Appendix 11.1 Conditions for intertemporal efficiency and optimality

Appendix 4.1 considered the conditions for efficiency and optimality in a timeless economy. To do that it explicitly analysed an economy in which two individuals each consumed two commodities, each of which was produced by two firms, each using two kinds of input, capital and labour. We noted that having just two of everything simplified the analysis without any loss in regard to the essentials. In the same spirit, here we consider just two periods of time, which we label 0 and 1. Considering the Appendix 4.1 economy for just two periods would still produce a model with lots of variables and symbols. To keep things as simple as possible, while not overlooking anything essential, we will have each commodity produced by just one firm, and we will assume that the only input to production is capital. As will be seen, the result is still a fairly complicated, or at least cluttered, model, and in much of the literature intertemporal analysis works with models that involve aggregations of various kinds. We will look at some of the widely employed specialisations of the basic model at the end of this appendix. It is worth looking at the general model, as doing so makes clear the implicit assumptions in the aggregated models.

You may find it useful to take a look again at the appendices to Chapter 4 before working through this appendix and the next, so as to refresh your memory in regard to notation etc.

A11.1.1 Intertemporal rates of substitution and transformation

There are two individuals A and B, each of whom consumes two commodities *X* and *Y*, in each of two periods 0 and 1. Their utility functions are

$$U^{A} = U^{A}(X^{A}_{0}, X^{A}_{1}, Y^{A}_{0}, Y^{A}_{1}) \qquad \text{and}$$

$$U^{B} = U^{B}(X^{B}_{0}, X^{B}_{1}, Y^{B}_{0}, Y^{B}_{1})$$

where a superscript identifies an individual and a subscript a period. For derivatives, the notation to be used here is an extension of that used in Appendix 4.1 so as to indicate period. Thus, for example, we write U^{A}_{X0} for $\partial U^{A}/\partial X^{A}_{0}$, A's marginal utility with respect to the consumption of *X* in period 0.

We can now define intratemporal – within period – and intertemporal – across period – marginal rates of utility substitution. Thus, for examples, A's intra-temporal MRUS with respect to *X* and *Y* in period 0 is

$$\text{MRUS}^{A}_{X0,Y0} \equiv \frac{dY^{A}_{0}}{dY^{A}_{0}}$$

for $\mathrm{d}X^{A}_{1} = \mathrm{d}Y^{A}_{1} = 0$, while A's intertemporal MRUS with respect to *X* in period 0 and *X* in period 1 is

$$\text{MRUS}^{A}_{X0,X1} \equiv \frac{dY^{A}_{1}}{dY^{A}_{0}}$$

for $dY_0^A = dY_1^A = 0$. These MRUSs have the same interpretation as previously. Thus, for example, $\text{MRUS}^A_{X0,X1}$ is the slope of an indifference curve in X_1^A/X_0^A space, multiplied by –1 so that MRUS is a positive number. Considering

$$dU^A = U^A_{X0}\,dX_0^A + U^A_{X1}\,dX_1^A = 0$$

for example, leads to

$$\text{MRUS}^A_{X0,X1} \equiv \frac{dY_1^A}{dY_0^A} = \frac{U^A_{X0}}{U^A_{X1}}$$

The full set of marginal rates of utility substitution for individual A is:

$$\text{MRUS}^A_{X0,Y0} \equiv \frac{U^A_{X0}}{U^A_{Y0}}, \text{MRUS}^A_{X1,Y1} = \frac{U^A_{X1}}{U^A_{Y1}}$$
$$\text{MRUS}^A_{X0,X1} \equiv \frac{U^A_{X0}}{U^A_{X1}}, \text{MRUS}^A_{X0,Y1} = \frac{U^A_{X0}}{U^A_{Y1}}$$
$$\text{MRUS}^A_{Y0,Y1} \equiv \frac{U^A_{Y0}}{U^A_{Y1}}, \text{MRUS}^A_{Y0,X1} = \frac{U^A_{Y0}}{U^A_{X1}} \tag{11.39}$$

An exactly equivalent set can be written for individual B.

