Lagrangian-based heuristics for production planning with perishable products, scarce resources, and sequence-dependent setup times

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Abstract

In this paper, we study a lot-sizing and scheduling problem apparent in the food industry that stemmed originally from the Brazilian meat production sector. More specifically, we consider a production environment in which various production lines share a set of scarce production resources. Therefore, only a subset of the existing production lines can simultaneously operate in each period under the limitations of the availability of resources. Moreover, we consider sequence-dependent setup times and costs, significant inventory holding costs, backlogging, and perishable products. The problem is formulated as a mixed integer programming model, and we propose four Lagrangian-based heuristics to find high-quality solutions for challenging instances. A computational study shows that proposed approaches are very competitive in solving the problem, outperforming methods already established in the literature.

Keywords: Lagrangian heuristic, production planning, scarce resources, perishable products.

MSC Classification: 90C11

1 Introduction

Production planning is an important management instrument that aims at the efficient usage of the available production resources. In many industrial environments, such as meat, soft drink, yogurt, and automotive companies (see Soler et al (2021b); Ferreira et al (2010); Kopanos et al (2010); Almeder and Almada-Lobo (2011), respectively), production planning involves decisions about which products must be produced in what quantities in each period (lot sizing) and the sequence in which those products are processed in the available production lines (scheduling). According to Almada-Lobo et al (2015), industries face serious problems in making these decisions in practice. Therefore, it is important to develop specialized mathematical models and solution approaches to deal with integrated lot sizing and scheduling problems (LSP) stemming from real-world scenarios.

In this paper, we deal with an LSP considering specific characteristics observed in Brazilian meat industries, namely the lot sizing and scheduling problem with multiple production lines sharing scarce productive resources and considering perishable products (LSP-PLSR). The produced items are perishable and can be stocked only for a limited period of time. Moreover, an important characteristic of this industry is that production lines share the same machines, tools, workstations, and workers, and because of the scarcity of these production resources, only a subset of production lines can simultaneously operate in any production period. The problem is also characterized by the existence of significant sequence-dependent setup times and costs, capacity constraints, machine eligibility, and high inventory holding and backlogging costs. Under these conditions, for each production period, the addressed problem consists in deciding: i) which production lines should be assembled; ii) which products should be produced and how much; and iii) the production sequences; with the objective of minimizing costs incurred in the process.

The specific problem was recently introduced in the literature, and was addressed in Soler et al (2021a,b) and Soler et al (2021c), as will be discussed in Section 3.2. The existing literature has already established that the problem in question is $\mathcal{NP}$-hard in the strong sense and that large-sized test instances based on real-world scenarios are computationally challenging. In order to address these challenges, we propose efficient solution approaches for this LSP. More specifically, we develop two Lagrangian approaches that decompose the original problem into easier-to-solve sub-problems. We also develop a heuristic procedure to find high-quality feasible solutions from the dual solutions obtained in the Lagrangian approaches. Finally, valid inequalities are proposed in order to improve the quality of the obtained dual bounds. The effectiveness of the proposed approaches is studied through extensive computational experiments, where the analysis involves benchmarking the performance of the proposed approaches to the performance of methods from the literature.

The main contribution of this paper to the literature consists of an innovative and customized application of Lagrangian approaches to solving a challenging real-world production planning problem. The proposed approaches consider the specific structure of the problem to develop novel and efficient valid inequalities that significantly improve the performance of the used methods and can be potentially used in other production planning problems involving sequence-dependent setup times. Even though the methodological approach follows a traditional framework, the developed methods outperform methods already established in the literature. Moreover, the computational study carried out with the Lagrangian heuristics reveals some properties of the problem in question.

The remainder of this paper is organized as follows. Section 2 presents a literature review covering successful Lagrangian approaches to deal with production planning problems, while Section 3 describes the addressed LSP and a mathematical model to represent it as well as the existing literature about the addressed problem. Section 4 is devoted to the presentation of the Lagrangian-based solution approaches, including the feasibility process and the valid inequalities to improve the Lagrangian bounds. The computational study is presented in Section 5. Finally, Section 6 presents the main conclusions and sketch perspectives for future works.

2 Literature review

The Lagrangian relaxation (LR) approach was originally proposed by Geoffrion (1974) in the 70's in order to solve integer and mixed integer programming problems. The development of the approach was motivated by the fact that various complicated problems become significantly easier to solve in the absence of some of "complicating" constraints. Basically, in LR approaches, the complicating constraints are included only in the objective function weighted by dual variables (also called Lagrangian multipliers) allowing the resulting (relaxed) problem, without involving these constraints, to be (hopefully) solved more effectively.

Naturally, there is more nuance than simply obtaining an easy-to-solve relaxed problem. In addition to the fact that the process of identifying which constraints to relax is very often not straightforward and problem-dependent, there is also the danger of relaxing too many constraints, which may result in, for example, very weak dual bounds that are not useful for the overall process. We refer the interested reader to Jans and Degraeve (2004); Akartunalı and Miller (2012) for a thorough elaboration on these matters (from the strength of various relaxations to their complexity to solve in practice) in the context of production planning.

As remarked by Gaudioso (2020), approaches combining Lagrangian relaxation and heuristic algorithms can be useful to tackle hard optimization problems inherent in production systems (see some specific examples in Toledo and Armentano (2006), Xiao et al (2015), and Fonseca et al (2019)). Usually, in such integrated approaches, a heuristic procedure is developed to yield feasible solutions exploiting the dual solutions obtained in the Lagrangian relaxation, making the most effective use of both heuristic and exact methods. In line with this, we propose LR approaches involving a construction heuristic to solve the LSP in hand. More specifically, we propose that the constraints linking decisions about lot sizing with scheduling decisions should be considered as complicating constraints. In the absence of these linking constraints, the resulting problem can be decomposed into separate lot sizing and parallel machine scheduling sub-problems. As will be discussed in Section 4, each resulting sub-problem is significantly smaller than the original problem and can be tackled efficiently.

As our primary focus is on building effective LR approaches, we discuss key approaches from the literature applied successfully to solve a range of production planning and related problems.

One of the earliest works we are aware of is the LR algorithm proposed in Trigeiro et al (1989) to address a capacitated lot sizing problem with setup times and costs. The capacity constraints are relaxed to decompose the problem into a set of independent, uncapacitated, single-item lot sizing sub-problems that can be effectively solved by dynamic programming. A heuristic procedure was also proposed to obtain feasible solutions, and the authors presented computational results able to find high-quality solutions for instances with up to 30 periods and 24 products.

A simultaneous lot sizing and scheduling problem in which only one type of product can be produced in each period was addressed by Fleischmann (1990). A Lagrangian scheme was proposed relaxing the constraints that ensure a single product type can be produced in each period. Similar to Trigeiro et al (1989), the resulting problem was decomposed into single-item, uncapacitated problems that were solved by dynamic programming. The algorithm used to solve the sub-problems was slightly modified to provide feasible solutions, and a computational study demonstrated promising results for instances with few products and up to 100 periods.

The problem considered by Fleischmann (1990) was later extended by Fleischmann (1994) to consider sequence-dependent setup costs. An LR approach was developed, using the same framework established in Fleischmann (1990), to provide dual bounds, and an independent heuristic procedure was proposed to find feasible solutions. The computational efficiency of the method was shown using instances with up to 10 products and 150 periods.

In Diaby et al (1992), Lagrangian-based approaches were combined with a branch-and-bound algorithm to solve a capacitated lot sizing and scheduling problem with limited overtime. More specifically, in each branch-and-bound iteration, the bounds were generated by Lagrangian relaxation. The authors evaluated different Lagrangian relaxations and developed specific algorithms to solve the obtained relaxed problems. Computational results on test instances with up to 99 items and 8 periods showed that the proposed algorithms are effective and indicated that the capacity constraints relaxation is superior to the demand constraints relaxation.

Millar and Yang (1994) developed Lagrangian relaxation and decomposition approaches to tackle a capacitated lot sizing problem with backorders. The logical constraints linking continuous and binary variables were relaxed so that the resulting problem could be decomposed into a transportation problem (formulated as LP) and an integer programming problem that could be solved by inspection, where further tricks such as creating auxiliary variables were also employed in addition to customized heuristics. Albeit using only small-sized test instances, computational analysis demonstrated that the proposed Lagrangian relaxation scheme performed better than the developed Lagrangian decomposition approach.

Tempelmeier and Derstroff (1996) addressed a multilevel lot-sizing problem considering setup times and multiple capacity constraints. An LR-based approach was proposed by relaxing the capacity constraints and the multilevel inventory balance constraints. The relaxed problem was thus decomposed into several uncapacitated single-item lot sizing problems that were solved by dynamic programming. The authors also developed a feasibility procedure combining the solution of uncapacitated single-item problems with a production transfer algorithm, and presented computational results for instances with up to 100 products and 16 periods.

An efficient LR approach was proposed by Toledo and Armentano (2006) to deal with a capacitated lot sizing problem on parallel machines. More specifically, the capacity constraints were relaxed so that the resulting problem is decomposed into single-item sub-problems solved by dynamic programming. The authors also proposed a three-phase heuristic procedure to obtain feasible solutions from the Lagrangian dual solutions. Fiorotto and de Araujo (2014) also proposed an LR approach for this specific problem using the classical facility location reformulation and relaxing the demand constraints. The resulting problem was decomposed into single period, single machine sub-problems, and a customized branch-and-bound algorithm was developed, along with a feasibility heuristic based on production transfer.

A capacitated lot sizing problem with setup times (but without setup costs) was addressed by Süral et al (2009). The authors considered a strong mathematical formulation for the problem and proposed a Lagrangian scheme relaxing the demand constraints. The decomposed single period sub-problems were formulated as a variation of the knapsack problem and solved by a previously established branch-and-bound algorithm. A primal heuristic was also developed, and computational experiments carried out for up to 24 products and 30 periods.

Wu et al (2013) proposed an LR approach for a multi-level lot sizing problem with backorders. They relaxed the capacity constraints to exploit decomposition into various single end product sub-problems. A commercial MIP solver was employed to solve the sub-problems while a relax-and-fix heuristic was developed to provide feasible solutions. The authors also discussed some alternative reformulations for the problem on hand.

An LR heuristic was proposed in Brahimi et al (2015) to address a two-level uncapacitated lot sizing problem with bounded inventory. More specifically, they relaxed the coupling constraints between the two considered levels, decomposing the relaxed problem into various well-solved single-item uncapacitated problems. Three smoothing heuristics were employed to build feasible solutions from the dual solutions, and computational results showed the superiority of the LR approach over a commercial MIP solver using instances with up to 40 periods and 90 items.

Wolosewicz et al (2015) proposed an LR heuristic to deal with an integrated lot sizing and fixed scheduling problem. Firstly, the authors proposed a new mathematical formulation for the problem by modeling paths of the conjunctive graph associated with the fixed sequence. Secondly, as widely explored in the literature, they relaxed the capacity constraints and obtained a resulting problem decomposed into a set of single-item uncapacitated problems. A smoothing heuristics based on production transfer between the different periods was developed to provide feasible solutions.

A hybrid Lagrangian-Simulated annealing-based heuristic was proposed in Xiao et al (2015) to tackle a parallel-machine lot sizing and scheduling problem from semiconductor manufacturing with sequence-dependent setup times and costs and machine eligibility constraints. A Lagrangian decomposition scheme was proposed in which auxiliary variables were introduced for production variables, along with coupling constraints linking them to the original production variables. The auxiliary variables were utilized to reorganize some original constraints, and the coupling constraints were relaxed. A feasibility procedure was developed and a simulated annealing-based heuristic was used to improve the quality of the obtained primal solutions.

