Network Models and Biproportional Apportionment for Fair Seat Allocations in the UK Elections

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Abstract

Systems for allocating seats in an election offer a number of socially and mathematically interesting problems. We discuss how to model the allocation process as a network flow problem, and propose a wide choice of objective functions and allocation schemes. We present biproportional apportionment, which is an instance of the network flow problem and is used in some European countries with multi-seat constituencies. We discuss its application to single seat constituencies and the inevitable consequence that seats are allocated to candidates with little local support. However, we show that variants can be selected, such as regional apportionment, to mitigate this problem. In particular, we introduce a parameter based family of methods which can be tuned to meet the public's demand for local and global "fairness". Using data from the 2010 UK General Election, we study a variety of network models and biproportional apportionments, and address conditions of existence and uniqueness.

Keywords: Fair seat allocation, networks and graphs, biproportional matrices, integer programming.

1 Introduction

Most countries in Europe use some form of *proportional representation* (PR) as a means of allocating members of parliament at both local and national level. The UK has joined in recently and a number of elections (e.g., European, Scottish) use forms of PR; although there is strong resistance to bringing it into the election for members of parliament (MPs) to Westminster, as the 2011 referendum on the Alternative Vote (AV) system indicates (White, 2011). Currently, UK elections are fought on the *first past the post* (FPTP) system, where the winner takes all in each constituency. In a multi-party election, this can skew results significantly from proportionality. For example, in the 2005 general election, the Labour party won 57% of the seats but only 36% of the votes. Conversely, in 2010 the Liberal Democrats received 23% of the votes but only 9% of the seats (Thrasher et al., 2011).

FPTP is also alleged to be responsible for effectively disenfranchising many voters, as the demographics of some constituencies means that they almost never change hands. For example, Gower,

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Normanton and Makerfield have elected Labour MPs without exception since 1906. If the result is a foregone conclusion voter turnout can be expected to be adversely affected. A similar problem in Switzerland led a disgruntled voter to sue (successfully) providing impetus for a subsequent change in the local electoral law (Balinski and Pukelsheim, 2006).

One of the main objections to PR for the UK parliamentary elections is that it breaks the link of MPs with individual constituencies: as well as being members of a party, MPs have traditionally represented the interests of individual voters in the towns or districts they have been elected to. Suppose that a voting area has n districts, where σ_i seats are to be allocated to district $i = 1, \ldots, n$, and a list system typically allocates the σ_i seats proportionally to party shares in i. If $\sigma_i = 1$ as in the UK, then this simply becomes FPTP, whereas using a single transferable vote reduces to AV. If we are to devise a model of PR which retains a constituency link, a balance must be made between global and local voting patterns. In particular, it should accommodate the strong support for nationalist parties in certain regions and the consistent levels of support for other smaller parties.

Any electoral system implicitly attempts to solve an optimization problem: given a set of votes, one allocates seats to parties based on their proportionate strength at regional or national level while minimizing a particular objective function[1]. For some systems, such as FPTP, the minimization part of the problem is trivial; however, explicitly framing electoral systems in the language of optimization offers insight. In particular, we choose to interpret electoral systems as instances of network flow, taking care when translating continuous models to the integer problem underlying the allocation of MPs. (Pukelsheim et al., 2012) provides an excellent and very recent review of network models in this area for the interested reader. In our work, we look at a wide range of objective functions that we can attempt to optimize, chosen to promote criteria that seem reasonable for PR to achieve. If the proportional strengths are calculated at constituency level we simply recover FPTP. If proportionality at national level is too great a leap, our methods offer a halfway house that may prove more satisfactory to the general populace than AV.

For one particular choice of objective function, network flow can be viewed as a well known linear algebra problem, namely biproportional apportionment (BPA), which has been proposed as a tool to reform PR systems with multi-seat districts by many research groups such as BAZI[2]. BPA applies a global scaling, which means that each individual influences the result in every constituency. It was first proposed as a system for proportional representation in (Balinski and Demange, 1989) and has since been adopted successfully in a number of legislatures (Balinski and Pukelsheim, 2006; Maier et al., 2010), though only in multi-seat constituencies. In (Balinski, 2008),the author suggests that there is nothing to stop its implementation in elections in single-seat constituencies in the USA, although this also exploits the fact that only two parties would be involved. We believe that this paper is the first to look at BPA and network flow models in the UK context, and we investigate in detail the feasibility of BPA as a means of distributing seats in an election where each constituency has a single representative.

There are two stages to the allocation, whether we use BPA or network flow. First we use the national results to determine an appropriate proportional distribution of seats amongst parties. Next we must divide the constituencies amongst the parties to match this distribution. We observe that each of the stages can be implemented in many different ways. For example, in the first stage we could insist that the distribution of seats to parties matches FPTP or (up to rounding) precisely matches the proportion of votes won at the national level. One feature of our proposed allocation is that this choice can be tuned to lie anywhere between these two extremes to match the public appetite for proportionality.

[1] This objective function may only be present implicitly.

[2] `www.uni-augsburg.de/bazi`

The main flexibility in the second stage comes in the choice of objective function; however, we can also apply a hierarchical approach to try and deal sensitively with regional voting patterns. We will expound this approach in more detail in our analysis. We will show that broad classes of allocation methods work (in the sense of existence and uniqueness of solutions) under only the mildest of assumptions.

To test our models, we use the voting data from the May 2010 General Election (The Electoral Commission, 2013). Note that we excluded Northern Ireland from our electoral map, due to significant difference in constituency sizes and the popular parties in comparison with the rest of the UK. For simplicity and consistency, we also assigned the votes for the Speaker in Buckingham to the Conservative party and amalgamated Green party votes although they are different parties in different nations. Since we primarily aim to highlight the possibility of applying such a system and justify its use, we leave treating smaller parties and other implementation details to the electoral decision makers. Biproptional apportionment is solved in MATLAB, and all network optimization problems are implemented and solved using FICO Xpress 7.3. Finally, we note that the technical report version of this paper with more examples and high-quality pictures is available online (Akartunalı and Knight, 2012).

2 Models for Seat Allocation

Consider an election over the set of m constituencies (set denoted by I), contested by the set of n political parties (set denoted by J). Suppose that each party $j \in J$ gets a_{ij} votes in the constituency $i \in I$, and let x_{ij} indicate the number of seats allocated to party $j \in J$ in the constituency $i \in I$.