Consider the production of commodity *X*. The production function for period 0 is

$$X_0 = X_0(K_0^X)$$

where K_0^X is the amount of capital existing at the beginning of period 0, and hence the amount employed in the production of *X* during period 0. We make the assumption that the commodity *X* is such that it can either be consumed or added to the capital stock for the production of *X*. This greatly simplifies the analysis. Actually, of course, what happens is that the producers of *X* use some of the proceeds from sales of *X* to buy capital equipment. Our assumption short-circuits this and avoids the need to introduce further notation for capital goods and the price thereof. The output of *X* is, then, either sold for consumption, X^{C}, or invested, $X^{\mathrm{I}} = K_1^X - K_0^X$, where K_1^X is the capital stock at the beginning of period 1. Thus, we have

$$\begin{aligned} X_0^C &= X_0(K_0^X) - X_0^I = X_0(K_0^X) - (K_1^X - K_0^X) \\ &= X_0(K_0^X) - K_1^X + K_0^X \end{aligned} \tag{11.40a}$$

for period 0, and proceeding in the same way for period 1 gives

$$\begin{aligned} X_1^C &= X_1(K_1^X) - X_0^I = X_1(K_1^X) - (K_2^X - K_1^X) \\ &= X_1(K_1^X) - K_2^X + K_1^X \end{aligned} \tag{11.40.b}$$

We are interested in the marginal shifting of the consumption of X as between periods 0 and 1 by a marginal change in the level of investment in period 0. We define the, intertemporal, marginal rate of transformation for X_0^C and X_1^C as

$$\mathrm{MRT}_{0,I}^{X} \equiv \frac{dX_1^C}{dX_0^C}$$

From equations 11.40

$$\mathrm{d}X_0^C = X_{K0}\mathrm{d}K_0^X - \mathrm{d}K_1^X + \mathrm{d}K_0^X$$

and

$$\mathrm{d}X_1^C = X_{K1}\mathrm{d}K_1^X - \mathrm{d}K_2^X + \mathrm{d}K_1^X$$

where $X_{K0} = \partial X_0/\partial K_0^X$ is the marginal product of capital in period 0, and similarly for X_{K1}. With $\mathrm{d}K_0^X = \mathrm{d}K_2^X = 0$

$$\frac{dX_1^C}{dX_0^C} = \frac{X_{K1}dK_1^X + dK_1^X}{-dK_1^X}$$

so that

$$\mathrm{MRT}_{0,I}^{X} \equiv \frac{dX_1^C}{dX_0^C} = 1 + X_{K1} \tag{11.41a}$$

and similarly

$$\mathrm{MRT}_{0,I}^{Y} \equiv \frac{dY_1^C}{dY_0^C} = 1 + Y_{K1} \tag{11.41b}$$

A11.1.2 Efficiency conditions

The problem to be considered to derive the intertemporal efficiency conditions is

$$\text{Max } U^{\mathrm{A}} = U^{\mathrm{A}}(X_0^A, X_1^A, Y_0^A, Y_1^A)$$

subject to

$$U^{\mathrm{B}}(X_0^{\mathrm{B}}, X_1^{\mathrm{B}}, Y_0^{\mathrm{B}}, Y_1^{\mathrm{B}}) = Z$$

$$X_0(K_0^{\mathrm{X}}) - (K_1^{\mathrm{X}} - K_0^{\mathrm{X}}) = X_0^{\mathrm{A}} + X_0^{\mathrm{B}}$$

$$X_1(K_1^{\mathrm{X}}) - (K_2^{X} - K_1^{\mathrm{X}}) = X_1^{\mathrm{A}} + X_1^{\mathrm{B}}$$

$$Y_0(K_0^{Y}) - (K_1^{Y} - K_0^{Y}) = Y_0^{\mathrm{A}} + Y_0^{\mathrm{B}}$$

$$Y_1(K_1^{Y}) - (K_2^{Y} - K_1^{Y}) = Y_1^{\mathrm{A}} + Y_1^{\mathrm{B}}$$

Here, Z is some arbitrary fixed level for B's utility, and the opening and closing stocks of capital in each line of production are taken as given. The allocation problem concerns the commodity consumptions of A and B in each period, and the amount of investment in each line of production.

