i=0}^{N_{\mathbb{H}}} R_i^\top A_i^{-1} R_i A \equiv M_{ad} A$$

Condition number estimates

Theorem (A., Houston, Pennesi, & Süli 2019)

Assuming that the elements $\mathcal{D} \in \mathcal{T}_{\mathcal{H}}$ are convex, the following result holds:

$$K(P_{ad}) \lesssim \max_{1 \leq j \leq N_H} \left(\frac{\overline{\rho}_j}{\underline{\rho}_j} \right) \left(\frac{p^2}{q} \frac{H}{h} + \frac{p^2}{q^2} \frac{H^2}{h\mathbb{H}} \right) (N_{\mathbb{S}} + 2).$$

- The proof is based on the abstract theory of Schwarz methods. Toselli & Widlund 2004

- If the diffusion coefficient ρ is constant on each coarse element $\mathcal{D}_j$, $j = 1, \ldots, N_H$, then the condition number is independent of the jump in ρ.

- For uniform diffusion, with $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$, the above result may be generalised to non-nested meshes:

$$K(P_{ad}) \lesssim \left(\frac{p^4}{q^2} \frac{H^2}{h^2} \right) (N_{\mathbb{S}} + 2).$$

A. & Houston 2011, A., Houston, & Smears 2016, Smears 2017, Dryja & Krzyzanowski, 2016

Sketch of the proof

Assumption 1: Local Stability

There exists an $\alpha \in (0, 2)$ such that

$$\mathcal{A}_h(R_i^\top v_i, R_i^\top v_i) \leq \alpha \mathcal{A}_i(v_i, v_i) \quad \forall\, v_i \in V_i,\ i = 0, 1, \ldots, N_H. \qquad [\alpha = 1]$$

Assumption 2: Strengthened Cauchy-Schwarz Inequality

There exist constants $\epsilon_{ij} \in [0, 1]$, for $1 \leq i, j \leq N_{\mathbb{H}}$, such that

$$|\mathcal{A}_h(R_i^\top v_i, R_j^\top v_j)| \leq \epsilon_{ij} \mathcal{A}_h(R_i^\top v_i, R_i^\top v_i)^{1/2} \mathcal{A}_h(R_j^\top v_j, R_j^\top v_j)^{1/2}$$

for all $v_i \in V_i$, $v_j \in V_j$. $\qquad [\Theta(\mathcal{E}) \leq (N_{\mathbb{S}} + 1)]$

Assumption 3: Stable Decomposition

Each $u_h \in V_h$ admits a decomposition of the form $u_h = \sum_{i=0}^{N_{\mathbb{H}}} R_i^\top u_i$, $u_i \in V_i$, $i = 0, \ldots, N_{\mathbb{H}}$, such that

$$\sum_{i=0}^{N_{\mathbb{H}}} \mathcal{A}_i(u_i, u_i) \leq C_\sharp^2 \mathcal{A}_h(u_h, u_h).$$

$$\Rightarrow \quad K(P_{ad}) \lesssim C_\sharp^2 \alpha (\Theta(\mathcal{E}) + 1).$$

Toselli & Widlund 2004

Assumption 3: Stable Decomposition

Approximation Property (Uniform Diffusion)

For any $v_h \in V_h$, there exists $v_H \in V_H$ such that

$$\begin{aligned} \|v_h - R_0^\top v_H\|_{L^2(\Omega)} &\lesssim \frac{H}{q}\|v_h\|_{DG}, \\ |v_h - R_0^\top v_H|_{H^1(\mathcal{T}_h)} &\lesssim \|v_h\|_{DG}. \end{aligned}$$

- Given $v_h \in V_h$, there exists $\mathcal{H}(v_h) \in H_0^1(\Omega)$ such that