In the UK electoral system, each constituency is allocated exactly one seat, hence $x_{ij} \in \{0, 1\}$ for all i and j. The current allocation system of FPTP ensures that the winner in a constituency simply takes the seat. Our aim is to prescribe a fairer allocation of the x_{ij} incorporating overall votes regionally/nationally. Our proposal is to choose an objective function $f(x)$, that is minimized when some criteria based on fairness are met subject to certain constraints placed on x.

Let q_{ij} be the "fair seat allocation" in constituency i to party j. An obvious choice is

$$q_{ij} = \frac{a_{ij}}{\sum_{j' \in J} a_{ij'}}$$

This fair share is going to be part of the objective function, and is therefore crucial for the optimal allocation. Note that one can also define a normalized version of this, as follows:

$$\widehat{q}_{ij} = \frac{a_{ij}}{\max_{j' \in J} a_{ij'}}$$

$\widehat{q}_{ij}$ simply denotes the ratio of a particular party's vote to the highest vote of any party in the constituency, and there will be always a party j' in each constituency with $\widehat{q}_{ij'} = 1$. We also note the ranking of party j in constituency i, denoted by r_{ij}, is an alternative measure of fairness.

Based on these measures, we propose the following objective functions as targets for minimization to achieve a fair seat allocation. This list is by no means exhaustive.

1. $f_1(x) = \sum_{i \in I} \sum_{j \in J} (1 - q_{ij}) x_{ij}$

2. $f_2(x) = \sum_{i\in I} \sum_{j\in J} (1 - \widehat{q}_{ij}) x_{ij}$

3. $f_3(x) = \sum_{i\in I} \sum_{j\in J} (1/q_{ij}) x_{ij}$

4. $f_4(x) = \sum_{i\in I} \sum_{j\in J} (r_{ij} - 1) x_{ij}$

5. $f_5(x) = \sum_{i\in I} \sum_{j\in J} |x_{ij} - q_{ij}|$

6. $f_6(x) = \sum_{i\in I} \sum_{j\in J} |x_{ij} - \widehat{q}_{ij}|$

7. $f_7(x) = \max_{i\in I, j\in J} |x_{ij} - q_{ij}|$

8. $f_8(x) = \max_{i\in I, j\in J} |x_{ij} - \widehat{q}_{ij}|$

Note that the first four functions consider only the tuples (i, j) that are given a seat allocation at the end. On the other hand, the function 5 to 8 consider all tuples: these are ℓ_1 and ℓ_∞ norms, respectively. A significant observation in our context is that since all variables are binary, ℓ_2 is redundant. We also note that (Serafini and Simeone, 2012) discusses the ℓ_∞ case as presented here in $f_7(x)$. The advantage of $f_2(x)$ over $f_1(x)$ is that it considers zero penalty when the winner of a constituency is given the seat, which might be more preferable in some electoral settings due to its emphasis on the winner. The difference of $f_3(x)$ is that the penalties are anti-proportional to the amount of votes received, hence making a low-ranked party virtually impossible to win a seat, again a possible choice of electorates. In a similar fashion, $f_4(x)$ aims to avoid low-ranked parties to win seats, though it doesn't differentiate between amount of votes and only considers ranking. $f_7(x)$ and $f_8(x)$ are different from others in the sense that only the "extreme case" is considered, i.e., if electorate is simply sensitive about an extreme winner/loser, then these functions would be more appropriate to use. We also note that electorate might consider a number of these criteria and hence a multi-objective approach is the best approach to their needs. Finally, we note the recent paper of (Pukelsheim et al., 2012) as an excellent review of network modelling approaches for various electoral problems including seat allocation and political districting, and the work of (Gaffke and Pukelsheim, 2008*a*) and (Gaffke and Pukelsheim, 2008*b*) treating the fairness problem by convex integer optimization and duality to structure algorithms.

We also consider the objective function $f_9(x) = \sum_{i\in I} \sum_{j\in J} x_{ij}(-\ln(q_{ij}) - 1)$. This can be viewed as a measure of entropy and it is well known that solving the network flow problem with this objective function is equivalent to solving BPA (Lamond and Stewart, 1981; Rote and Zachariasen, 2007).

The usual choice of entropy measure is $\sum_{i\in I} \sum_{j\in J} x_{ij} \left(\ln \frac{x_{ij}}{a_{ij}} - 1\right)$. Since the x_{ij} can take only binary values in our model, this is equivalent to $f_9(x)$ (the scaling by $\sum_{j'\in J} a_{ij'}$ in the definition of q_{ij} makes no difference).

Having chosen an objective function to minimise we must then determine our constraints. Obviously each constituency must be assigned to one party. We insist that a seat can only be assigned to a party that has a candidate standing there. We also need to fix the number of seats each part should be awarded. As with the objective function, we have a number of choices depending on what we consider to be fair.

The simplest idea is to allocate s_j seats to party j so that

$$\frac{s_j}{S} \approx \frac{\sum_{i\in I} a_{ij}}{\sum_{i\in I} \sum_{j'\in J} a_{ij'}} \tag{1}$$

where S is the total number of seats (i.e., $\sum_{j \in J} s_j$). We can define the (probably) fractional seat allocation to the party j as:

$$s_j = \frac{\sum_{i \in I} a_{ij}}{\sum_{i \in I} \sum_{j' \in J} a_{ij'}} S.$$

Another alternative to this measure is that we can define it based on constituencies, as follows (since each constituency has a single seat):

$$s_j = \sum_{i \in I} q_{ij}.$$

Aiming for such a level of proportionality in seat allocation inevitably leads to a radical upheaval from FPTP and one has to balance the apparent fairness of the s_j calculated on national levels of support against the fairness of imposing an MP who has little local support: with $m = 632$, as in the UK, a party would be awarded a constituency even if it only wins around 0.2% of the vote nationally and this low level of popularity may also be reflected locally. Thus it may be desirable to manipulate the vector s before allocating constituencies to to respect local trends. In our experiments we show a simple way of computing the s_j that goes some way towards this goal. We note that it is common for electoral systems to impose minimum levels of popular support before representation is permitted. The precise method for calculating is a choice to be made by policy makers (and indirectly by the public). We note that however we choose a "fair" seat allocation, our methods will still produce an apportionment; and that this method can be changed incrementally to suit the public's appetite.