The Lagrangian is

$$\begin{aligned}
L = U^A = {} & U^A(X_0^A, X_1^A, Y_0^A, Y_1^A) \\
& + \lambda_1[U^B(X_0^B, X_1^B, Y_0^B, Y_1^B) - Z] \\
& + \lambda_2[X_0(K_0^X) - K_1^X + K_0^X - X_0^A - X_0^B] \\
& + \lambda_3[X_1(K_1^X) - K_2^X + K_1^X - X_1^A - X_1^B] \\
& + \lambda_4[Y_0(K_0^Y) - K_1^Y + K_0^Y - Y_0^A - Y_0^B] \\
& + \lambda_5[Y_1(K_1^Y) - K_2^Y + K_1^Y - Y_1^A - Y_1^B]
\end{aligned}$$

giving the first-order conditions:

$$\frac{\partial L}{\partial X_0^A} = U_{X0}^A - \lambda_2 = 0 \tag{11.42a}$$

$$\frac{\partial L}{\partial X_1^A} = U_{X1}^A - \lambda_3 = 0 \tag{11.42b}$$

$$\frac{\partial L}{\partial Y_0^A} = U_{Y0}^A - \lambda_4 = 0 \tag{11.42c}$$

$$\frac{\partial L}{\partial Y_1^A} = U_{Y1}^A - \lambda_5 = 0 \tag{11.42d}$$

$$\frac{\partial L}{\partial X_0^B} = \lambda_1 U_{X0}^B - \lambda_2 = 0 \tag{11.42e}$$

$$\frac{\partial L}{\partial X_1^B} = \lambda_1 U_{X1}^B - \lambda_3 = 0 \tag{11.42f}$$

$$\frac{\partial L}{\partial Y_0^B} = \lambda_1 U_{Y0}^B - \lambda_4 = 0 \tag{11.42g}$$

$$\frac{\partial L}{\partial Y_1^B} = \lambda_1 U_{Y1}^B - \lambda_5 = 0 \tag{11.42h}$$

$$\frac{\partial L}{\partial K_1^X} = \lambda_2 + \lambda_3 X_{K1} \lambda_3 = 0 \tag{11.42i}$$

$$\frac{\partial L}{\partial K_1^Y} = \lambda_4 + \lambda_5 X_{K1} \lambda_5 = 0 \tag{11.42j}$$

From the eight conditions on consumption, 11.42a to 11.42h, using the MRUS definitions from Section A11.1.1, we have:

$$\mathrm{MRUS}^{A}_{X0,Y0} = \mathrm{MRUS}^{B}_{X0,Y0} = \lambda_2/\lambda_4 \qquad (11.43a)$$

$$\mathrm{MRUS}^{A}_{X1,Y1} = \mathrm{MRUS}^{B}_{X1,Y1} = \lambda_3/\lambda_5 \qquad (11.43b)$$

$$\mathrm{MRUS}^{A}_{X0,X1} = \mathrm{MRUS}^{B}_{X0,X1} = \lambda_2/\lambda_3 \qquad (11.43c)$$

$$\mathrm{MRUS}^{A}_{X0,Y1} = \mathrm{MRUS}^{B}_{X0,Y1} = \lambda_2/\lambda_5 \qquad (11.43d)$$

$$\mathrm{MRUS}^{A}_{Y0,Y1} = \mathrm{MRUS}^{B}_{Y0,Y1} = \lambda_4/\lambda_5 \qquad (11.43e)$$

$$\mathrm{MRUS}^{A}_{Y0,X1} = \mathrm{MRUS}^{B}_{Y0,X1} = \lambda_4/\lambda_3 \qquad (11.43f)$$

Note that these are an extended version of the consumption efficiency conditions for the intratemporal allocation problem – the two individuals must have the same MRUS for all possible pairs of commodities. In saying this we are treating the same physical commodity at two different dates as two different commodities – we are, for example, treating X in period 0 as a different commodity from X in period 1.