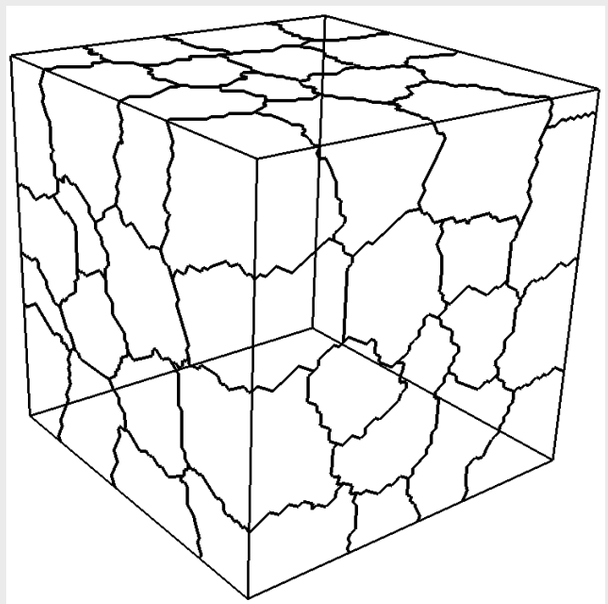
$$\begin{aligned} \|v_h - \mathcal{H}(v_h)\|_{L^2(\Omega)} &\lesssim \frac{h}{p}\|\sigma^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}_h)}, \\ |\mathcal{H}(v_h)|_{H^1(\Omega)} &\lesssim \|v_h\|_{DG}. \end{aligned}$$

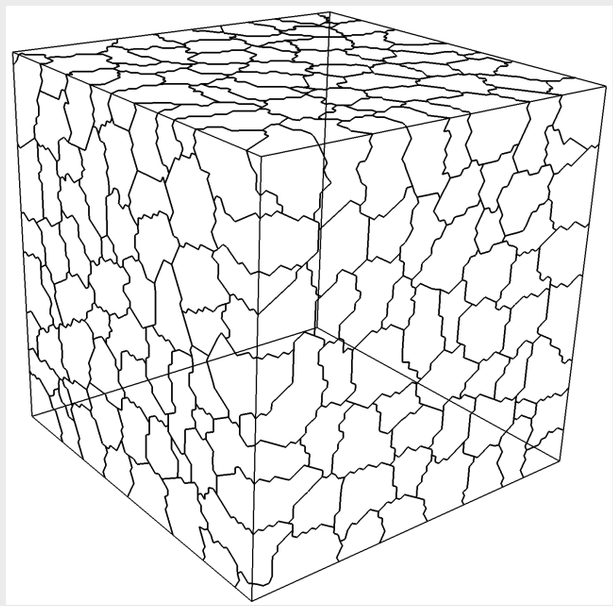
$\Rightarrow$ Proof based on a duality argument. Smears 2014, A., Houston, & Smears 2016, Smears 2017

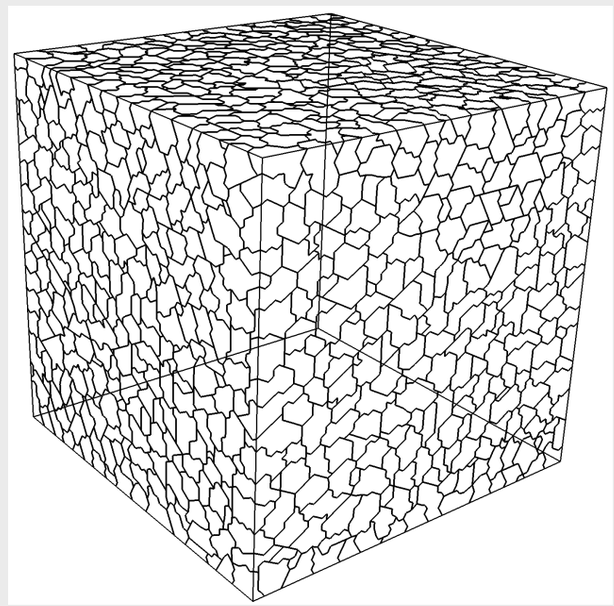
- For non-uniform ρ, analogous results on each $\mathcal{D} \in \mathcal{T}_H$ must be established.
- This involves employing an adjoint (Neumann) problem on each $\mathcal{D} \in \mathcal{T}_H$.