In addition to how to define s values, another important aspect is the fractionality of these seat allocations. In practice, a common way to handle fractional s_j values is to round them to the nearest integer according to the *largest remainder rule*, i.e., round down all s_j values first and then round up the remaining fractional parts from the largest to the smallest fraction, until $\sum_{j \in J} s_j = S$. We will refer to this rounding with the notation $|\bullet|_{LRR}$. Alternatively, rather than constraining s_j to a specific value, we can also accept allocations where the seats awarded to party j lies in an interval $[\underline{s}_j, \overline{s}_j]$. A particularly simple example is to choose $\underline{s}_j = \lfloor s_j \rfloor$ and $\overline{s}_j = \lceil s_j \rceil$. In electoral settings a commonly used alternative to the largest remainder rule is to employ the d'Hondt method (Balinski and Young, 1978), also known as the Jefferson method. In the d'Hondt method, the votes cast for each party are divided by $1, 2, \ldots, m$ and the values are tabulated. A seat is then assigned to each party for each of the m largest such values.

Given $f(x)$ and s our network optimization problem with integer variables is as follows:

$$\begin{aligned}
\min \quad & f(x) & & \qquad (2)\\
\text{s.t.} \quad & \sum_{j \in J} x_{ij} = 1 & i \in I & \qquad (3)\\
& \underline{s}_j \le \sum_{i \in I} x_{ij} \le \overline{s}_j & j \in J & \qquad (4)\\
& x_{ij} \in \{0, 1\} & i \in I, j \in J & \qquad (5)
\end{aligned}$$

Note that in this basic form, the problem is simply an assignment problem and it has the integer solution property when the constraints (5) are relaxed, under general assumptions such as that the objective function is linear or convex (Ahuja et al., 1993). Furthermore, this problem has the advantage that variables are limited to values between 0 and 1. We will discuss these aspects further in the coming sections.

In the next section we will describe BPA and look at the solutions for various choices of s before returning to the general network flow problem. The equivalence of the two problems means that the insights we gain in studying BPA inform our understanding of the network flow formulation and vice versa. In particular, existence and uniqueness results can be understood more clearly by looking at the two different facets of the same problem. Note that in total, we consider $m = 632$ constituencies, and present details we have used for parties in the appendix.

3 Biproportional Apportionment

Before applying BPA to the General Election data we consider the general problem. Suppose $A \in \mathbb{R}^{m\times n}$, $t \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$ are all nonnegative and that $\|t\|_1 = \|s\|_1$. The problem of BPA is to find diagonal matrices D_1 and D_2 (whose diagonals are positive) such that the ith row sum of $X = D_1AD_2$ is t_i and the jth column sum of X is s_j. The problem has many applications (including interpreting economic data (Bacharach, 1970), understanding traffic circulation (Lamond and Stewart, 1981), matching protein samples (Daszykowski et al., 2009) and ordering nodes in a graph (Knight, 2008)), particularly when A is square and X is doubly stochastic. Existence and uniqueness of solutions is well understood (Brualdi, 1968) and relates to the nonzero pattern of A. We can calculate X by a very straightforward iterative process: given starting vectors $r_1 \in \mathbb{R}^m$ and $c_1 \in \mathbb{R}^n$ we form the sequence of vectors

$$r_{k+1} = \frac{t}{Ac_k}, \quad c_{k+1} = \frac{s}{A^T r_{k+1}}, \tag{6}$$

where the division of vectors is applied componentwise.[3] If a solution exists, then the iterates converge linearly: $\mathrm{diag}(r_k) \to D_1$, $\mathrm{diag}(c_k) \to D_2$. It is usual to set all of the elements of r_1 and c_1 to 1. We will do so, too, and we use e to denote a vector of ones (whose dimension should be clear from context).

In many cases, such as in the electoral setting, the entries of t, s and X (not necessarily those of A) must be integers. In this case the BPA problem is to find D_1 and D_2 so that when we round the entries of D_1AD_2 we form X. Simply rounding the continuous solution to BPA is rarely the answer and in the integer case (the unrounded) D_1AD_2 is usually a long way from the continuous solution.

Algorithms for computing X in the integer case generally start by applying a number of steps of (6). In our experiments, we have found that only one or two such steps are needed. Notice that since the entries of r_2 are simply the reciprocals of the row sums of A,

$$\mathrm{diag}(r_2)A\,\mathrm{diag}(c_1) = \mathrm{diag}(r_2)A = Q,$$

explaining why $f_9(x)$ is equivalent to the usual formulation of the entropy measure for BPA. (Maier et al., 2010) describe a number of algorithms for the integer problem including a discrete version of (6) based on alternating iterative vector apportionment of the multipliers to achieve the desired row and column sums. We have adapted their algorithm to take advantage of the fact that in our case $x_{ij} \in \{0, 1\}$ to remove the guesswork needed to find initial estimates of the multipliers at each step. The algorithm can be seen in (Akartunalı and Knight, 2012). At each step we calculate a range of multipliers which give a rounding to the desired sum, from which we take the midpoint. Typically, we need no more than 50 steps of the integer algorithm to find the solution. This can be reduced with a more aggressive choice of multiplier, at an endpoint of the interval, though at the

[3] Note that in many places, the algorithm is expressed in terms of a sequence of matrices $A_0 = A, A_1, A_2, \ldots$ where $A_k = \mathrm{diag}(r_k)A\mathrm{diag}(c_k)$.

risk of failure as it has a tendency to create ties in rounding (a phenomenon we have never seen with midpoint multipliers).

The question of how to round has been discussed by a number of authors. However, we use the standard rounding rule, assuming that the computations have been performed in binary floating point arithmetic.[4]

Conditions for existence of an apportionment are given by (Balinski and Demange, 1989), where the authors consider the more general problem of finding apportionments satisfying inequality constraints. When we want equality, as we do in our case, the conditions are almost exactly the same as those established by (Brualdi, 1968) when rounding is not used, and are stated below. The difference is simply that a strong inequality becomes weak. We make use of the following definition.

Definition 3.1 *The sets $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$ are a* **reducible partition** *of $A \in \mathbb{R}^{m \times n}$ if $a_{ij} = 0$ for all $i \in I$ and $j \notin J$.*

Note that we can use a reducible partition to induce a permutation of A of the form

$$\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}$$

where $A_1 = A(I, J)$. $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ forms a reducible partition for any A.