Carvalho and Nascimento (2016) addressed a capacitated multi-plant lot sizing problem. A Lagrangian heuristic was proposed by relaxing the capacity constraints of the plants and a commercial MIP solver was used to solve the relaxed problem. Construction and improvement heuristics were proposed by exploring transfers of production among plants and/or periods. Finally, the improvement heuristic was hybridized with a path-relinking heuristic to enhance the quality of the solutions found.

A parallel-machine lot sizing problem with carbon emission constraints and machine purchasing was considered by Wu et al (2018). Relaxing the capacity constraints, carbon emission constraints and machine purchasing constraints, the resulting problem was decomposed into a set of single-item uncapacitated sub-problems. A relax-and-fix heuristic was employed to find initial feasible solutions for the original problem and a knowledge-guided improvement heuristic was developed. The efficiency of the LR heuristic was computationally investigated using instances with up to 35 products, 24 periods, and 15 machines. For these instances, the LR dual bounds are competitive with a Dantzig–Wolfe decomposition and a column generation method.

As' ad et al (2020) also addressed a lot sizing problem considering carbon emission constraints, as well as perishable products and some operational characteristics apparent in cold-chain industries (such as poultry, dairy, and vaccines). The authors proposed an LR approach consisting of the relaxation of the carbon emission constraints. In this way, the resulting problem could be solved by a dynamic programming algorithm previously established in the literature. A bisection-based algorithm was proposed to solve the associated dual problem. A computational study was presented considering small-sized test instances that were solved optimally in order to identify some managerial insights.

To sum up, we observe that most commonly the Lagrangian relaxation is applied to capacity constraints in production planning problems, though this may vary depending on the specific characteristics of the problem on hand and identifying which constraints to relax, as discussed earlier, is always critical to the success of the process. As we also demonstrate in this brief review, it is also customary to build customized heuristics such as smoothing, in order to exploit the dual bounds in the most effective way to generate feasible solutions.

Moreover, we observe that in the literature until the 2000s, Lagrangian approaches were usually employed to solve classical problems without considering many characteristics observed in real-world applications. In those problems, when some constraints are relaxed, the resulting problem can be solved by polynomial algorithms. It is also common to utilize simple heuristic procedures to repair the dual solutions to achieve feasibility. Recently, after the enhancement of commercial MIP solvers (such as Cplex and Gurobi), Lagrangian heuristics have been commonly developed to solve more complex problems in which the main characteristics of the supply chain are considered. In these works, the relaxed problems are usually $\mathcal{NP}$-hard but can be efficiently tackled using MIP solvers. Nowadays, it is also common the proposition of hybrid methods combining Lagrangian relaxation with meta-heuristic approaches to obtain feasible solutions. Finally, despite the existence of alternative methods to deal with the dual problem, most works dealing with production planning problems observed in practice use subgradient optimization. This fact aligns with some computational tests we have experimented with throughout this research.

3 Problem presentation

In this section, we describe, model, and present the works related to the problem addressed in this paper, i.e., the lot sizing and scheduling problem with multiple production lines sharing scarce productive resources and considering perishable products (LSP-PLSR).

3.1 Problem description and modeling

The LSP-PLSR is a production planning problem that considers a set of products J (indexed by i and j), a set of heterogeneous and capacitated production lines L (indexed by l), a set of production resources K (indexed by k), and a finite planning horizon of T periods (indexed by t and p). The problem also assumes dynamic and deterministic demands, with d_{jt} denoting the demand of product j in period t. Backlogging is allowed, i.e., the demands can be met with delay, and for each product j backlogged, a time-independent backlogging cost, denoted by b_j, is incurred per period. On the other hand, when the products are produced in advance of demand, they are charged a time-independent unit inventory holding cost h_j for product j.

In the LSP-PLSR, the production lines need to be assembled at the beginning of each period by assigning the necessary production resources. In order to assemble line l, r_{kl} units of resource k are required. The production environment is constrained such that the available amount of resource k, denoted by R_k, is not sufficient to assemble all production lines simultaneously, requiring a decision on which production lines to assemble in each period. If production line l is assembled in period t (incurring a setup cost of ac_l), then it operates in that period with a production capacity C_{lt}. W.l.o.g., we also assume that each product can only be produced on a specific production line, i.e., if we let P_l be the set of products produced in line l, then $P_{l_1} \cap P_{l_2} = \emptyset, \forall l_1, l_2 \in L$ with $l_1 \neq l_2$. This assumption is based on the common practice of specialized production lines in food companies (as discussed in Soler et al (2021b)) and also dictates which products can be produced in a period. We explore this feature in the feasibility procedure described in Section 4.4. We remark that our proposed heuristics can also be applied to a setting where each product can be produced on more than one production line.

Each production line admits a unique configuration, obtained by allocating the necessary productive resources. Therefore, we only need to decide which production lines to assemble in each period, without considering alternative configurations for each line.

We also consider sequence-dependent setup times and costs, denoted by st_{lij} and sc_{lij}, respectively, when the production of item i is followed by the production of product j on line l. If item j is produced in a period, a minimum m_j units must be produced, with each unit consuming a_{lj} units of the capacity of the production line l. Finally, in the LSP-PLSR we suppose that the produced items are perishable, and hence, each item j can remain in stock no longer than its shelf life, i.e., sl_j periods.

As we noted in Section 2, the LSP-PLSR was previously studied in Soler et al (2021b,a,c), and in particular, Soler et al (2021c) explored different modeling techniques to yield nine mathematical models for the LSP-PLSR. The extensive computational study presented by the authors indicated that a strong formulation can be obtained from the traditional CLSD model, as proposed in Haase (1996), by making the binary production variables explicit, using the facility location reformulation for the lot sizing decisions, and the single commodity flow constraints to eliminate production sub-sequences. Therefore, in this paper, we add the perishability constraints into the FL-CLSD$^{w}_{SCF}$ model of Soler et al (2021c), and use this as the base model for the proposed Lagrangian approaches.

To describe the model, we define a parameter hb_{jtp} to indicate the unit inventory holding (if $t \leq p$) or backlogging (if $t > p$) cost incurred when item j is produced in period t to meet the demand of period p. Using the parameters introduced earlier, the parameter hb_{jtp} can be defined as

$$hb_{jtp} = \begin{cases} (p-t)h_j, & \text{if } t \leq p \\ (t-p)b_j, & \text{if } t > p \end{cases}.$$

Finally, we define the decision variables of the model, as follows:

x_{ljtp} Amount of item j produced on line l in period t to meet the demand of period $p = 1, \dots, \min\{t + sl_j, T\}$;

δ_{lt} 1, if line l is assembled in period t; 0, otherwise;

w_{ljt} 1, if item j is produced on line l during period t; 0, otherwise;

y_{ljt} 1, if item j is the first item produced on line l and period t; 0, otherwise;

z_{lijt} 1, if there is change of production from item i to j on line l during the period t; 0, otherwise;

V_{lijt} Auxiliary variable used in the single commodity flow constraints to eliminate sub-sequences. This variable can be interpreted as the amount of a hypothetical commodity that is transferred on line l from item i to j in period t.

The mathematical model, which we will refer to as FL-CLSD$^{w}_{SCF}$, is as follows.

$$\text{Min} \sum_{l,j,t,p} hb_{jtp} x_{ljtp} + \sum_{l,i,j,t} sc_{lij} z_{lijt} + \sum_{l,t} ac_l \delta_{lt} \tag{1}$$

$$\text{s.t.} \quad \sum_{p=\max\{1,t-sl_j\}}^{T} x_{ljpt} = d_{jt}, \qquad \forall l, j \in P_l, t \tag{2}$$

$$m_j w_{ljt} \le \sum_{p} x_{ljtp} \le \frac{C_{lt}}{a_{lj}} w_{ljt}, \qquad \forall l, j \in P_l, t \tag{3}$$

$$\sum_{j \in P_l} w_{ljt} \le |P_l| \delta_{lt}, \qquad \forall l, t \tag{4}$$

$$\sum_{l} r_{kl} \delta_{lt} \le R_k, \qquad \forall k, t \tag{5}$$

$$\sum_{j \in P_l, p} a_{lj} x_{ljtp} + \sum_{i,j} st_{lij} z_{lijt} \le C_{lt}, \qquad \forall l, t \tag{6}$$

$$w_{ljt} = y_{ljt} + \sum_{i \in P_l} z_{lijt}, \qquad \forall l, j \in P_l, t \tag{7}$$

$$\sum_{j \in P_l} y_{ljt} \le 1, \qquad \forall l, t \tag{8}$$

$$y_{ljt} + \sum_{i \in P_l \setminus \{j\}} z_{lijt} \ge \sum_{i \in P_l \setminus \{j\}} z_{ljit}, \qquad \forall l, j \in P_l, t \tag{9}$$

$$\sum_{j \in P_l} V_{l0jt} \le |P_l|, \qquad \forall l, t \tag{10}$$

$$V_{l0jt} \le |P_l| y_{ljt}, \qquad \forall l, j \in P_l, t \tag{11}$$

$$\sum_{i \in P_l \cup \{0\}} V_{lijt} - \sum_{i \in P_l} V_{ljit} = y_{ljt} + \sum_{i \in P_l} z_{ljit}, \qquad \forall l, j \in P_l, t \tag{12}$$

$$V_{lijt} \le (|P_l| - 1) z_{lijt}, \qquad \forall l, i, j \in P_l, t \tag{13}$$

$$x_{ljtp} \ge 0, \qquad \forall l, j \in P_l, t, p \tag{14}$$

$$w_{ljt} \in \{0,1\}, \qquad \forall l, j \in P_l, t \tag{15}$$

$$\delta_{lt} \in \{0,1\}, \qquad \forall l, t \tag{16}$$

$$y_{ljt} \in \{0,1\}, \qquad \forall l, j \in P_l, t \tag{17}$$

$$z_{lijt} \in \{0,1\}, \qquad \forall l, i \in P_l, j \in P_l, t \tag{18}$$

$$V_{lijt} \ge 0, \qquad \forall l, i \in P_l, j \in P_l, t. \tag{19}$$

The objective function (1) consists of minimizing the total costs associated with inventory holding, backlogging, sequence-dependent setups, and assembly of the production lines. Constraints (2) ensure the fulfillment of the customer demands. The right inequality of constraints (3) ensures that each type of item can only be produced if the production line is set up for the respective item, while the left inequality introduces minimum lot sizes. Constraints (4) guarantee that production occurs only on assembled production lines, while (5) are the capacity constraints related to the production resources. Constraints (6) are the capacity constraints of the assembled production lines, and constraints (7) ensure that the produced items are properly sequenced. Constraints (8) establish that only one item can be the first item produced in each production line and period, while constraints (9) ensure the flow balance for sequencing of lots. Constraints (10) to (13) are the single commodity flow (SCF) constraints to eliminate sub-sequences in the production sequences. Finally, constraints (14) to (19) define the domains of variables.