Example 1: 3D cube

- $\Omega = (0,1)^3$, $\rho = 1$, $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$ (tetrahedra); nested coarse mesh.



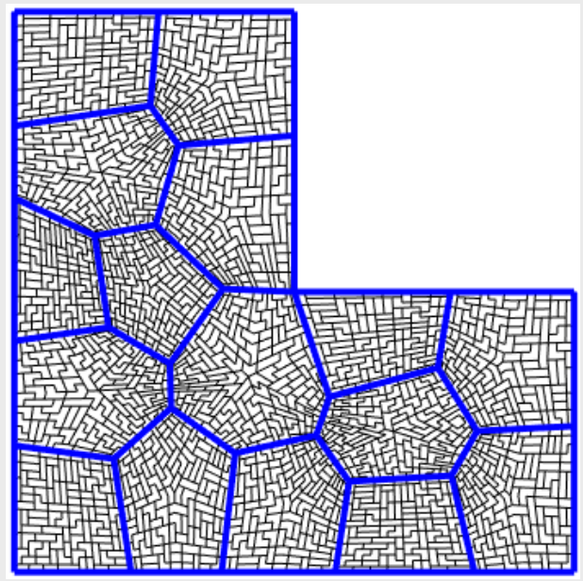


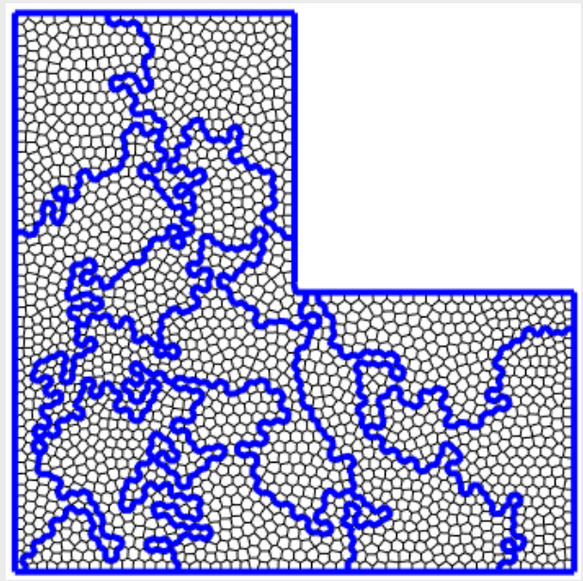


	$N_h \rightarrow$					
$\downarrow N_H$	384	3072	24576	196608	1572864	12582912
48	107 (85)	411 (156)	1497 (294)	6216 (580)	25791 (1089)	94276 (2012)
384	-	136 (95)	499 (169)	1878 (311)	7407 (584)	28762 (1106)
3072	-	-	146 (96)	480 (165)	1904 (306)	7861 (578)
24576	-	-	-	144 (94)	491 (164)	1973 (306)
196608	-	-	-	-	144 (94)	496 (164)
1572864	-	-	-	-	-	145 (94)
$K(A_h)$	734 (166)	2859 (289)	11407 (507)	45618 (933)	182199 (1649)	708509 (3012)

Example 2: rough diffusion

- Ω is an L-shaped domain, $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$ (polygons); nested coarse mesh.