Theorem 3.1 *(Balinski and Demange, 1989) Suppose $A \in \mathbb{R}^{m \times n}$ is a nonnegative matrix, $t \in \mathbb{N}^m$ and $s \in \mathbb{N}^n$. Then there exist nonnegative diagonal matrices D_1 and D_2 such that if $X = round(D_1 A D_2)$ then $Xe = t$ and $X^T e = s$ if and only if*

$$\sum_{j \in J} s_j \geq \sum_{i \in I} t_i \tag{7}$$

for any reducible partition of A (with equality if $a_{ij} = 0$ for all $i \notin I$ and $j \in J$).

In terms of single seat constituencies, the consequence of this theorem is that any reasonable choice of s will do.

Corollary 3.1 *So long as no party is awarded more seats than it has candidates who win votes, then an apportionment exists for an election for m single member constituencies contested by n parties for any allocation of seats $s \in \mathbb{Z}_+^n$ such that $\sum_i s_i = m$.*

Proof. Clearly, a total of m seats must be allocated to parties. Now suppose a reducible partition, (I, J) exists that allows a permutation of the matrix of votes into the form

$$\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}$$

where $A_1 \in \mathbb{R}^{k \times l}$. Since parties cannot be awarded seats where they did not receive votes, $\sum_{j \notin J} s_j \leq m - k$ so

$$\sum_{j \in J} s_j \geq k = \sum_{i \in I} t_i$$

(as $t = e$). If $A_2 = 0$ then $\sum_{j \in J} s_j \leq k$, thus (7) becomes an equality. □

While existence criteria can be unambiguously stated, uniqueness is not necessarily guaranteed. In particular, in any election there is the possibility of ties: consider how any system, whether FPTP or proportional, fairly allocates seats in an election where all parties earn the same number of votes in every constituency. (Maier et al., 2010) analysed realistic data for districts with multiple representatives and found no instances of non-uniqueness. We revisit uniqueness in the context of network flow later in the paper and find that for single seat constituencies a judicious choice of objective function seems to prevent a threat of multiple solutions.

[4]Namely "round to nearest, ties to even" as prescribed by IEEE 754.

4 Biproportional Apportionment in A General Election

We have applied BPA to the May 2010 General Election data. To determine a fair share of seats we simply summed the total votes for each party/independent nationally and applied the d'Hondt method. Roughly speaking, a seat is awarded by this method for every 45,000 votes won. No minimum threshold of national support was stipulated at this stage and of the roughly 130 parties and 300 independent candidates who participated in the election, a total of nine parties won enough votes to be assigned seats, hence we set $n = 9$. Table 1 compares the seat assignments when the d'Hondt method is applied nationally against the actual FPTP allocation. These give two possible sets of candidate values for s while t is simply a vector of ones.

	Con	Lab	LD	UK	SNP	BNP	G	PC	ED
FPTP	307	258	57	0	6	0	1	3	0
D'Hondt	238	191	151	20	10	12	6	3	1

Table 1: FPTP and D'Hondt Allocation of Parliamentary seats.

With the choices of m and n as described, we can then form the matrix A (of a_{ij} values) sized 632×9. Note that with the allocation methods we have used, incorporating the votes and seat allocations of unrepresented parties in A and s would make no difference to our results, simply resulting in additional zero columns in X. The results of applying BPA with s prescribed by the d'Hondt values is compared with the actual result in Figure 1. Each constituency is coloured according to the party awarded the seat (see the appendix for the colour coding).

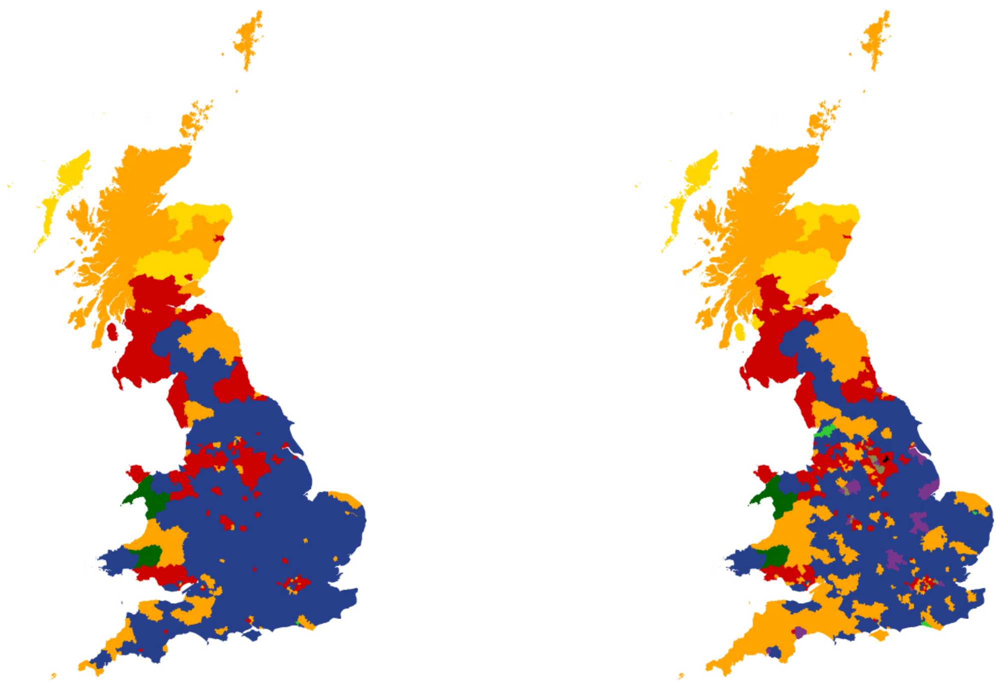

Figure 1: Allocation of constituencies according to FPTP (left) and BPA (right).

The effects can be characterised as taking the excess seats of the two main parties and redistributing them among the smaller parties (in fact, the Conservatives gain two seats from Labour—Southampton and Bolton West, both highly marginal—and the Liberal Democrats lose Norwich

South to the Greens.) Table 2 quantifies the number of seats assigned to each party in terms of their ranking in constituency votes (for example, 83 of the BPA Liberal Democrats were runners up according to FPTP). The sum of each column in Table 2 matches the d'Hondt values in Table 1, as intended.