We remark the SCF constraints were introduced by Gavish and Graves (1978) to eliminate sub-tours in formulations for the traveling salesman problem and were used by Guimarães et al (2014) and Soler et al (2021c) to eliminate sub-sequences in the production planning context. The SCF approach requires polynomial numbers of constraints and variables. In the proposed model, assuming that each production line produces the same number of different products, the SCF approach has $\left(L + 2J + J^2\right) T$ constraints and requires J^2T specific continuous variables. For a test instance with 14 periods, 10 production lines, and 110 products, these expressions correspond to 172,620 constraints and 169,400 variables. According to the results presented by Guimarães et al (2014) and Soler et al (2021c), the SCF constraints outperform the widely used MTZ constraints proposed by Miller et al (1960).

3.2 Related works

To conclude this section, we review the existing literature on production planning problems similar to the LSP-PLSR.

Jain and Palekar (2005) addressed a multi-level lot sizing problem in which various machines are connected to form different production lines. Moreover, production lines can operate in a fraction of the periods and in-process inventory is not allowed. The problem was formulated as a mixed integer programming problem and heuristic approaches were developed to provide high-quality feasible solutions in reasonable running times for real-size instances. The problem studied by Jain and Palekar (2005) differs from LSP-PLSR because sequencing decisions are not considered and the products are not of perishable nature.

The LSP-PLSR was originally introduced in Soler et al (2021b). The authors adapted the mixed integer programming model proposed in Haase (1996) to consider the specific characteristics of the problem. More specifically, variables and constraints were introduced to control the age of the products in stock and to decide which production lines should be assembled in each period. A relax-and-fix heuristic was presented in which the original problem could be decomposed into easier-to-solve sub-problems. The computational results showed that the proposed heuristic was able to find high-quality feasible solutions and competitive dual bounds, outperforming a commercial MIP solver and other traditional relax-and-fix heuristics.

Soler et al (2021c) studied the problem emphasizing the achievement of strong mathematical formulations. The perishability aspect was not considered and the authors proposed nine mathematical models exploring different techniques to model the setup, the sequencing, and the lot sizing decisions. An extensive computational study showed that an efficient model is obtained from the model proposed in Soler et al (2021b) by making explicit the binary setup variables, using the single commodity flow constraints to eliminate any sub-sequences, and the facility location reformulation to define the continuous lot sizing variables. In this paper, we introduce the perishability aspect in the best model proposed in Soler et al (2021c) and use it as the basis line for the proposed Lagrangian approaches.

Finally, Soler et al (2021a) proposed construction and improvement heuristics for the LSP-PLSR. More specifically, in the construction heuristic, an aggregated model was used only to determine the production lines to be assembled in each period. Next, these decisions were fixed in the original problem, and then, the resulting sub-problem decomposed by production lines. A local search heuristic, induced by local branching constraints, and a stochastic fix-and-optimize heuristic were proposed to improve the initial feasible solutions found by the constructive procedure. A computational study found that the proposed solution methods outperformed the relax-and-fix heuristic introduced by Soler et al (2021b).

From the literature, we observe that large-sized test instances commonly observed in industrial settings (especially with more than 80 products, 10 periods, and 10 production lines) are computationally challenging, requiring the development of new solution approaches. We also observe a lack of specialized exact methods to obtain strong primal and dual bounds.

4 Lagrangian based heuristics

This section is devoted to the elaboration on the proposed Lagrangian solution approaches. Firstly, we introduce a Lagrangian scheme in which we dualize the constraints that link the lot sizing and the sequencing decisions. Secondly, exploring the same decomposition framework of the original problem, we develop a Lagrangian decomposition approach characterized by the duplication of the production variables x_{ljtp}. Finally, we discuss dual problems and propose feasibility procedures.

4.1 A Lagrangian Relaxation scheme

The Lagrangian Relaxation (LR) scheme proposed in this paper is obtained from the dualization of constraints (6) and (7) that link the lot sizing and the scheduling decisions in the FL-CLSD$^{w}_{SCF}$ model. To present the LR approach, let $\lambda = (\lambda_{lt}) \geq 0$ and $\alpha = (\alpha_{ljt})$ be the dual variables (i.e., Lagrangian multipliers) associated with constraints (6) and (7), respectively. The Lagrangian sub-problem $S(\lambda, \alpha)$ can then be formally defined as

$$\begin{aligned} Z(\lambda,\alpha) = \text{Min} & \sum_{l,j,t,p} (hb_{ljtp} + \lambda_{lt} a_{lj}) x_{ljtp} + \sum_{l,j,t} \alpha_{ljt} w_{ljt} + \sum_{l,t} ac_l \delta_{lt} \\ & + \sum_{l,i,j,t} (sc_{lij} + \lambda_{lt} st_{lij} - \alpha_{ljt}) z_{lijt} - \sum_{l,j,t} \alpha_{ljt} y_{ljt} \end{aligned} \tag{20}$$

$$
\begin{aligned}
&- \sum_{l,t} \lambda_{lt} C_{lt} \\
&\text{s.t. } (x, w, \delta, y, z) \in X
\end{aligned}
$$

where $X = \{(x, w, \delta, y, z) | (2) - (5), (8) - (19)\}$. Note that the last term of the objective function is a constant for any given λ. We observe that in the Lagrangian sub-problem $S(\lambda, \alpha)$, the decisions about which lines to assemble and the size of the production lots do not depend on the sequencing decisions. Therefore, $S(\lambda, \alpha)$ can be decomposed into a lot sizing problem with choices of production lines to assemble, denoted by $LC(\lambda, \alpha)$, and a scheduling problem $SC(\lambda, \alpha)$, as follows.

$\bullet LC(\lambda, \alpha)$:

$$
\begin{aligned}
Z^{LC}(\lambda, \alpha) = \text{ Min } &\sum_{l,j,t,p} (hb_{ljtp} + \lambda_{lt} a_{lj}) x_{ljtp} + \sum_{l,j,t} \alpha_{ljt} w_{ljt} + \sum_{l,t} ac_l \delta_{lt} \qquad (21) \\
&\text{s.t. } (x, w, \delta) \in \{(2) - (5), (14) - (16)\}
\end{aligned}
$$

$\bullet SC(\lambda, \alpha)$:

$$
\begin{aligned}
Z^{SC}(\lambda, \alpha) = \text{ Min } &\sum_{l,i,j,t} (sc_{lij} + \lambda_{lt} st_{lij} - \alpha_{ljt}) z_{lijt} - \sum_{l,j,t} \alpha_{ljt} y_{ljt} \qquad (22) \\
&\text{s.t. } (y, z) \in \{(8) - (13), (17) - (19)\}
\end{aligned}
$$

Using this decomposition and for any $\lambda \geq 0$ and α, we have that

$$
Z(\lambda, \alpha) = Z^{LC}(\lambda, \alpha) + Z^{SC}(\lambda, \alpha) - \sum_{l,t} \lambda_{lt} C_{lt}.
$$

Our computational experiments find that problem $LC(\lambda, \alpha)$ is significantly easier-to-solve than the original problem using a MIP solver. Moreover, the convergence of the solver's algorithm can be accelerated by using the branching rule suggested in Soler et al (2021b), i.e., by prioritizing branching the δ_{lt} variables before w_{ljt}. On the other hand, problem $SC(\lambda, \alpha)$ can be further decomposed into $|L|$ single-machine scheduling problems. We denote the scheduling problem associated with line l by $SC^l(\lambda, \alpha)$. Each sub-problem $SC^l(\lambda, \alpha)$ considers only $|P_l| \leq |J|$ different types of products.

Note that, for any Lagrangian multipliers $\lambda \geq 0$ and α, the optimal value $Z(\lambda, \alpha)$ of the Lagrangian sub-problem $S(\lambda, \alpha)$ is a dual bound for the original problem FL-CLSD$^w_{SCF}$. In this paper, we propose two approaches to improve the quality of the bounds obtained by the LR approach. Firstly, we offer a slight reformulation of the original capacity constraints and prove that the dualization of the reformulated constraints provides tighter dual bounds than those obtained by dualizing the original constraints. Secondly, we develop valid inequalities for the original problem that are employed in the Lagrangian sub-problem to obtain better dual bounds and facilitate the construction of high-quality feasible solutions. The reformulated capacity constraints are stated as follows:

$$\sum_{j \in P_l, p} a_{lj} x_{ljtp} + \sum_{i,j} st_{lij} z_{lijt} \leq C_{lt} \delta_{lt}, \forall l, t. \tag{23}$$

Constraints (23) simply state that the capacity of production line l can only be used in period t if this line is assembled. The relaxation of constraints (23), rather than (6), generates a reformulated Lagrangian sub-problem, denoted by $S^{'}(\lambda, \alpha)$, as follows:

$$\begin{aligned} Z^{'}(\lambda, \alpha) = \text{Min} \ & \sum_{l,j,t,p} (hb_{ljtp} + \lambda_{lt} a_{lj}) x_{ljtp} + \sum_{l,j,t} \alpha_{ljt} w_{ljt} + \sum_{l,t} ac_l \delta_{lt} \\ & + \sum_{l,i,j,t} (sc_{lij} + \lambda_{lt} st_{lij} - \alpha_{ljt}) z_{lijt} - \sum_{l,j,t} \alpha_{ljt} y_{ljt} \\ & - \sum_{l,t} \lambda_{lt} C_{lt} \delta_{lt} \\ & \text{s.t. } (x, w, \delta, y, z) \in X \end{aligned} \tag{24}$$

We observe that for any Lagrangian multipliers $\lambda \geq 0$ and α, the optimal value $Z^{'}(\lambda, \alpha)$ of the reformulated Lagrangian sub-problem is greater than or equal to the value $Z(\lambda, \alpha)$. This is formally presented in Proposition 1.

Proposition 1. *For any $\lambda = (\lambda_{lt}) \geq 0$ and $\alpha = (\alpha_{ljt})$,*

$$Z^{'}(\lambda, \alpha) \geq Z(\lambda, \alpha).$$

Proof. Firstly, we remark that the feasible region for the original ($S(\lambda, \alpha)$) and of the reformulated ($S^{'}(\lambda, \alpha)$) Lagrangian sub-problems are identical, i.e., they are both $(x, w, \delta, y, z) \in X$. Let $L(x, w, \delta, y, z)$ and $L^{'}(x, w, \delta, y, z)$ be the objective function value of the original ($S(\lambda, \alpha)$) and of the reformulated ($S^{'}(\lambda, \alpha)$) Lagrangian sub-problems, respectively, for a fixed vector of (x, w, δ, y, z). Since $0 \leq \delta_{lt} \leq 1$ (due to (16)), $\lambda_{lt} \geq 0$ (by definition) and $C_{lt} \geq 0$ (by definition), it follows that

$$-\sum_{l,t} \lambda_{lt} C_{lt} \delta_{lt} \geq -\sum_{l,t} \lambda_{lt} C_{lt}.$$

Hence,

$$L^{'}(x, w, \delta, y, z) \geq L(x, w, \delta, y, z) \tag{25}$$

Since both problems have the same feasible region of X, (25) ensures that $Z^{'}(\lambda, \alpha) \geq Z(\lambda, \alpha)$. □

As $S(\lambda, \alpha)$ is an uncapacitated lot sizing and scheduling problem, the absence of capacity constraints allows solutions with high production quantities, and consequently, the production lines tend to be assembled for a reduced number of periods. These dual solutions far from feasibility may provide weak dual bounds. Hence, in order to improve dual bounds, we develop valid inequalities that represent sequence-independent capacity constraints for the original problem that can be considered in the lot sizing part of the Lagrangian sub-problem ($LC(\lambda, \alpha)$), as follows.