If we had explicitly shown a labour input to production, we would have obtained intratemporal production efficiency conditions, the same as those in Chapter 4, for each line of production in each period.

The necessary conditions relating to investment, 11.42i and 11.42j, can be written

$$1 + X_{K1} = \lambda_2/\lambda_3 \quad (11.44a)$$

and

$$1 + Y_{K1} = \lambda_4/\lambda_5 \quad (11.44b)$$

and comparing these with 11.43c and 11.43e, using the definitions for the marginal rates of transformation provided in Section A.11.1.1 above, gives

$$MRUS^{A}_{X0,X1} = MRUS^{B}_{X0,X1} = MRT^{X}_{0,1} \quad (11.45a)$$

$$MRUS^{A}_{Y0,Y1} = MRUS^{B}_{Y0,Y1} = MRT^{Y}_{0,1} \quad (11.45b)$$

For each commodity, the intertemporal MRUS has to equal the MRT.

In the chapter, the conditions for intertemporal efficiency were stated, following the practice in much of the literature, in terms of rates of return to investment in different lines of production and the consumption discount rate. To demonstrate the equivalence of the conditions derived here with that statement of the conditions requires some definitions and an assumption.

Taking the assumption first. Assume that for both individuals

$$\mathrm{MRUS}_{X0,Y0} = \mathrm{MRUS}_{X1,Y1} \tag{11.46}$$

which by 11.43a and 11.43b implies

$$\frac{\lambda_2}{\lambda_4} = \frac{\lambda_3}{\lambda_5}$$

or

$$\frac{\lambda_2}{\lambda_3} = \frac{\lambda_4}{\lambda_5}$$

which by 11.44a and 11.44b gives:

$$X_{K1} = Y_{K1} \tag{11.47}$$

Efficiency requires the equalisation of the marginal product of capital in each of the lines of production.

This is equivalent to requiring equality of rates of return to investment, as in equation 11.3 in the chapter. The rate of return to investment is defined as difference between the increase in the next period consumption pay-off and the associated increase in current investment, expressed as a proportion of the increase in investment. In terms of *X*, the definition is

$$\delta_X \equiv \frac{dX_1^C - dX_0^1}{dX_0^1}$$

where the increase in investment dX_0^I entails an equal decrease consumption dX_0^C. Substituting –dX_0^C for dX_0^I in the definition

$$\delta_X \equiv \frac{dX_1^C - \left(-dX_0^C\right)}{-dX_0^C} = \frac{dX_1^C - dX_0^C}{-dX_0^C} = \frac{dX_1^C}{dX_0^C} - 1$$

which, using 11.41a, gives:

$$\delta_X = X_{K1}$$

Marginal products and rates of return are the same things. Hence, 11.47 can be written, as in the chapter, in terms of rates of return as:

$$\delta_X = \delta_Y \qquad (11.48)$$

Consider A's intertemporal marginal utility rate of substitution for commodity *X* and define as A's consumption discount rate for commodity *X*:

$$r^A_{X0,X1} \equiv \text{MRUS}^A_{X0,X1} - 1 \qquad (11.49a)$$

We can also define

$$r^A_{Y0,Y1} \equiv \text{MRUS}^A_{Y0,Y1} - 1 \qquad (11.49b)$$

for A, and

$$r^{B}_{X0,X1} \equiv \text{MRUS}^{B}_{X0,X1} - 1 \tag{11.49c}$$

and

$$r^{B}_{Y0,Y1} \equiv \text{MRUS}^{B}_{Y0,Y1} - 1 \tag{11.43d}$$

for B. With these definitions we can restate the intertemporal MRUS conditions from 11.43 in terms of commodity consumption discount rates as

$$r^{A}_{X0,X1} = r^{B}_{X0,X1} \tag{11.49e}$$

$$r^{A}_{Y0,Y1} = r^{B}_{Y0,Y1} \tag{11.49f}$$

in the same manner as equation 11.2 in the body of the chapter.