	Con	Lab	LD	UK	SNP	BNP	G	PC	ED	TOTAL
1st	236	191	56		6		1	3		493
2nd	2		83	1	4					90
3rd			12			2				14
4th				18		10	5		1	34
5th				1						1

Table 2: FPTP Rank of BPA.

While nearly 80% of constituencies retain the same MP as with FPTP, the fact that the candidate who comes fourth or fifth can become the MP may not sit well with some voters: the result of the AV referendum in 2011 suggests an overwhelming resistance from the populace to PR in Westminster elections, and such a radical realignment of parties may suggest that BPA is unpalatable. However, BPA gives us a freedom that other methods of proportional representation, such as STV and AV, do not have when applied to single seat constituencies: we can tune s so its redistributive nature matches the public appetite.

We first note that FPTP is itself a biproportional apportionment for a particular choice of s.

Theorem 4.1 *Suppose BPA is used in an election with single seat constituencies where the party allocation vector, s, uses the FPTP results and that there are no ties for first place in any constituency. Then the resulting allocation exactly matches that of FPTP.*

Proof. Suppose A is the matrix of votes and let $D_1 = \text{diag}(r)$ where $1/r_i = 2\max_j a_{ij}$. Then precisely one entry in each row of $X = \text{round}(D_1 A)$ equals one: the entry corresponding to the largest value in row i of A. Thus a BPA matching FPTP exists.

Furthermore, in this case the BPA is unique. For suppose there existed diagonal matrices R and C such that $Y = \text{round}(RAC)$ satisfied the marginals provided by FPTP and $X \neq Y$. Since $Xe = Ye$ and the entries of X and Y are binary, there must exist sets of indices I and J of equal length (k, say) such that

$$x_{i_k j_k} = 1, \quad x_{i_k j_{k+1}} = 0, \quad y_{i_k j_k} = 0, \quad y_{i_k j_{k+1}} = 1,$$

where $j_{k+1} = j_1$. Since we know that $a_{i_k j_k}$ is the largest element in row i_k we end up with the sequence of inequalities amongst the column scalings

$$c_{j_1} < c_{j_2} < \cdots < c_{j_{k+1}} = c_{j_1},$$

hence no such Y exists. $\square$

Suppose d is the vector of party assignments according to a d'Hondt apportionment and f is that given by FPTP. Let $0 \leq \alpha \leq 1$; then we can mitigate the effects of our original model of PR by choosing

$$s(\alpha) = \text{round}(\alpha f + (1-\alpha)d)$$

where we choose a rounding that ensures that $\|s(\alpha)\|_1$ matches the number of seats being contested.

α	0	0.25	0.5	0.75	1
1st	493	526	563	595	632
2nd	90	73	48	28	0
3rd	14	8	6	3	0
4th	34	24	14	6	0
5th	1	1	1	0	0

Table 3: Variance of FPTP Rank with α.

An illustration of allocations with $\alpha = 0.75, 0.5, 0.25$ can be found in (Akartunalı and Knight, 2012). One measure of the effect of changing s is given in Table 3 where we indicate the total number of seats won according to FPTP rank for a range of values for α. The low ranking of some allocations is exclusively due to the need to assign constituencies to smaller parties.

Note that if we choose $\alpha > 0.5$ then, subject to resolving ties in rounding favourably, any party that wins a seat through FPTP will win a seat through BPA. Thus we can guarantee that any constituency election that is dominated by local issues (sleaze and the need to elect a Speaker are two of the diverse examples from recent General Elections) will not be swamped by the national mood.

4.1 Multilevel Biproportional Apportionment

Comparing the results of BPA and FPTP, one can see that large areas of the country are unaffected by the reallocation. In particular, the Conservative and Labour parties remain tightly wedded to their traditional heartlands: Gower, Normanton and Makerfield remain Labour seats under BPA. In essence, BPA finds that the simplest way to deal with the iniquities of FPTP is to remove the surplus seats. However, this means that regional imbalances remain: Scotland still has only a single Conservative MP and the South and East of England are almost Labour-free outside London; both factors that run counter to proportionality. Frustrated constituents can console themselves that their vote has made a difference somewhere in the country, but this effect is rather intangible.

To counter this, one can incorporate a hierarchy of seats into the allocation. At the bottom level of the hierarchy (level N) one has the individual constituencies and at the top (level 0) is the country as a whole. The lth level is a partition of all the constituencies into p_l sets $S_1^{(l)}$, $S_2^{(l)} \ldots, S_{p_l}^{(l)}$, such that if two seats belong to the same set in level l then they are in the same sets at levels $0, 1, \ldots, l-1$. An allocation is then made down the levels until every constituency is assigned an MP.

One can use BPA at every level: target column sums coming from the party allocations at the next higher level and row allocations from the sizes of the partition of a set at the next lower level. However, for expedience and simplicity, we suggest another approach.

We propose to add a single extra layer to the hierarchy which contains eleven regions commonly used in electoral maps. These are listed (along with the number of seats in each region) in an appendix. We first apportion seats to parties in each of the regions using the d'Hondt method, pooling the votes from the constituencies. Next we apply BPA to each of the regions in turn. The results with the May 2010 election data are shown in Table 4.

Notice that we have lost the strict proportionality at a national level that BPA guarantees. Due to rounding effects, the d'Hondt method is biased against the smaller parties, although this bias

	Con	Lab	LD	UK	SNP	BNP	G	PC
1st	213	171	53		6		1	3
2nd	25	30	83	1	6			1
3rd	1		15			1		
4th				12		7	1	
5th				2				
	239	201	151	15	12	8	2	4

Table 4: Allocation of constituencies according to regional BPA.

is slight when compared to FPTP. However, as well as enforcing a regional spread of some of the parties we have reduced the number of seats being handed to poorly supported candidates. This property could be enhanced by adding extra levels to the hierarchy, or by simply working with a larger number of sets at level 1. Our hierarchical approach ensures that the conditions of Theorem 3.1 hold at every level, so we can be sure an allocation exists.

Of course, the two variants of BPA we have described can be combined; and one can even add additional constraints (for example, a minimum threshold that parties must achieve locally/nationally to be awarded seats). The main aim of this paper is to show the viability of BPA and we fear that looking at ever more intricate allocation methods will obfuscate this aim. One benefit of using BPA, however it is implemented, is that once the scaling factors r and c are calculated it is straightforward for anyone to validate the results by confirming that the entries of X are the correct scalings of A.