Proposition 2. *Let ST_{lj} be the minimum setup time to produce item j on line l, and STm_l be the maximum among the minimum setup times associated to line l, i.e.,*

$$ST_{lj} = \min_{i \in P_l, i \neq j} \{st_{lij}\}, \quad \forall l, j \in P_l; \quad \text{and} \quad STm_l = \max_{j \in P_l} \{ST_{lj}\}, \quad \forall l.$$

Then, sequence-independent setup capacity (SISC) inequalities (26) are valid for the original problem FL-CLSD$^{w}_{SCF}$.

$$\sum_{j \in P_l, p} a_{lj} x_{ljtp} + \sum_{j \in P_l} ST_{lj} w_{ljt} \leq C_{lt} + STm_l, \forall l, t. \tag{26}$$

Proof. We note, $\forall l$ and $\forall t$

$$\sum_{i,j \in P_l} st_{lij} z_{lijt} \geq \sum_{i,j \in P_l} ST_{lj} z_{lijt} = \sum_{j \in P_l} ST_{lj} \left(\sum_{i \in P_l} z_{lijt} \right) = \sum_{j \in P_l} ST_{lj} \left(w_{ljt} - y_{ljt} \right)$$

$$= \sum_{j \in P_l} ST_{lj} w_{ljt} - \sum_{j \in P_l} ST_{lj} y_{ljt} \geq \sum_{j \in P_l} ST_{lj} w_{ljt} - STm_l \tag{27}$$

where the first inequality holds due to $ST_{lj} \leq st_{lij}$ (by definition), the first equation is a simple rearrangement, the second equation holds due to constraints (7), the third equation is a rearrangement, and finally, the last inequality is due to the following relationship established by the definition of parameter STm_l (i.e., $STm_l \geq ST_{lj}$) and constraints (8) and (17):

$$\sum_{j \in P_l} ST_{lj} y_{ljt} \leq \sum_{j \in P_l} STm_l y_{ljt} = STm_l \sum_{j \in P_l} y_{ljt} \leq STm_l$$

Substituting (27) into the original capacity constraints (6), we have

$$C_{lt} \geq \sum_{j \in P_l, p} a_{lj} x_{ljtp} + \sum_{i,j} st_{lij} z_{lijt} \geq \sum_{j \in P_l, p} a_{lj} x_{ljtp} + \sum_{j \in P_l} ST_{lj} w_{ljt} - STm_l \tag{28}$$

□

In Section 5 we study the impact of using SISC inequalities (26) with respect to the quality of the dual bounds as well as the feasible solutions built.

4.2 A Lagrangian Decomposition scheme

The Lagrangian Decomposition (LD) technique was proposed by Guignard and Kim (1987) aiming to allow the decomposition of complex problems into easier-to-solve sub-problems. The LD approach is particularly suitable to deal with problems resulting from the integration of two or more well-studied problems. In traditional Lagrangian relaxation approaches some constraints of the original problem are relaxed, being only considered in the objective function of the Lagrangian sub-problem. On the other hand, in LD approaches all the original constraints appear in the Lagrangian sub-problem. The LD approach consists of creating identical copies of some decision variables and using one of these copies in each set of constraints. Additional constraints enforcing the equality between the copy and original variables are introduced in the model and dualized.

In this paper, we propose an LD scheme in which the obtained Lagrangian sub-problem can also be decomposed into a lot sizing problem with decisions about the lines to be assembled and various single-machine scheduling problems. A similar LD approach was studied in Xiao et al (2015) to solve a parallel machine lot sizing and scheduling problem from the semiconductor manufacturing industry. In our LD approach, we define copy variables A_{ljtp} for the production variables x_{ljtp}, and present the reformulated model, denoted by $M^{'}$, as follows:

$$\begin{aligned}
\text{Min} \quad & (1) \\
\text{s.t.} \quad & (2)-(5), (8)-(19) \\
& \sum_{j \in P_l, p} a_{lj} A_{ljtp} + \sum_{i,j} st_{lij} z_{lijt} \leq C_{lt}, \ \forall l, t && (29) \\
& \sum_{p} A_{ljtp} \leq \frac{C_{lt}}{a_{lj}} (y_{ljt} + \sum_{i \in P_l} z_{lijt}), \quad \forall l, j \in P_l, t && (30) \\
& x_{ljtp} = A_{ljtp}, \qquad \forall l, j, t, p && (31) \\
& A_{ljtp} \geq 0, \qquad \forall l, j, t, p && (32)
\end{aligned}$$

Capacity constraints (29) use the copy variables A_{ljtp} instead of the original variables x_{ljtp}, while constraints (30) ensure that the production of each item occurs only if that item appears in the production sequence of the respective line. Finally, constraints (31) enforce the equality between the copy (A_{ljtp}) and original (x_{ljtp}) variables, while constraints (32) define the variable domain of A_{ljtp}. We observe that constraints (30) together with constraints (31) and the right side of constraints (3) guarantee that, for each l, j and t, $w_{ljt} = 1$ if and only if $y_{ljt} + \sum_j z_{lijt} = 1$. Therefore, the constraints (7) are excluded in the model $\mathrm{M}^{'}$.

The proposed LD approach consists of dualizing constraints (31). For this purpose, let $\beta = (\beta_{ljtp})$ be the dual variables associated with these constraints. The Lagrangian sub-problem $S_{LD}(\beta)$ is then as follows:

$\bullet S_{LD}(\beta)$:

$$\begin{aligned}
Z_{LD}(\beta) = \text{Min} \ & \sum_{l,j,t,p} (hb_{jtp} + \beta_{ljtp}) x_{ljtp} + \sum_{l,t} ac_l \delta_{lt} \\
& + \sum_{l,i,j,t} sc_{lij} z_{lijt} - \sum_{l,j,t,p} \beta_{ljtp} A_{ljtp} && (33) \\
\text{s.t.} \ & (2)-(5), (8)-(19), (29), (30)
\end{aligned}$$

Now, we note that the Lagrangian sub-problem $S_{LD}(\beta)$ can be decomposed into a lot sizing problem with decisions about assembling the production lines, denoted by $\mathrm{LC}_{LD}(\beta)$, and a parallel machine scheduling problem, denoted by $\mathrm{SC}_{LD}(\beta)$, as defined below.

• $LC_{LD}(\beta)$:

$$Z^{LC_{LD}}(\beta) = \text{Min} \sum_{l,j,t,p} (hb_{ljtp} + \beta_{ljtp})x_{ljtp} + \sum_{l,t} ac_l \delta_{lt} \quad (34)$$
$$\text{s.t. } (2) - (5), (14) - (16)$$

• $SC_{LD}(\beta)$:

$$Z^{SC_{LD}}(\beta) = \quad \text{Min} \sum_{l,i,j,t} sc_{lij} z_{lijt} - \sum_{l,j,t,p} \beta_{ljtp} A_{ljtp} \quad (35)$$
$$\text{s.t. } (8) - (13), (17) - (19), (29), (30), (32)$$

Problem $LC_{LD}(\beta)$ has the same feasible region as the problem $LC(\lambda, \alpha)$. On the other hand, beyond sequencing decisions, problem $SC_{LD}(\beta)$ also considers some lot sizing aspects. However, similar to problem $SC(\lambda, \alpha)$, problem $SC_{LD}(\beta)$ can also be decomposed into $|L|$ single machine scheduling problems.

Using the proposed decomposition of the Lagrangian sub-problem, we have that, for each Lagrangian multiplier $\beta = (\beta_{ljtp})$, the optimal value of the Lagrangian sub-problem associated with our LD approach is given by

$$Z_{LD}(\beta) = Z^{LC_{LD}}(\beta) + Z^{SC_{LD}}(\beta).$$

To strengthen the dual bounds obtained from our LD approach, we include constraints (36) in $SC_{LD}(\beta)$ to enforce the satisfaction of customer demands.

$$\sum_{p=\max\{1, t-sl_j\}}^{T} A_{ljpt} = d_{jt}, \ \forall l, j \in P_l, t \quad (36)$$

Finally, we remark that SISC inequalities (26) can be used in the problem $LC_{LD}(\beta)$.

4.3 Dual problems

4.3.1 Dual problem for the LR approach

For each pair of dual variables $\lambda = (\lambda_{lt}) \geq 0$ and $\alpha = (\alpha_{ljt})$, the optimal value $Z(\lambda, \alpha)$ of the Lagrangian sub-problem is a dual (lower) bound for the original problem. Therefore, the dual problem associated with the LR approach is given by

$$\max \ \ Z(\lambda, \alpha)$$

$$\text{s.t.}\quad \lambda_{lt} \geq 0, \quad \forall l, t \tag{37}$$
$$\alpha_{ljt} \in \mathbb{R}, \; \forall l, j, t.$$

In this paper, we use the well-established subgradient algorithm (Held et al (1974)) to solve the dual problem (37). More specifically, we start with all dual variables set to zero, i.e., $\lambda_{lt}^1 = 0, \forall l, t$ and $\alpha_{ljt}^1 = 0, \forall l, j, t$. Then, for each iteration $iter \geq 2$, we update the dual variables according to (38) and (39).

$$\lambda_{lt}^{iter} = \max\{0, \lambda_{lt}^{iter-1} + step^{iter-1} \, [u_{\lambda^{iter-1}}]_{lt}\}, \quad \forall l, t, \tag{38}$$
$$\alpha_{ljt}^{iter} = \alpha_{ljt}^{iter-1} + step^{iter-1} \, [v_{\alpha^{iter-1}}]_{ljt} \, , \; \forall l, j, t. \tag{39}$$

In (38) and (39), $x^{iter}, z^{iter}, \delta^{iter}, w^{iter}$, and y^{iter} are the solutions obtained by solving the Lagrangian sub-problem $S(\lambda^{iter}, \alpha^{iter})$, while $step^{iter}$ is the incumbent step size, and $u_{\lambda^{iter}}$ and $v_{\alpha^{iter}}$ are the subgradients associated with constraints (6) and (7), respectively, i.e.,

$$[u_{\lambda^{iter}}]_{lt} = \sum_{j \in P_l, p} a_{lj} x_{ljtp}^{iter} + \sum_{i,j} st_{lij} z_{lijt}^{iter} - C_{lt} \delta_{lt}^{iter}, \forall l, t, \tag{40}$$
$$[v_{\alpha^{iter}}]_{ljt} = w_{ljt}^{iter} - y_{ljt}^{iter} - \sum_{i \in P_l} z_{lijt}^{iter}, \forall l, j, t. \tag{41}$$

Finally, for each iteration $iter$, the step size is updated according to (42), where $\overline{Z}$ is an upper bound for the original problem, $Z(\lambda^{iter}, \alpha^{iter})$ is the incumbent lower bound (the optimal value for Lagrangian sub-problem), and the denominator is the Euclidian norm of the subgradient vector.

$$step^{iter} = \theta^{iter-1} \frac{\overline{Z} - Z(\lambda^{iter-1}, \alpha^{iter-1})}{|| \, [u_{\lambda^{iter-1}}, v_{\alpha^{iter-1}}] \, ||^2}. \tag{42}$$

In (42), θ^{iter} is a control parameter satisfying $0 < \theta^{iter} < 2$. This rule to update the step size was proposed by Polyak (1969). As discussed by Held et al (1974), this rule ensures the convergence of the method to $\overline{Z}$ or obtaining λ^{iter} and α^{iter} so that $Z(\lambda^{iter}, \alpha^{iter}) \geq \overline{Z}$. Moreover, as already observed by Carvalho and Nascimento (2016), we empirically noted that better dual bounds are obtained if the step size is reduced systematically. Therefore, we impose that $\lim_{iter \to \infty} \theta^{iter} = 0$.