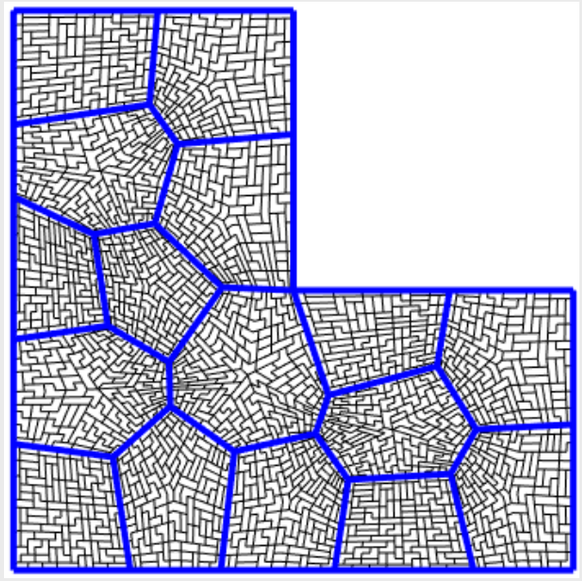
- Mesh aligned: $\rho|_{\mathcal{D}_j} = 1$ for odd j; $\rho|_{\mathcal{D}_j} = \rho_e$ otherwise, $\mathcal{D}_j \in \mathcal{T}_H$.

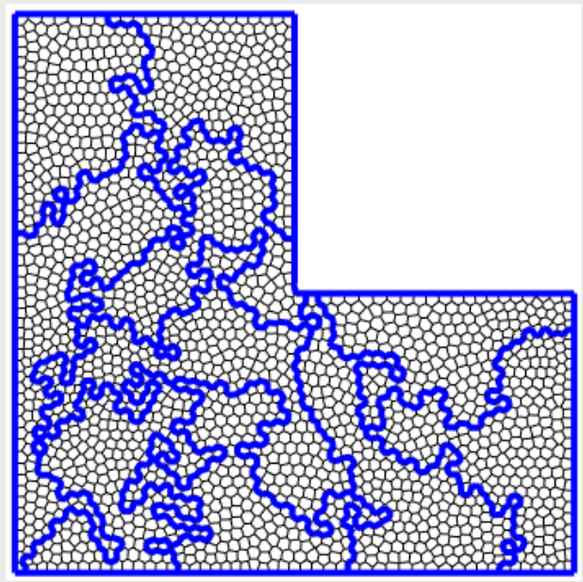
Convex Coarse Eles	$\rho_e \rightarrow$ 10	10^2	10^3	10^4	10^5	10^6
$p = 1$	$2.31 \cdot 10^2$	$2.33 \cdot 10^2$	$2.34 \cdot 10^2$	$2.34 \cdot 10^2$	$2.34 \cdot 10^2$	$2.34 \cdot 10^2$
$p = 2$	$6.12 \cdot 10^2$	$6.14 \cdot 10^2$	$6.14 \cdot 10^2$	$6.12 \cdot 10^2$	$6.11 \cdot 10^2$	$6.10 \cdot 10^2$

Non-Convex Coarse Eles	$\rho_e \rightarrow$ 10	10^2	10^3	10^4	10^5	10^6
$p = 1$	$9.65 \cdot 10^2$	$1.15 \cdot 10^3$	$1.20 \cdot 10^3$	$1.21 \cdot 10^3$	$1.21 \cdot 10^3$	$1.21 \cdot 10^3$
$p = 2$	$2.47 \cdot 10^3$	$2.86 \cdot 10^3$	$3.18 \cdot 10^3$	$3.25 \cdot 10^3$	$3.26 \cdot 10^3$	$3.26 \cdot 10^3$

Example 2: rough diffusion

- Ω is an L-shaped domain, $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$ (polygons); nested coarse mesh.