5 Properties of Different Objective Functions

To gain additional insight into the process, particularly with respect to uniqueness of apportionments, we return to the general problem of apportionment through network flow. We first observe that the choice of objective function is critical. In Figure 2 we show the apportionments when we solve equations (2)–(5) with different objectives. We have computed the s_j using (1) and used $\underline{s}_j = \lfloor s_j \rfloor$ and $\overline{s}_j = \lceil s_j \rceil$ in (4). This results in a small change from the party allocations given by d'Hondt.

While the first three objective functions give roughly similar results, comparable with BPA, the ℓ_∞-based measure $f_7(x)$ completely transforms the picture. To understand the connections between objective functions we first give a couple of simple results.

Corollary 5.1 $f_5(x) \equiv f_1(x)$.

This follows from the fact that we can pick only one party (say j') in each constituency i, i.e., $x_{ij'} = 1$, and hence the objective function's value for i is simply $2(1 - q_{ij'})$ (since $\sum_{j \in J, j \neq j'} q_{ij} = 1 - q_{ij'}$). Therefore, $f_5(x) = \sum_{i \in I} \sum_{j \in J} 2(1 - q_{ij}) x_{ij}$.

Corollary 5.2 $f_6(x) \equiv f_2(x)$.

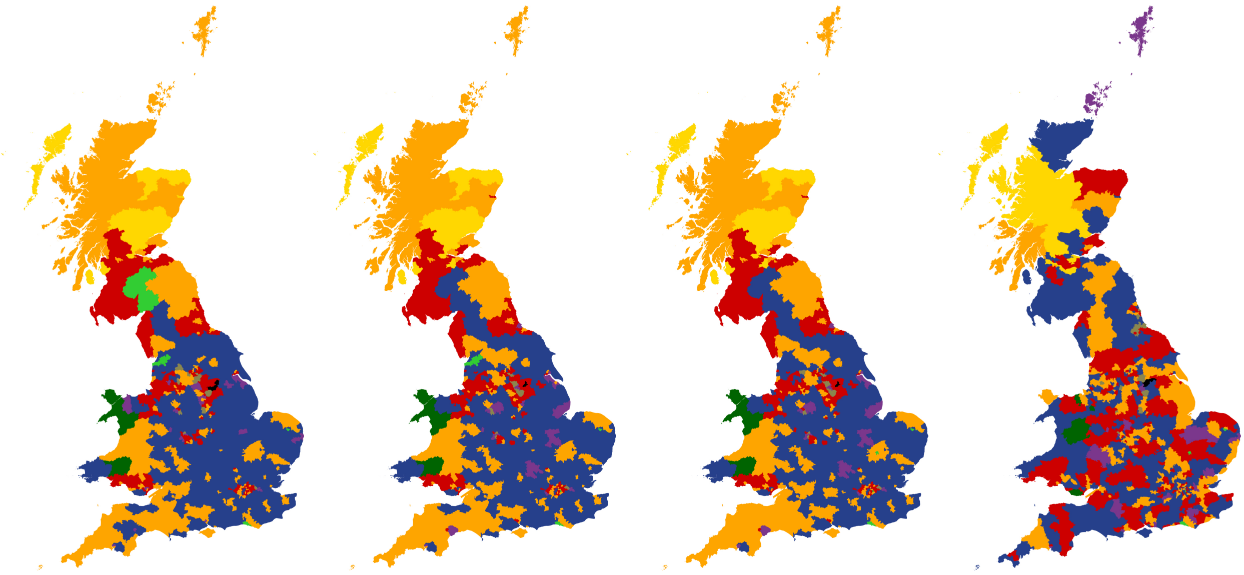

Figure 2: Allocation of constituencies according to (from left to right) $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_7(x)$.

This follows from the fact that when we pick one party (say j') in constituency i, i.e., $x_{ij'} = 1$, then the objective function's value for i is: $(1 - \widehat{q}_{ij'}) + \sum_{j \in J, j \neq j'} \widehat{q}_{ij}$. Therefore, $f_6(x) = \sum_{i \in I} \sum_{j \in J} (1 + \sum_{j' \in J} \widehat{q}_{ij'} - 2\widehat{q}_{ij}) x_{ij}$, where $1 + \sum_{j' \in J} \widehat{q}_{ij'}$ is simply a constant.

Any seat allocation system should produce a unique solution for a given election, and this uniqueness property is even more significant than the fairness. Next, we will present some simple numerical examples to discuss solution uniqueness of the objective functions presented in previous section. Recall that each row of a vote matrix represents a constituency whereas each column represents a party. For simplicity, we will assume fair seat allocation to a party follows the largest remainder rule.

Example 1. Suppose votes for an election with 3 constituencies and 3 parties as in V_1.

$$V_1 = \begin{bmatrix} 5 & 1 & 4 \\ 1 & 5 & 4 \\ 5 & 2 & 3 \end{bmatrix} \qquad X_{1,1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad X_{1,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By the largest remainder rule, fair seat allocation dictates that all parties earn a seat, where the first and third party got each 11 votes total, and the second party got 8 votes. The objective functions $f_4(x)$ and $f_8(x)$ will reach multiple solutions as presented in $X_{1,1}$ and $X_{1,2}$, with $f_4(x) = 1$ and $f_8(x) = 1$. The functions $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_7(x)$ and $f_9(x)$ all have a unique solution given by $X_{1,1}$. The optimal objective function values are $f_1(x) = \frac{8}{5}$, $f_2(x) = \frac{1}{5}$, $f_3(x) = 6.5$, $f_7(x) = \frac{3}{5}$ and $f_9(x) = \ln(10) - 3$. We note that the two previous corollaries imply that $f_5(x)$ and $f_6(x)$ also have unique solutions; we omit this trivial result here and in the following discussion. □

We note that the objective function $f_8(x)$ has more than 2 solutions, since any solution x satisfying the row and column equations also satisfies $f_8(x) = 1$ for this problem. This is the key weakness of this function, as it loses sensitivity whenever a winner in a constituency is not given a seat, making the objective function equal 1 and the rest of the problem becomes irrelevant. This insensitivity is natural for ℓ_∞ (or "minimax") solutions, as also pointed out by (Serafini and Simeone, 2012) for $f_7(x)$ (though $f_7(x)$ is much more successful at generating unique solutions than $f_8(x)$, as we will discuss in the upcoming examples). This uniqueness problem can be dealt with by using strongly optimal solutions and unordered lexico minima, and we refer the interested reader to (Serafini and Simeone, 2012) for details.