4.3.2 Dual problem for the LD approach

The dual problem associated with the LD approach is as follows:

$$\max \quad Z_{LD}(\beta)$$
$$\text{s.t.}\; \beta_{ljtp} \in \mathbb{R}, \; \forall l, j, t, p. \tag{43}$$

Applying the subgradient algorithm to solve the dual problem (43), we again start with the dual variables set to zero ($\beta_{ljtp}^1 = 0, \; \forall, j, t, p$) and, for each iteration $iter \geq 2$, the variables β are updated according to (44).

$$\beta_{ljtp}^{iter} = \beta_{ljtp}^{iter-1} + step^{iter-1}[g_{\beta^{iter-1}}]_{ljtp}, \forall l, j, t, p. \tag{44}$$

In (44), $g_{\beta^{iter}}$ are the associated subgradient vector defined according to (45), where x^{iter} and A^{iter} are the solutions obtained by solving the Lagrangian sub-problem $S_{LD}(\beta)$; $step^{iter}$ is the incumbent step size, and the expression $x^{iter} - A^{iter}$ represents the incumbent subgradient. Under these conditions, the step size is updated similarly to the procedure of Section 4.3.1.

$$[g_{\beta^{iter}}]_{ljtp} = x_{ljtp}^{iter} - A_{ljtp}^{iter}, \forall l, j, t, p. \tag{45}$$

4.4 Feasibility procedure and Lagrangian heuristics

The feasibility procedure, which is used to obtain feasible solutions from the dual variable values, explores a specific property of FL-CLSD$_{SCF}^{w}$. The sets P_l, $l \in L$ are disjoint, i.e., each product is produced only on its respective production line, and only constraints (5) link different production lines through binary variables δ_{lt}, $l \in L, t \in T$. Therefore, in each subgradient iteration $iter$, fixing the value of each variable δ_{lt} in the feasible value δ_{lt}^{iter} obtained by solving the Lagrangian sub-problem ($S(\lambda^{iter}, \alpha^{iter})$ and $S_{LD}(\beta^{iter})$ for LH and LD approaches, respectively), the resulting problem can be decomposed into $|L|$ single machine lot sizing and scheduling problems, denoted by FP$_l(\delta^{iter})$, $l = 1, \ldots, |L|$.

Each problem FP$_l(\delta^{iter})$ considers only $|P_l|$ different products rather than $|J| = \sum_l |P_l|$ products considered in the original problem. Moreover, FP$_l(\delta^{iter})$ considers all original constraints concerning its respective products except constraints (5) that are satisfied by construction, i.e., each problem FP$_l(\delta^{iter})$ considers the relaxed constraints concerning to production line l. Therefore, a feasible solution for the original problem is obtained simply by grouping the solutions found for problems FP$_l(\delta^{iter})$, $l \in L$.

We also observe that for each $l \in L$, we only solve the problem FP$_l(\delta^{iter})$ in the iterations of the subgradient method in which the vector $\Delta_l^{iter} = (\delta_{l1}^{iter}, \ldots, \delta_{l|T|}^{iter})$ differs from the feasible pattern obtained in the previous iteration (Δ_l^{iter-1}). Preliminary computational experiments indicated that few of these vectors change in each iteration. Finally, since there is no data dependency between the problems FP$_l(\delta^{iter})$, $l = 1, \ldots, |L|$, a parallel computing approach can be used to solve all necessary sub-problems simultaneously.

In this paper, we use a commercial MIP solver and the branching rule proposed in Oliveira and Santos (2017) to solve the sub-problems FP$_l(\delta^{iter})$, $l = 1, \ldots, |L|$. Since the optimal solutions to these problems are not required to build a feasible solution, we have limited the running time for each sub-problem.

Our feasibility procedure is presented in Algorithm 1, to be used as a subroutine in the Lagrangian-based heuristics presented in Algorithm 2. In Algorithm 1, the input parameters are (i) the current subgradient iteration $iter$; (ii) the current and the previous dual solutions δ^{iter} and δ^{iter-1}; and (iii) the objective function values of each sub-problem FP$_l(\delta^{iter-1})$, denoted by $Z_l^{'}$. In the first iteration, $iter = 1$, we consider $\delta^{iter-1} = \delta^0 = (0, \ldots, 0)$ and $Z_l^{'} = 0$, for all l. Next, for each l, we define the vectors Δ_l

and $\overline{\Delta}_l$ to represent the current and previous assembly patterns of the production line l respectively and check if Δ_l differs from $\overline{\Delta}_l$, and if so, we update $Z_l^{'}$ by solving the sub-problem $\text{FP}_l(\delta^{iter})$. Finally, we update the value of the current feasible solution.

We observe that Algorithm 1 obtains a feasible solution in an iteration *iter* if all sub-problems $\text{FP}_l(\delta^{iter})$, $l = 1, \ldots, |L|$, are feasible. Our computational experiments found that the dual solutions δ^{iter} obtained in the proposed Lagrangian schemes usually provide feasible solutions from the first subgradient iteration onwards. However, we highlight that if a feasible solution can not be obtained for a problem $\text{FP}_l(\delta^{iter})$, we just set $Z_l^{'}$ to ∞ and the heuristic proceeds normally. It is reasonable to expect that in a later iteration, a feasible solution will probably be obtained.

Algorithm 1 Feasibility procedure($iter$, δ^{iter}, δ^{iter-1}, $Z^{'}$)

Initialization: $\overline{Z} \leftarrow 0$;
for $l = 1 \ldots, |L|$ **do**
  $\Delta_l \leftarrow (\delta_{l1}^{iter}, \ldots, \delta_{l|T|}^{iter})$;
  $\overline{\Delta}_l \leftarrow (\delta_{l1}^{iter-1}, \ldots, \delta_{l|T|}^{iter-1})$;
  if $\Delta_l \neq \overline{\Delta}_l$ **then**
    if $\text{FP}_l(\delta^{iter})$ is feasible **then**
      $Z_l^{'} \leftarrow Z_{FP_l(\delta^{iter})}$;
    else
      $Z_l^{'} \leftarrow \infty$;
    end if
  end if
  $\overline{Z} \leftarrow \overline{Z} + Z_l^{'}$;
end for

To conclude this section, Algorithm 2 presents an overview of the proposed Lagrangian approaches. In this algorithm, the input parameters are the maximum running time ($Time^{\max}$) and the maximum number of iterations ($iter^{\max}$). Both approaches start with all dual variables set at zero. Next, while the stopping criteria are not reached, we first solve the associated Lagrangian sub-problem using the decomposition schemes presented in Sections 4.1 and 4.2 and obtain a dual bound denoted by $\underline{Z}$. Using the solution obtained by solving the Lagrangian sub-problem we can easily compute the subgradient vectors (see equations (40) and (41) for the LR approach and equation (45) for LD). Then, we compute a feasible solution using Algorithm 1 and obtain a primal bound $\overline{Z}$. Finally, we update the step size, the dual variables, and the processing time to start a new iteration.

5 Computational Analysis

This section presents a computational study, carried out using a data set from the literature, with the aim of evaluating the efficiency and effectiveness of the proposed Lagrangian approaches. Firstly, in Section 5.1 we describe the computational environment and test instances. Secondly, in Section 5.2 we present computational results

Algorithm 2 Lagrangian based heuristics (LR and LD)

Initialization: $iter \leftarrow 1$; $time \leftarrow 0$; $\overline{Z} \leftarrow \infty$; $Z \leftarrow 0$;
LR: $\lambda \leftarrow 0$, and $\alpha \leftarrow 0$;
LD: $\beta \leftarrow 0$;
while $time < Time^{\max}$ **and** $iter < iter^{\max}$ **and** $\overline{Z} > Z$ **do**
  Solve the Lagrangian sub-problem and obtain the dual bound Z;
  LR: $S(\lambda, \alpha)$
  LD: $S_{LD}(\beta)$
  Compute the subgradient;
  LR: u_λ and v_α according to (40) and (41), respectively;
  LD: g_β according to (45);
  Use Algorithm 1 to obtain a primal bound $\overline{Z}$;
  Update the step size $step$ according to (42);
  Update the dual variables;
  LR: $\lambda \leftarrow \max\{0, \lambda + step \cdot u_\lambda\}$; and
    $\alpha \leftarrow \alpha + step \cdot v_\alpha$
  LD: $\beta \leftarrow \beta + step \cdot g_\beta$
  Update the running time ($time$);
  Update the iteration: $iter \leftarrow iter + 1$;
end while

and analysis regarding the proposed methods, and finally, benchmarking our methods with algorithms from the literature is provided in Section 5.3.

5.1 Computational setting and test instances

The mathematical model and the solution approaches considered in this paper were implemented in C++ language using the Concert Technology library of the IBM ILOG Cplex 20.1 Optimization solver. The computational experiments were executed on a computer with two Intel Xeon 2.8 GHz (10 cores, 2 threads/core, and 25 MB SmartCache) processors and 128 GB DDR3 1866 MHz RAM memory.

We consider the data set proposed in Soler et al (2021b) in line with industry practices. The test instances are divided into 5 classes, each with 20 test instances, representing small to large-sized companies. Table 1 presents the main characteristics (number of periods, production lines, products, and resources) of the data set following our notation. Table 1 also presents the values adopted for some auxiliary parameters used to compute the other necessary parameters.

We use the notation $U[a, b]$ to indicate that an integer value was randomly chosen in the interval $[a, b]$ using uniform probability distribution. The test instances have the following specifications.

- $C_{lt} = 480$, i.e., minutes in a production day of 8 hours;
- $st_{lij} \in U[15, 45]$ i.e., the setup times randomly range from 15 to 45 minutes, and $sc_{lij} = 2st_{lij}$;
- For each product j, there is only one production line $\gamma_j \in U[1, |L|]$ that can produce it. Hence, the set of products that can be produced on line l is $P_l = \{j|\gamma_j = l\}$;

Table 1 Parameter values used in each class of the test instances.

Class	$\|T\|$	$\|L\|$	$\|J\|$	$\|K\|$	φ^d	φ_k^r	φ_0^r	ϕ^b	ϕ^e
1	10	7	45	5	100	0.8	0.6	0	0
2	10	10	80	6	100	0.8	0.6	0	0
3	14	10	90	6	90	0.6	0.5	0	0
4	12	10	110	7	90	0.6	0.55	100	150
5	14	10	110	7	90	0.6	0.55	50	150

Columns $|T|, |L|, |J|$ and $|K|$ represent the number of periods, production lines, products, and resources, respectively. φ^d, φ_k^r, φ_0^r, ϕ^b, and ϕ^e present the values adopted for the auxiliary parameters used to compute the customer demands, the available resources, and the costs to assemble the production lines.

- $d_{jt} \in U\left[0, \frac{C_{lt} - \min\limits_{i,j}\{st_{lij}\}\gamma - \varphi^d}{|P_l|}\right]$, with φ^d as specified in Table 1;
- $a_{lj} = 1$ and $m_j = 2$, i.e., we consider fixed processing times and minimum lot sizes;
- $h_j \in U[1, 10]$ and $b_j = 10h_j$;
- $r_{kl} \in U[0, 2]$ when $k > 1$ and $r_{1l} \in U[5, 10]$. In Soler et al (2021b), the authors considered that $k = 1$ represents the workers, while $k = 2, \cdots, |K|$ represent other resources, such as machines, tools, and workstations;
- $R_{kt} = \max\left\{\max\limits_{l=1,\dots,L}\{r_{kl}\}, \varphi_k^r \sum\limits_{l\in[L]} r_{kl}\right\}$, with φ_k^r as specified in Table 1, to represent the maximum percentage of production lines that can simultaneously operate in each period;
- $ac_l = \sum_k rc_k r_{kl}$, where $rc_k \in U[\phi^b, \phi^e]$ (ϕ^b and ϕ^e as specified in Table 1). In other words, the cost to assemble the line l is the sum of the costs of the required resources.