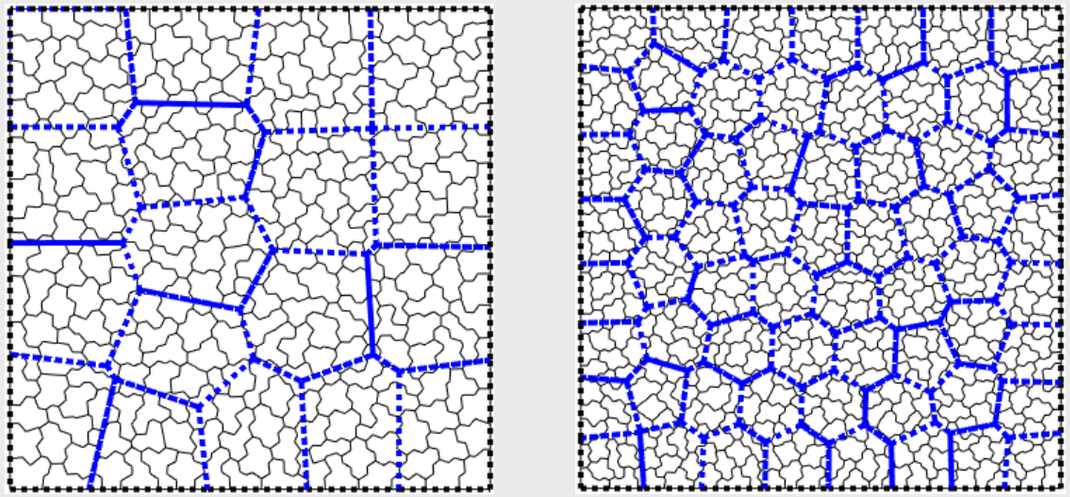
- Non-aligned: $\rho|_{\kappa_j} = 1$ for odd j; $\rho|_{\kappa_j} = \rho_e$ otherwise, $\kappa_j \in \mathcal{T}_h$.

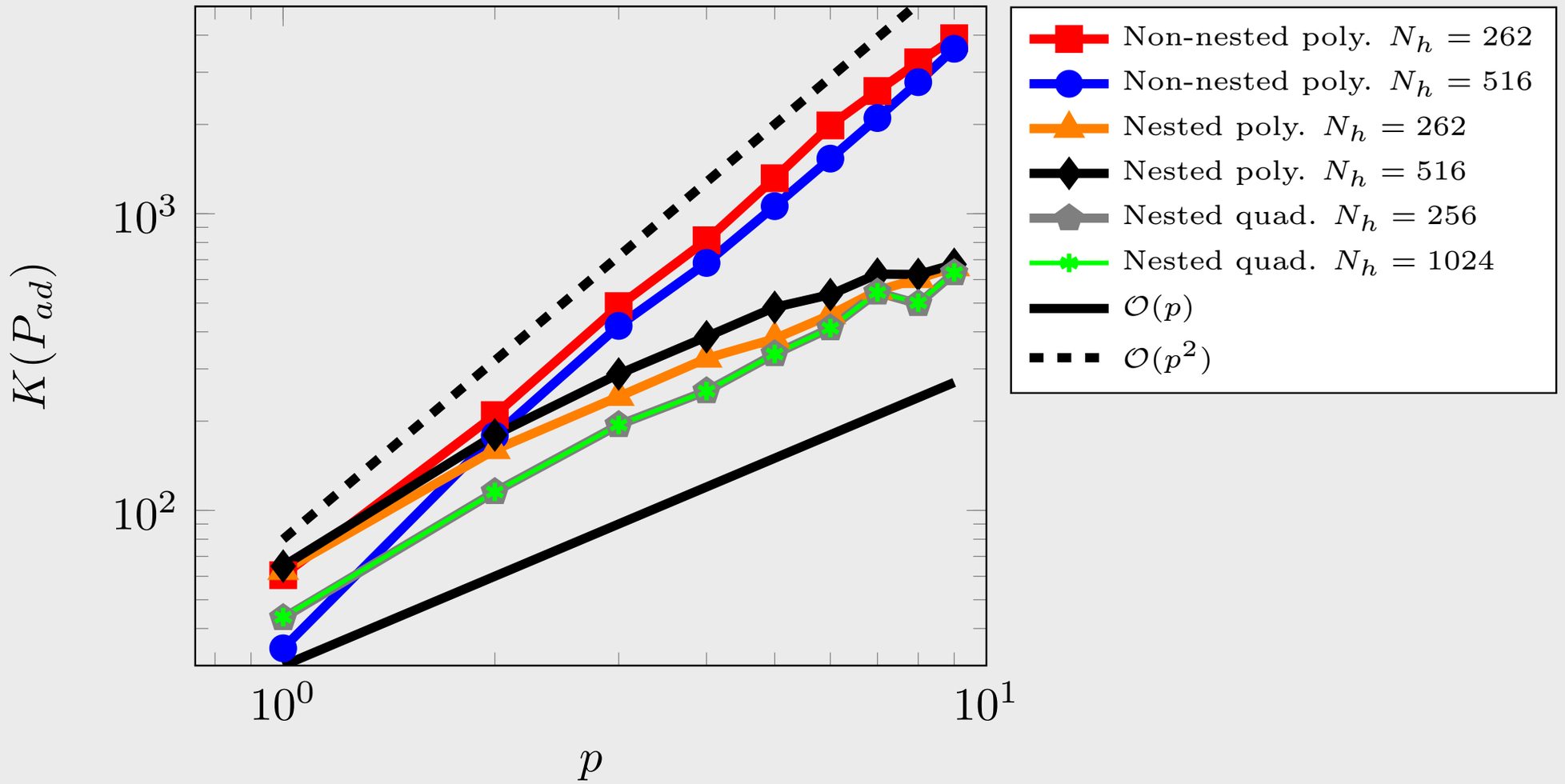
Convex Coarse Eles	$\rho_e \rightarrow$ 10	10^2	10^3	10^4	10^5	10^6
$p = 1$	$2.92 \cdot 10^2$	$1.04 \cdot 10^3$	$7.73 \cdot 10^3$	$7.42 \cdot 10^4$	$7.39 \cdot 10^5$	$7.38 \cdot 10^6$
$p = 2$	$6.54 \cdot 10^2$	$2.26 \cdot 10^3$	$1.71 \cdot 10^4$	$1.65 \cdot 10^5$	$1.64 \cdot 10^6$	$1.64 \cdot 10^7$