Example 2. Consider an election with 2 constituencies and 3 parties, with votes presented as in V_2 (first two parties deserving one seat each):

$$V_2 = \begin{bmatrix} 9 & 8 & 1 \\ 9 & 8 & 0 \end{bmatrix} \qquad X_{2,1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad X_{2,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

For objective functions $f_2(x)$, $f_4(x)$, $f_8(x)$ and $f_9(x)$, the optimal seat allocation is not unique, obtainable with $X_{2,1}$ and $X_{2,2}$. On the other hand, the objective function $f_1(x)$ has a unique optimal seat allocation as given by $X_{2,1}$, and $f_3(x)$ and $f_7(x)$ have a unique optimal seat allocation given by $X_{2,2}$. $\square$

This example raises the question of "which objective function provides a better/fairer seat allocation", as they do not necessarily provide the same allocation even when they generate a unique seat allocation. This, in turn, gives a decision maker different options to choose from, e.g., a society can have a different perspective on fairness in this context and hence choose their preferred objective function.

Example 3. Assume votes for an election with 3 constituencies and 3 parties is given by V_3:

$$V_3 = \begin{bmatrix} 60 & 1 & 39 \\ 41 & 40 & 19 \\ 1 & 60 & 39 \end{bmatrix} \qquad X_{3,1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad X_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For objective functions $f_7(x)$ and $f_8(x)$, the optimal seat allocation is achieved with both $X_{3,1}$ and $X_{3,2}$. On the other hand, the objective functions $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$ and $f_9(x)$ have a unique optimal seat allocation, given by the matrix $X_{3,1}$. $\square$

Example 4. We consider an election with 3 constituencies and 4 parties, with votes stated in V_4.

$$V_4 = \begin{bmatrix} 6 & 4 & 5 & 0 \\ 6 & 4 & 2 & 3 \\ 0 & 4 & 0 & 11 \end{bmatrix} \qquad X_{4,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad X_{4,2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For all objective functions except $f_4(x)$, the optimal seat allocation is not unique and can be obtained with both of the solutions stated in $X_{4,1}$ and $X_{4,2}$. On the other hand, the function $f_4(x)$ has a unique optimal allocation given by $X_{4,1}$. $\square$

Example 5. Consider the votes V_5 for an election with 2 constituencies and 3 parties:

$$V_5 = \begin{bmatrix} 3 & 4 & 3 \\ 4 & 5 & 1 \end{bmatrix} \qquad X_{5,1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad X_{5,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

For objective functions $f_1(x)$, $f_4(x)$ and $f_8(x)$, the optimal seat allocation can be obtained with either $X_{5,1}$ or $X_{5,2}$. On the other hand, for objective functions $f_2(x)$, $f_3(x)$, $f_7(x)$ and $f_9(x)$, the optimal seat allocation is unique, given by $X_{5,1}$. $\square$

Note that we present more simple examples with different scenarios in (Akartunalı and Knight, 2012). As these examples indicate, different objective functions generate unique results in different cases, and none of these objective functions seem in particular superior to the others in this aspect, although it is clear that $f_8(x)$ consistently generates multiple solutions. Similarly, $f_4(x)$ generates often multiple solutions, although it is not very clear from these small examples: in particular, one can create an instance such as Example 4 to argue its usefulness, although a matrix with less rows than columns is unrealistic in an election setting (we will discuss this in more realistic cases in computational results). From a social point of view, one can easily argue that each of these objective functions has its own merits and use of them in combination could provide the "fairest" seat allocation. For example, in case of Example 7 (though unrealistic), both solutions seem at first look as good as each other, but having a thought on number of actual winners in a constituency is a strong argument in favor of the objective function $f_4(x)$ and hence the unique solution it provides. Finally, note that an electoral system might combine a number of these criteria in a multi-objective approach.

The only theoretical uniqueness result we are aware of stems from the *max algebra* literature, as discussed in detail in (Burkard and Butkoviç, 2003) and (Burkard et al., 2009). The uniqueness of the linear assignment problem with a cost matrix $A \in \mathbb{R}^{n \times n}$ is proven to be equivalent to the matrix A being *strongly regular* (or the max algebraic system $A \bigotimes x = b$ has a unique solution). However, this result is limited to square matrices only, and therefore offers virtually no applicability for a general election setting as stated in our problem. We are not aware of any other uniqueness result in a general setting, however we note this as a possibility for extension in the future.

To gain additional insight into uniqueness, we tested the different objective functions presented using the UK election setting (excluding the Northern Ireland for reasons previously mentioned). We used FICO Xpress 7.3 to implement and solve the network optimization problems. First, we generated 1,000 random election results (with [0.1,0.3] of votes v_{ij} being zero, to be comparable with the last election results), and optimized each of the objective functions (except $f_5(x)$ and $f_6(x)$ due to equivalence result presented before). After the optimal solution x^* is found, we add the following cover cut (see e.g. (Nemhauser and Wolsey, 1999)) before re-solving:

$$\sum_{\substack{i \in I, j \in J \\ \text{s.t. } x^*_{ij}=1}} x_{ij} \leq |I| - 1$$

This will ensure that the first found solution is eliminated from the solution space and hence a different solution will be found, whether with the same optimal value or not, hence showing us uniqueness of the solution x^*. From 1,000 instances, the objective functions $f_4(x)$ and $f_8(x)$ had multiple optimal solutions for each of the 1,000 instances, whereas $f_7(x)$ achieved a unique optimal solution for 19 of the instances but failed to do so for the remaining 981 instances. On the other hand, the objective functions $f_1(x)$ and $f_2(x)$ had a unique optimal solution for each of the 1,000 instances, whereas $f_3(x)$ and $f_9(x)$ failed to do so only for one instance each (not for the same instance, though). Details are presented in the first row of the Table 5.