A more detailed description of the data set and a discussion about the relationship between the test instances and real-world practical scenarios can be found in Soler et al (2021b).

In this section, we frequently report the percent deviation from the objective function value to the known dual bound, denoted by *gap*, as a quality measure of the computational performance of the studied methods. More specifically, for each test instance σ, the gap_σ is computed according to (46), where OF_σ and DB_σ are the obtained objective function and dual bound values, respectively.

$$gap_\sigma = 100 \cdot \frac{OF_\sigma - DB_\sigma}{OF_\sigma}. \tag{46}$$

5.2 Results and analysis of the proposed methods

In this section, we study the computational performance of the proposed Lagrangian approaches and evaluate the impact of the SISC inequalities introduced in Proposition 2. For this purpose, we compare the following computational experiments:

1. M: MIP solver Cplex to solve FL-CLSD$^{w}_{SCF}$ introduced in Section 3;
2. LR: Lagrangian relaxation approach (without SISC);
3. LR^{SISC}: Lagrangian relaxation approach with SISC;
4. LD: Lagrangian decomposition scheme (without SISC); and
5. LD^{SISC}: Lagrangian decomposition scheme with SISC.

To ensure a fair comparison, we fixed the maximum running time to 3,600 seconds for all considered solution methods.

For each experiment and class of instances, Table 2 presents the obtained average gap (Ave%), best gap (Best%), worst gap (Worst%), the variance of the gaps (Var%), and the average running time (Time). For the Lagrangian approaches, the table also presents the average number of performed iterations (It).

From the results presented in Table 2, we observe that for small-sized test instances (classes 1 and 2), the high-performance MIP solver Cplex (experiment M) was able to find optimal solutions for all test instances very quickly. For the same instances, approaches LR^{SISC} and LD^{SISC} found mostly optimal solutions, albeit significantly slower than Cplex. Moreover, for these instances, LR and LD approaches performed poorly.

On the other hand, for medium to large-sized test instances (classes 3, 4, and 5), we observe that the Lagrangian approaches with SISC constraints (i.e., LR^{SISC} and LD^{SISC}) significantly outperformed the MIP solver in all the performance metrics used. LR^{SISC} and LD^{SISC} in particular presented lower gap variances (G$_{var}$), indicating a reduction in the observed number of outliers (i.e., difficult test instances with high gaps). For these classes, LR and LD demonstrated competitive results compared with the MIP solver performance.

Next, we analyze the impact of the SISC inequalities. Comparing LR to LR^{SISC} (as well as LD to LD^{SISC}), we observe that the SISC versions presented significantly lower average, best, and worst gaps, as well as gap variances, in comparison to their counterparts without SISC. These results indicate that LR^{SISC} (and likewise LD^{SISC}) was able to find better feasible solutions and dual bounds. Moreover, the presence of SISC constraints reduced the occurrence of outliers. However, observing the average number of performed iterations (IT) and the average running times (RT), we conclude that the iterations of SISC versions are more computationally expensive than those of their respective counterparts. This stems from the presence of SISC constraints making the lot sizing part of the Lagrangian sub-problem more challenging. However, this slower execution is aligned with high-quality bounds. Therefore, SISC were highly effective in improving both proposed Lagrangian procedures.

Figure 1 presents the performance profiles of the considered approaches (according to Dolan and Moré (2002)): the x-axis presents the relative deviation from the minimum observed objective function value, and the y-axis presents the percentage of test instances. For each relative deviation θ, we associate the percentage of test instances for which the deviation from the obtained objective function value to the minimum observed objective function value is less than or equal to θ. We note that LR and LD were able to find the best objective function values for about 10% of the test instances, while M and LD^{SISC} did for about 40% and LR^{SISC} did for about 60% of test instances. Moreover, LR^{SISC} and LD^{SISC} performed best with maximum

Table 2 Results obtained for the MIP solver and the proposed Lagrangian heuristics.

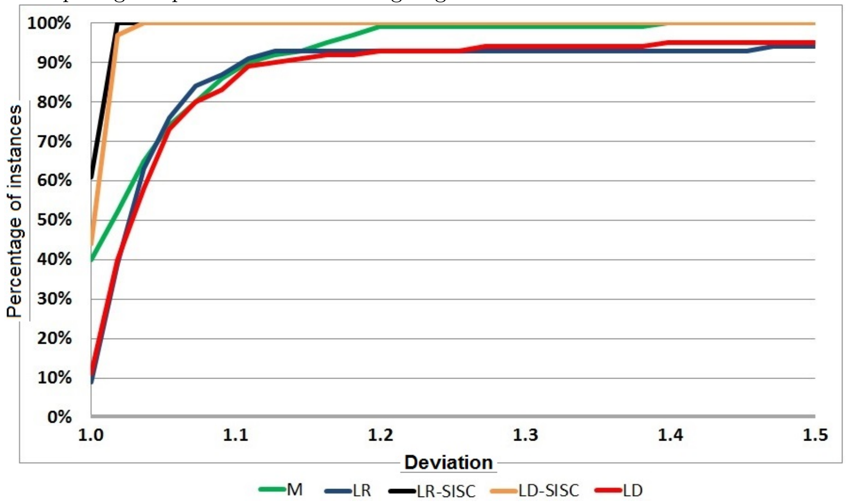
		Class 1	Class 2	Class 3	Class 4	Class 5	Average
M	Ave%	0.00	0.00	12.51	9.54	10.89	6.59
	Best%	0.00	0.00	1.09	2.28	4.70	1.61
	Worst%	0.00	0.00	41.32	18.85	21.74	16.38
	Var%	0.00	0.00	109.98	25.95	25.80	32.35
	Time	66	352	3600	3600	3600	2243
LR	Ave%	3.52	1.42	14.96	8.27	11.01	7.84
	Best%	0.00	0.00	1.10	2.65	4.06	1.56
	Worst%	31.19	5.94	67.02	16.23	57.25	35.53
	Var%	47.75	3.90	323.17	11.97	138.45	105.05
	It	100	100	64	44	33	68
	Time	695	2760	3600	3600	3600	2851
LR^{SISC}	Ave%	0.13	0.17	6.94	5.04	5.13	3.48
	Best%	0.00	0.00	0.97	1.46	1.96	0.88
	Worst%	1.35	1.30	23.69	9.42	9.11	8.97
	Var%	0.10	0.09	27.39	6.36	4.13	7.61
	It	100	100	31	16	13	52
	Time	775	3230	3600	3600	3600	2961
LD	Ave%	2.10	1.49	14.69	10.27	9.82	7.67
	Best%	0.01	0.01	1.10	2.65	4.32	1.62
	Worst%	9.34	5.93	62.01	32.47	22.60	26.47
	Var%	5.59	3.26	291.98	42.48	24.65	73.59
	It	99	55	32	21	13	44
	Time	2674	3600	3600	3600	3600	3415
LD^{SISC}	Ave%	0.13	0.19	7.10	5.33	5.41	3.63
	Best%	0.00	0.01	0.97	1.61	2.18	0.95
	Worst%	1.35	1.30	23.27	10.18	9.78	9.18
	Var%	0.09	0.09	30.23	7.49	4.58	8.50
	It	96	45	12	11	10	35
	Time	3018	3600	3600	3600	3600	3484

Ave%/Best%/Worst% present the average/best/worst gaps, respectively; Var% is the variance of the obtained gaps; *It* is the average number of performed iterations; and *Time* is the average running time.

deviations less than 1.05, followed by M with around 1.2, and LR and LD both above 1.5.

Figure 2 presents the obtained average GAPs against the observed average running times, considering all test instances, for the analyzed methods. Firstly, we observe that in the absence of SISC constraints, the Lagrangian approaches, LR and LD, were not able to outperform M. Secondly, concerning the quality of the obtained solutions, LR^{SISC} and LD^{SISC} outperform the other considered approaches, presenting significant smaller GAPs. The great performance of the SISC constraints is due to the high quality of the dual solutions obtained in the lot sizing part of the Lagrangian

Fig. 1 Comparing the performance of the Lagrangian heuristics and the MIP solver.



sub-problems ($LC(\lambda, \alpha)$ and $LC_{LD}(\beta)$ for LR^{SISC} and LD^{SISC}, respectively). More specifically, SISC constraints allow the consideration of realistic capacity constraints without linking the lot sizing and scheduling sub-structures. This fact avoids producing infeasible lots and enables the determination of good and feasible patterns for assembling the production lines. Therefore, as the feasibility procedure fixes the dual decisions about assembling the lines, high-quality feasible solutions are built. It is also important to note that the consideration of SISC constraints did not significantly increase the running times.

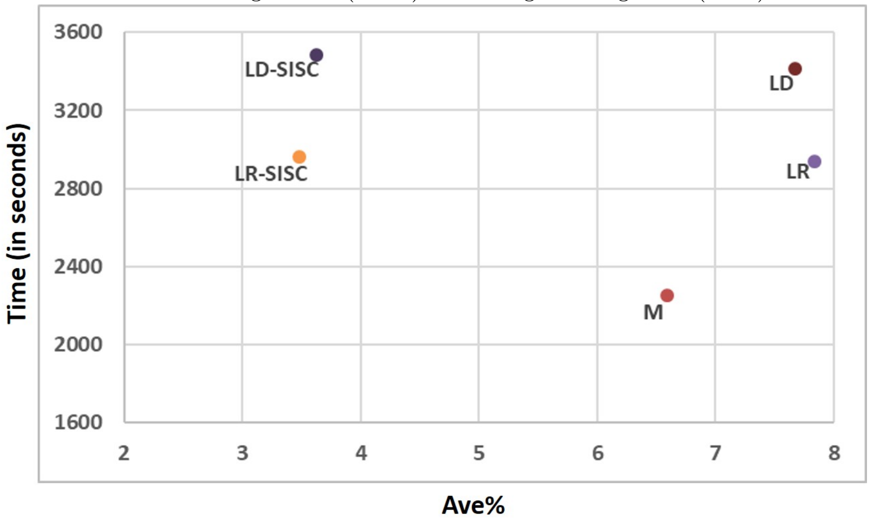
Finally, we also observe that M presented the lowest average running time. This fact occurred due to the great performance of M approach for the small-sized test instances from classes 1 and 2. For these classes, M quickly found the optimal solutions. We observed that LR^{SISC} and LD^{SISC} were also able to find optimal solutions for classes 1 and 2, however, these methods spent much more time for proving the optimality of the solutions, i.e., the dual bounds progress more slowly than in approach M for these small test instances.

5.3 Comparison with methods from the literature

In this section, we compare the performance of the most promising Lagrangian approach proposed in this paper (LR^{SISC}) with the methods established in the literature for the LSP-PLSR. More specifically, we analyze the relax-and-fix heuristic (RFH) of Soler et al (2021b), and the decomposition-based constructive heuristic (D), the local branching (LB), and the fix-and-optimize (FO) heuristics of Soler et al (2021a). As heuristic approaches do usually not offer dual bounds, we used the dual bounds obtained in this paper to compute the gaps for all methods. The results are presented in Table 3 and Figure 3.