Non-Convex Coarse Eles	$\rho_e \rightarrow$ 10	10^2	10^3	10^4	10^5	10^6
$p = 1$	$8.02 \cdot 10^2$	$2.09 \cdot 10^3$	$1.68 \cdot 10^4$	$1.65 \cdot 10^5$	$1.64 \cdot 10^6$	$1.64 \cdot 10^7$
$p = 2$	$2.36 \cdot 10^3$	$6.10 \cdot 10^3$	$4.74 \cdot 10^4$	$4.62 \cdot 10^5$	$4.61 \cdot 10^6$	$4.61 \cdot 10^7$

Example 3: p-dependence

- $\Omega = (0,1)^2$, $\rho = 1$, $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$ (polygons).





W-cycle multigrid

- $\Omega = (0,1)^2$, $\rho = 1$, L: No of multigrid levels.
- $m = 3$ smoothing steps; AS smoother, with $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$ (polygons).

N_h/L	2	3	4	5	6	7	8	9	10
256	11	15	21						
1024	13	17	22	26					
4096	14	21	23	27	28				
16384	15	18	25	29	30	29			
65536	15	18	24	29	32	31	30		
262144	14	18	23	26	29	31	30	29	
1048576	14	17	23	26	29	31	32	32	30

Conclusions and perspectives

Conclusions

- Agglomeration strategies can be successfully incorporated within the design of solvers and preconditioners
- The proposed solvers are scalable and well suited to parallelisation.
- Numerical integration of polynomial functions on general shaped elements is slow, mainly because of the high number of function evaluations $\longrightarrow$ quadrature free method (not discussed today) [A., Houston, Pennesi, 18]

Ongoing

- Efficient agglomeration (3D).
- Parallelization.
- Application to industrial or geophysical problems.

Thank you for your attention!