Another interesting aspect was how different objective functions would handle s_j differentiation, i.e., given election results, the effects of varying s_j values (not necessarily perfected values such as using largest remainder rule but *any* values) and also the effects of alternative $(\underline{s}_j, \overline{s}_j)$ values (fixed as $\underline{s}_j = \overline{s}_j = |s_j|_{LRR}$, or in interval of $\underline{s}_j = \lfloor s_j \rfloor$ and $\overline{s}_j = \lceil s_j \rceil$). Using the last UK election results, we generated 1,000 random fair seat allocations to parties, ensuring not to violate the total number of arcs for each party so that the problems are feasible. As the results in Table 5 indicate, the objective functions $f_4(x)$ and $f_8(x)$, in line with the previous results, had multiple

	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_7(x)$	$f_8(x)$	$f_9(x)$
Differentiation of A	1,000	1,000	999	0	19	0	999
$(\underline{s}_j, \overline{s}_j) = (\lvert s_j\rvert_{LRR}, \lvert s_j\rvert_{LRR})$	1,000	1,000	1,000	0	363	0	1,000
$(\underline{s}_j, \overline{s}_j) = (\lfloor s_j \rfloor, \lceil s_j \rceil)$	1,000	999	1,000	0	212	0	1,000

Table 5: Number of unique solutions for 1,000 instances with different votes (first row) and with s_j differentiation (second and third rows)

optimal seat allocations for all cases. Although using $(\lfloor s_j \rfloor, \lceil s_j \rceil)$ increases the dimension of the problem by $|J|$ and hence solution space increases and theoretically one would expect more solutions and less uniqueness, the effect of this has been very minimal for most of the objective functions: There was only one instance out of 1,000 and only for $f_2(x)$ that resulted in multiple optimal solutions. However, $f_7(x)$ presents the more interesting case here (again, similar to previous tests) that observation of uniqueness significantly decreases with this dimension increase. Therefore, the uniqueness is in general more dependent on the matrix of votes as well as the objective function used, whereas for $f_7(x)$, the s_j differentiation also affects it fairly.

Example 6. Consider an election with 3 constituencies and 3 parties, votes as in V_6.

$$V_6 = \begin{bmatrix} 5 & 1 & 4 \\ 1 & 5 & 4 \\ 5 & 1 & 4 \end{bmatrix} \qquad X_{6,1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad X_{6,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All parties earn a seat, where the first party got 11 votes, the second party got 7 votes, and the third party got 12 votes. Note that for this particular example, no objective function provides a unique solution; all problems have $X_{6,1}$ and $X_{6,2}$ optimal. Note that this particular instance does not have a unique solution when biproportional apportionnment is used either, hence it indicates a very special case of tie break that needs to be handled specially when apparent. □

One important note to make here is that these examples are quite artificial in a real world election setting, where tie breaks would occur much less frequently. However, it is still obvious that the objective functions $f_4(x)$ and $f_8(x)$ are more prone to multiple solutions and hence less practical in an election setting. Finally, we refer to (Ahuja et al., 1993) for sensitivity analysis of networks, as this is also an interesting aspect regarding different levels of votes.

6 Conclusions

In this paper we have introduced a number of tuneable methods of proportional representation appropriate to single seat constituencies. We have shown that existence of allocations is guaranteed for any vote metric. Uniqueness is not always guaranteed: as with any other voting system, a tie breaking system must be employed when two parties get matching vote numbers. However, our simulations using realistic data show that for certain choices of objective function such ties are (almost) nonexistent. Single seat constituencies prove not to be an unsurmountable challenge for network flow models and the binary nature of some of the variables makes the properties of some objective functions more amenable to analysis.

Our various proposed voting systems offer a continuum between pure proportional representation at a global level through to FPTP. Indeed, if we constrain our party allocations to match those of FPTP, our methods reproduce the constituency allocations exactly.

At the same time as trying to introduce a degree of fairness in the sense of proportionality, any electoral system should be simple to explain, to implement, and to validate. These four criteria (and we could add more) are tricky to satisfy simultaneously. In particular, we admit that our models may fail some of the simplicity tests; however we feel that our focus on fairness outweighs any perceived limitations.

There is still an issue with how to deal with smaller parties. If the fourth or fifth ranked party is handed a constituency, it is likely to prove unpopular with the local electorate. We have shown how to mitigate this to some extent by manipulating the number of seats allocated to each party, or by allocating seats at a regional level. Moreover, our methods ensure that seats are only ever allocated to candidates in the place where they stand. However, it may be desirable to impose minimum thresholds on the number of votes a party must receive (at regional or national level) before they can be awarded seats. This is, of course, a common component of current PR systems worldwide.

In our work, we used the voting data from the May 2010 General Election. Naturally, this does not allow us to pick up any changes in voting patterns that a new system would produce. This is left for future research, where for example game theoretic approaches might address such interesting issues.

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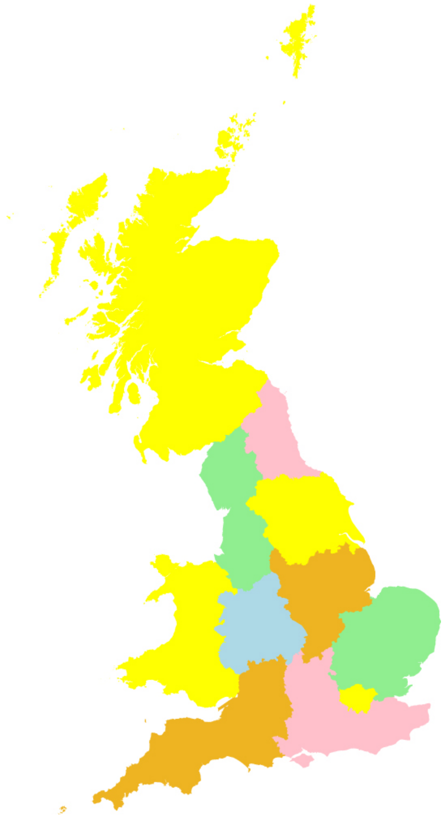
A Electoral Regions and Parties

Region	Number of seats
Scotland	59
Wales	40
South West	55
London	73
South East	84
East	58
West Midlands	59
East Midlands	46
Yorkshire and Humberside	54
North East	29
North West	75



Party	Code	Colour
Conservative	Con	blue
Labour	Lab	red
Liberal Democrat	LD	orange
UK Independence Party	UK	purple
Scottish National Party	SNP	yellow
British National Party	BNP	khaki
Green Party	G	light green
Plaid Cymru	PC	dark green
English Democrat	ED	black