In Figure 3, for each method in consideration, we present the box-plot of the gaps obtained for all test instances. We observe that for RFH and LR^{SISC}, the first quartile

Fig. 2 Relation between average GAPs (Ave%) and average running times (Time).



was equal to zero, indicating that optimal solutions were obtained for at least 25 test instances. On the other hand, the first quartiles for approaches D, LB, and FO were 0.44%, 0.33%, and 0.27%, respectively. LR^{SISC} presented the smallest median (2.82%), followed by D (4.06%), RFH (4.47%), FO (4.55%), and LB (4.61%). For all solution methods, we identify two outliers, corresponding to the two most challenging test instances from class 3. For these instances, LR^{SISC} provided the smallest gaps and consequently better feasible solutions than other methods.

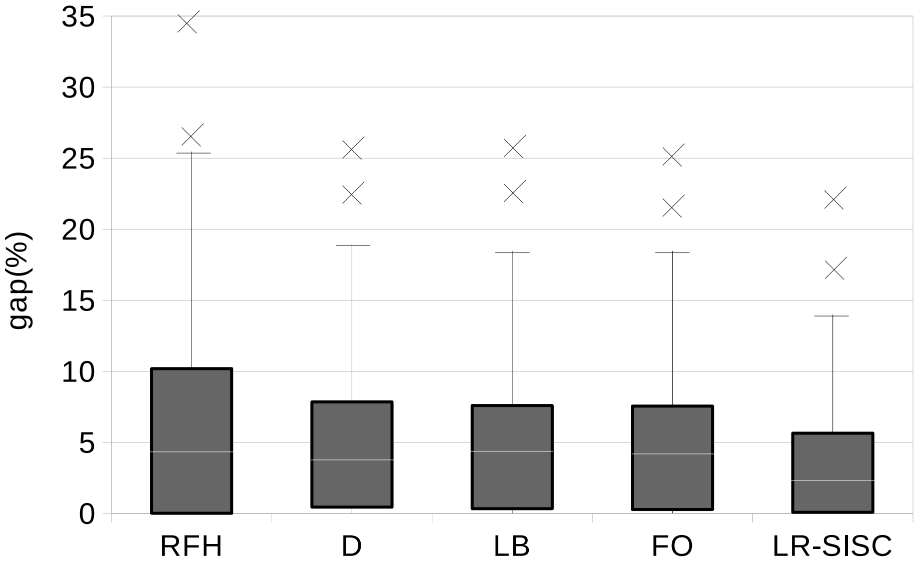
Comparing the results reported in Table 3 with those observed for LR^{SISC} (Table 2), we note that, for classes 1 and 2, RFH obtained better average gaps and consequently better feasible solutions. For these instances, the decomposition-based heuristic (D) presented the lowest running times. On the other hand, for medium to large-sized test instances (classes 3, 4, and 5), LR^{SISC} was able to find better feasible solutions than all the other methods in consideration. Moreover, LR^{SISC} presented the lowest gap variance (Var%) indicating that, for each class of instances, the solution performance is more stable and less volatile.

In general, we conclude that LR^{SISC} outperforms the methods from the literature concerning the quality of the obtained feasible solutions. Finally, LR^{SISC} has the additional significant advantage that it is also able to provide dual bounds, while the heuristic approaches from the literature do not offer this.

5.4 Sensitivity analysis

To conclude our computational study, we present a sensitivity analysis to study the impact of changing some key parameter values. Due to the high computational time required to solve all test instances in a range of scenarios, this study is conducted using 5 randomly selected test instances from each class, totaling 25 test instances. The considered methods are the model (M), the Lagrangian relaxation approach (LR), and

Fig. 3 Box-plots of obtained gaps.



the Lagrangian relaxation approach with SISC constraints (LR^{SISC}). The maximum running time was fixed in 3600 seconds for all experiments.

5.4.1 Impact of the setup costs

The original test instances, described in section 5.1, consider the existence of setup costs proportional to the setup times. This assumption is based on the packaged meat production system. In this production environment, most setup procedures consist of cleaning machines and tools, requiring the utilization of chemical products and abundant water. Therefore, the considered setup costs represent the cleaning products and water consumption costs. However, we observe that in other production environments, it may be difficult to estimate the setup costs, because these mainly represent the opportunity cost of the setup times. Therefore, we perform additional computational tests considering different setup cost values to investigate the performance of the methods proposed in this paper to deal with problems arising in different industries.

More specifically, we consider the following scenarios:

- $sc_{lij} = 0, \forall l, i, j$, i.e., without setup costs. This scenario is considered to represent systems with ample production capacity, in which the opportunity cost of additional setups is essentially zero;
- $sc_{lij} = st_{lij}, \forall l, i, j$. This scenario represents systems with low opportunity costs of the setup time and/or environments with tangible setup costs; and
- $sc_{lij} = 2st_{lij}, \forall l, i, j$. This is the original scenario observed in some Brazilian meat companies, as discussed earlier.

Tables 4, 5, and 6 present the computational results considering approaches M, LR, and LR^{SISC}, respectively. In these tables, columns Ave% and Time present the

Table 3 Results obtained for the methods from the literature.

		Class 1	Class 2	Class 3	Class 4	Class 5	Average
RFH	Obj	50974	63628	131224	194178	251472	138295
	Ave%	0.09	0.05	9.69	7.36	13.19	6.07
	Best%	0.00	0.00	0.97	1.55	4.06	1.32
	Worst%	1.35	1.56	34.73	13.94	24.43	15.20
	Var%	0.09	0.17	62.34	15.99	32.33	22.19
	Time	48	3600	3600	3600	3600	2890
D	Obj	51111	64089	128768	193358	236797	134825
	Ave%	0.34	0.74	8.40	7.01	7.99	4.90
	Best%	0.01	0.01	1.66	3.13	2.64	1.49
	Worst%	2.38	3.62	25.41	14.15	14.74	12.06
	Var%	0.31	0.82	34.35	11.11	8.68	11.05
	Time	19	88	3416	3481	3544	2110
LB	Obj	51032	63938	128969	194126	235898	134793
	Ave%	0.19	0.52	8.55	7.36	7.66	4.86
	Best%	0.01	0.01	1.66	3.55	3.36	1.72
	Worst%	1.49	2.10	25.41	14.25	13.47	11.34
	Var%	0.13	0.30	34.12	11.02	6.92	10.50
	Time	1611	3455	3613	3614	3428	3144
FO	Obj	51017	63829	128611	193731	235752	134588
	Ave%	0.18	0.38	8.32	7.20	7.60	4.74
	Best%	0.00	0.00	1.65	3.55	3.36	1.71
	Worst%	1.05	1.56	25.04	14.15	13.44	11.05
	Var%	0.11	0.17	33.36	9.52	6.83	10.00
	Time	686	3600	3600	3600	3600	3017

Note: Obj presents the average objective function values.

average GAPs and the average running times, respectively, while columns Obj, Scost, Hcost, Bcost, and Acost present the average of the obtained objective function values, setup costs, inventory holding costs, backlogging costs, and costs to assemble the production lines, respectively.

Considering the M approach, Table 4 evidenced that, for small and medium-sized test instances from classes 1, 2, and 3, the problem becomes more difficult as the setup costs increase. However, for large-sized test instances from classes 4 and 5, the problem becomes more difficult without setup costs.

For the LR approach, Table 5 shows that, for instances from classes 1 and 2, better feasible solutions were obtained using higher setup costs ($sc = 2st$). On the other hand, for classes 4 and 5, smaller average GAPs were observed in the absence of setup costs. Analyzing the LR^{SISC} approach, Table 6 shows that, for all classes, smaller GAPs were obtained when the setup costs were not considered.

Comparing the considered solution approaches, LR^{SISC} was able to provide better feasible solutions for instances from classes 3, 4, and 5 and for all considered setup cost values. For classes 1 and 2, LR^{SISC} and M found the same feasible solutions for all considered setup cost values, however LR^{SISC} spent more computational time than M to solve these small-sized test instances. We observe that LR^{SISC} is a promising solution approach to deal with problems with different setup cost values. On average, this approach outperforms M and LR for all considered scenarios.

Finally, we observe that the structure of the obtained feasible solutions slightly varies when the setup costs change. In general, when the setup costs increased, the inventory holding costs slightly increased and the costs to assemble the production lines decreased slightly. This fact indicates that when setup costs are relevant, the lot sizes increase to reduce the number of setups and the number of periods in which the production lines need to be assembled.

Table 4 Sensitivity analysis with different setup cost values using approach M.

		M						
		Ave%	Time	Obj	Scost	Hcost	Bcost	Acost
C1	$sc = 0$	0.01	15	40068	0	20744	19324	0
	$sc = st$	0.01	46	44045	3957	20765	19324	0
	$sc = 2st$	0.01	56	47966	7790	20848	19328	0
C2	$sc = 0$	0.01	104	51766	0	26330	25436	0
	$sc = st$	0.01	255	59093	7248	26407	25438	0
	$sc = 2st$	0.01	316	66231	13975	26974	25282	0
C3	$sc = 0$	9.72	3600	118648	0	61522	57126	0
	$sc = st$	16.02	3600	137280	8570	67076	61634	0
	$sc = 2st$	19.11	3600	152686	17000	65719	69968	0
C4	$sc = 0$	17.63	3600	194472	0	62328	72582	59563
	$sc = st$	13.52	3600	196658	8273	64080	64434	59870
	$sc = 2st$	13.74	3600	206974	15913	66416	66396	58249
C5	$sc = 0$	13.22	3600	222852	0	71803	70374	80674
	$sc = st$	8.43	3600	223868	9541	71689	63052	79585
	$sc = 2st$	9.47	3600	236069	18496	74088	65322	78163
Ave	$sc = 0$	8.12	2184	125561	0	48545	48968	28047
	$sc = st$	7.60	2220	132189	7518	50003	46776	27891
	$sc = 2st$	8.47	2234	141985	14635	50809	49259	27282

5.4.2 Impact of the minimum lot sizes

In the proposed model, constraints (3) introduce minimum production lot sizes. More specifically, when a setup is performed for a product j, at least m_j units of this product should be produced. This requirement was proposed by Fleischmann and Meyr (1997) in the context of the simultaneous lot sizing and scheduling problem to avoid setup state changes without product changes. More specifically, when the setup costs and/or setup times do not satisfy the triangle inequality, i.e., there exist products i, j, and k such that $sc_{lik} + sc_{lkj} < sc_{lij}$ and/or $st_{lik} + st_{lkj} < st_{lij}$, for some production line l, the

Table 5 Sensitivity analysis with different setup cost values using approach LR.

		LR						
		Ave%	Time	Obj	Scost	Hcost	Bcost	Acost
C1	$sc = 0$	1.75	421	40755	0	20555	20200	0
	$sc = st$	1.45	606	44670	3972	20896	19802	0
	$sc = 2st$	1.07	658	48474	7805	20691	19978	0
C2	$sc = 0$	4.00	1372	54022	0	25608	28414	0
	$sc = st$	5.16	2583	61371	7240	25714	28416	0
	$sc = 2st$	2.37	2896	67943	14200	26025	27718	0
C3	$sc = 0$	14,04	3600	128558	0	60532	68026	0
	$sc = st$	17.81	3600	139492	8511	59255	71726	0
	$sc = 2st$	15.09	3600	142137	17038	60137	64962	0
C4	$sc = 0$	6.30	3600	179972	0	57744	61622	60606
	$sc = st$	12.04	3600	195184	8088	59138	67472	60486
	$sc = 2st$	11.53	3600	201686	15894	59528	66822	59441
C5	$sc = 0$	5.42	3600	210563	0	65499	64382	80681
	$sc = st$	7.81	3600	222504	9436	66746	65682	80640
	$sc = 2st$	9.14	3600	235440	18473	69210	68496	79260
Ave	$sc = 0$	6.30	2519	122774	0	45987	48529	28257
	$sc = st$	8.86	2798	132644	7449	46350	50620	28225
	$sc = 2st$	7.84	2871	139136	14682	47118	49595	27740

optimal solution can present a setup for the intermediate product k without effective production of this product. Menezes et al (2011) observed that non-triangular setup times or costs are observed in the chemical, dyeing, and pharmaceutical industries, requiring minimum lot sizes to be imposed.

In the application studied in this paper, setup costs and times usually satisfy the triangle inequality. Therefore, in the original test instances, we adopted small minimum lot sizes ($m_j = 2, \forall j$). To study the impact of this parameter, we performed additional computational experiments relaxing this requirement, i.e., adopting $m_j = 0, \forall j$. We observe that as the used test instances have triangular setup costs and times, the same optimal solutions were obtained in both experiments. However, slight differences were observed in the computational performance of the proposed methods.

Considering the M approach, optimal solutions were obtained for all instances from classes 1 and 2 for both considered minimum lot sizes. Moreover, no significant differences were observed in the computational performance for these classes. For class 3, an average GAP of 19.56% was obtained for $m = 0$, and 19.11% for $m = 2$. Similar computational performances were observed for class 4. For class 5, average GAPs of 8.63% and 9.47% were observed for $m = 0$ and $m = 2$, respectively. Considering all test instances, the M approach presented the same running time for both adopted minimum lot sizes and similar average GAPs of 8.43% and 8.47% for $m = 0$ and $m = 2$, respectively.

Table 6 Sensitivity analysis with different setup cost values using approach LR^{SISC}.

		LR^{SISC}						
		Ave%	Time	Obj	Scost	Hcost	Bcost	Acost
C1	$sc = 0$	0.00	213	40068	0	20744	19324	0
	$sc = st$	0.01	696	44045	3956	20764	19324	0
	$sc = 2st$	0.01	749	47966	7780	20858	19328	0
C2	$sc = 0$	0,00	1096	51764	0	26328	25436	0
	$sc = st$	0.01	3252	59094	7250	26406	25438	0
	$sc = 2st$	0.12	3389	66305	14160	26709	25436	0
C3	$sc = 0$	3.94	3600	114894	0	59926	54968	0
	$sc = st$	7.20	3600	123521	8596	60089	54836	0
	$sc = 2st$	8.63	3600	132075	17051	60159	54864	0
C4	$sc = 0$	2.52	3600	172983	0	57836	54398	60749
	$sc = st$	5.36	3600	181257	8285	57844	54380	60749
	$sc = 2st$	5.85	3600	189510	16335	58201	54416	60557
C5	$sc = 0$	1.89	3600	202709	0	66286	55512	80912
	$sc = st$	3.49	3600	212277	9565	66315	55686	80710
	$sc = 2st$	3.83	3600	221845	18976	66672	55516	80681
Ave	$sc = 0$	1.67	2422	116484	0	46224	41928	28332
	$sc = st$	3.21	2950	124039	7531	46284	41933	28292
	$sc = 2st$	3.69	2988	131540	14860	46520	41912	28248

In the LR^{SISC} approach, the average running times observed for class 1 were 698 seconds with $m = 0$ and 750 seconds with $m = 2$, representing an increase of 7.5% when minimum lot sizes are imposed. However, optimal solutions were reached for 4 and 5 test instances for $m = 0$ and $m = 2$, respectively. For class 2, similar performances were observed in both cases. For classes 3 and 5, slightly lower GAPs were obtained when adopting $m = 0$. On the other hand, for class 4, average GAPs of 6.48% and 5.85% were observed for $m = 0$ and $m = 2$, respectively. In general, the minimum lot sizes have a low impact in the LR^{SISC} method. Considering all classes, the observed average GAPs were 3.64% and 3.69%, while the average running times were 2981 and 2988 seconds for $m = 0$ and $m = 2$, respectively.

The most relevant impact of the minimum lot sizes was observed in the LR approach. Except for class 5, the method found better feasible solutions for $m = 2$ than for $m = 0$. For example, for class 1, the average GAP was 3.37% adopting $m = 0$ and only 1.07% adopting $m = 2$. The average running times were similar for this class, being 631 and 658 seconds for $m = 0$ and $m = 2$, respectively. For classes 2 and 3, slightly smaller average GAPs were obtained for $m = 2$. In class 4, we observed the most discrepant average GAPs, being 16.22% for $m = 0$ and only 11.53% for $m = 2$. The absence of minimum lot sizes provides better results only for class 5, where the observed average GAPs were 7.98% and 9.14% adopting $m = 0$ and $m = 2$, respectively. On average, the obtained GAPs were 9.09% for $m = 0$ and 7.84% for $m = 2$, while the running times were 2860 and 2871 seconds, respectively.

Comparing the proposed methods, for both scenarios, LR^{SISC} outperforms M and LR concerning the quality of the obtained feasible solutions for classes 3, 4, and 5. However, for classes 1 and 2, M provided the shortest running times as highlighted earlier. In general, only the LR approach presented significantly different computational performance without minimum lot sizes.

5.4.3 Impact of the backlogging costs

To study the impact of the backlogging costs on the studied problem, beyond the original test instances with $b_j = 10h_j, \forall j$ we perform computational tests using alternative backlogging costs. More specifically, we considered two additional configurations, $b_j = 5h_j, \forall j$, and $b_j = 15h_j, \forall j$, to represent low, medium, and high backlogging costs. Table 7 presents the average results observed for the 25 selected test instances.

In all considered approaches, the average GAPs increased when the backlogging increased, while the running times presented insignificant variations. As observed in other scenarios, the LR^{SISC} approach significantly outperformed the LR and M approaches concerning the quality of the obtained feasible solutions. Regarding the structure of the obtained solutions, we observe that when parameters b_j increase, the holding costs also increase indicating that the production should be anticipated to avoid high backlogging costs. Moreover, we do not observe significant variations in the incurred setup costs and costs to assemble the production lines.

Table 7 Sensitivity analysis with respect parameters b_j.

		Ave%	Time	Obj	Scost	Hcost	Bcost	Acost
M	$b = 5h$	5.53	2240	114696	14706	47640	24714	27636
	$b = 10h$	8.47	2234	141985	14635	50809	49259	27282
	$b = 15h$	9.38	2229	166635	14695	52120	72304	27515
LR	$b = 5h$	5.36	2943	113783	14655	44957	26304	27867
	$b = 10h$	7.84	2871	139136	14682	47118	49595	27740
	$b = 15h$	9.61	2857	163957	14648	47773	73732	27857
LR^{SISC}	$b = 5h$	2.97	3024	110247	14821	44891	22439	28096
	$b = 10h$	3.69	2988	131540	14860	46520	41912	28248
	$b = 15h$	4.24	3029	152769	14857	47043	62571	28298

5.4.4 Impact of the setup times

To conclude this section, we evaluate the impact of changing the setup times, since they are key parameters for the proposed SISC constraints. Beyond the original setup times randomly chosen in [15,45], we experiment with setup times increased by 40% and 80%, ranging in the intervals [21, 63] and [27, 81], respectively. Table 8 presents the obtained average results emphasizing parameters to evaluate the computational performance of the considered methods, with column FS presenting the number of instances in which at least a feasible solution was obtained.

Firstly, we observe that the problem becomes much more challenging, for all considered algorithms, when increasing setup times. Secondly, we highlight that concerning the quality of the obtained solutions, LR^{SISC} significantly outperforms M and LR for all considered setup times. More specifically, LR^{SISC} provided smaller Gaps and solutions with smaller costs. Moreover, for setup times in the interval $[27, 81]$, LR^{SISC} was able to provide feasible solutions for all test instances, while LR found viable solutions for 22 test instances and M for 19 test instances, respectively. On the other hand, LR^{SISC} spent slightly longer computational time than M and LR.

In general, the obtained results show that SISC constraints work well for small to large setup times, allowing viable solutions to difficult problems to be obtained through a small increase in execution time.

Table 8 Sensitivity analysis with respect parameters st_{lij}.

		Ave%	Best%	Worst%	Obj	FS	Time
	M	6.59	1.61	16.38	141985	25	2243
$st \in [15, 45]$	LR	7.84	1.56	35.53	139136	25	2851
	LR^{SISC}	3.48	0.88	8.97	131540	25	2961
	M	23.56	11.09	36.80	271686	23	2456
$st \in [21, 63]$	LR	14.47	9.87	21.31	162744	25	2860
	LR^{SISC}	8.40	5.85	11.93	149293	25	3017
	M	39.19	34.39	45.56	476648	19	3070
$st \in [27, 81]$	LR	21.37	17.03	28.58	201229	22	2788
	LR^{SISC}	15.60	10.84	20.25	180142	25	3112

6 Conclusion and research perspectives

In this paper, we studied a lot sizing and scheduling problem, which originally stemmed from the Brazilian food industry. More specifically, the problem considers a production environment composed of various production lines sharing some scarce production resources and also involves sequence-dependent setup times and costs, backlogging, and perishable products. A mixed integer programming model was used to represent the problem and Lagrangian-based heuristics were proposed to find high-quality feasible solutions.

Firstly, we developed a Lagrangian relaxation scheme (LR) that consists of dualizing the constraints linking the lot sizing and the scheduling decisions. In this framework, the relaxed problem was decomposed into a lot sizing sub-problem and a parallel machine scheduling sub-problem, and the scheduling sub-problem was further decomposed into single-machine scheduling problems. To improve the quality of the obtained bounds, we also develop sequence-independent setup capacity (SISC) inequalities to be considered in the lot sizing part of the Lagrangian sub-problem. Secondly, we proposed a Lagrangian decomposition scheme (LD) in which identical copies of the continuous production variables were created and one of these copies was used in each set of constraints of the original problem. The constraints enforcing the equality between the original and copy variables were dualized, and the relaxed problem was also decomposed into a lot sizing sub-problem and various single-machine scheduling sub-problems. SISC were also employed. In both approaches, we used sub-gradient optimization to solve the associated dual problems. Finally, a MIP-based feasibility procedure was proposed to obtain feasible solutions from the dual solutions.

We presented a computational study using a data set previously established in the literature and composed of 100 test instances, representing small to large-sized real-world industries. The results indicated that including SISC inequalities was very effective in improving the quality of the obtained bounds in both Lagrangian schemes. Our computational experiments also demonstrated that our most effective method, the LR scheme with SISC, outperforms all the solution methods proposed in the literature for the specific problem.

We finally consider some potential directions for future work. We believe that there is further scope to investigate other exact solution approaches to exploit the structure of the problem. In particular, we plan to explore the use of Benders decomposition, column generation, and Branch-and-Price algorithms to make further use of decomposition, especially when the problem has further complications. Moreover, we observe in the literature a trend of considering different components of supply chain management in the same mathematical model. Such integrated approaches usually offer better solutions than the traditional hierarchical approaches. Hence, we plan on extending the model presented in this paper to consider the distribution planning aspect and raw material acquisition.

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