Using the definitions for commodity consumption discount rates, from 11.43c and 11.44a, replacing marginal product by rate of return gives

$$1 + r^{A}_{X0,X1} = 1 + r^{B}_{X0,X1} = 1 + X_{K1} = 1 + \delta_X$$

and similarly from 11.43e and 11.44b

$$1 + r^{A}_{Y0,Y1} = 1 + r^{B}_{Y0,Y1} = 1 + Y_{K1} = 1 + \delta_Y$$

from which

$$r^{A}_{X0,X1} = r^{B}_{X0,X1} = \delta_X$$

and

$$r^{A}_{Y0,Y1} = r^{B}_{Y0,Y1} = \delta_Y$$

which by the equality of rates of return, 11.48, is

$$r = \delta \qquad (11.50)$$

where sub- and superscripts can be dropped as consumption discount rates, across commodities and individuals, are required to be equalised along with rates of return, across commodities. Equation 11.50 here is the same as equation 11.4 in the body of the chapter.

It is important to be clear that although consumption discount rates and rates of return are often written without subscripts in the literature, as in equations 11.4 and 11.50 here and elsewhere in this text, they are *not* parameters. It is partly because getting to them via marginal rates of transformation and substitution may help to make this clear that we have done things that way.

The assumption 11.46 is that for each commodity pair, individuals indifferently exchange at the margin at a rate which is time-invariant. In terms of individuals facing given prices, this is, as will be made explicit in Appendix 11.2 below, the assumption that the relative prices of commodities are constant over time.

Note that there is another assumption that does the same job as 11.46. We could assume that for both consumers within both periods the two commodities are perfect substitutes for each other. In that case the intratemporal marginal rates of utility substitution are unity, i.e.

$$\text{MRUS}^{A}_{X0,Y0} = \text{MRUS}^{B}_{X0,Y0} = 1$$

and

$$\text{MRUS}^{A}_{X1,Y1} = \text{MRUS}^{B}_{X1,Y1} = 1$$

which by 11.43a and 11.43b gives

$$\frac{\lambda_2}{\lambda_4} = \frac{\lambda_3}{\lambda_5} = 1$$

so that

$$\frac{\lambda_2}{\lambda_3} = \frac{\lambda_4}{\lambda_5}$$

which gives equal marginal products, $X_{K1} = Y_{K1}$, by 11.44a and 11.44b. This is the assumption effectively adopted in the chapter, and in much of the literature, to simplify the exposition. See also Section A11.1.4.1, on aggregation over commodities, below.

A11.1.3 Optimality conditions

The relationship between the optimality problem and the efficiency problem, and between the necessary conditions arising in each case, is the same as in the static case examined in Appendix 4.1. Confirming this is left as an exercise for the reader – see Problem 1.

A11.1.4 Some specialisations

A11.1.4.1 Aggregation over commodities

In order to focus more on the intertemporal dimensions of the efficiency and optimality problems, we can specify them in terms of a single commodity, Q say, which can be either consumed or invested. Then the efficiency problem is

$$\text{Max } U^{\mathrm{A}}(C_0^A, C_1^A)$$

subject to

$$U^{\mathrm{B}}(C_0^B, C_1^B) = Z$$

$$Q_0(K_0) - (K_1 - K_0) = C_0^A + C_0^B$$

$$Q_1(K_1) - (K_2 - K_1) = C_1^A + C_1^B$$

For

$$L = U^{A}(C_0^A, C_1^A) + \lambda_1[U^{B}(C_0^B, C_1^B) - Z]$$

$$+ \lambda_2[Q_0(K_0) - K_1 + K_0 - C_0^A - C_0^B]$$

$$+ \lambda_3[Q_1(K_1) - K_2 + K_1 - C_1^A - C_1^B]$$

necessary conditions are

$$U^{A}_{C0} - \lambda_2 = 0$$

$$U^{A}_{C1} - \lambda_3 = 0$$

$$\lambda_1 U^{B}_{C0} - \lambda_2 = 0$$

$$\lambda_1 U^{B}_{C1} - \lambda_3 = 0$$

$$-\lambda_2 + \lambda_3 Q_{K1} + \lambda_3 = 0$$

from which it is now straightforward to derive the intertemporal efficiency condition as

$$\text{MRUS}^{\text{A}}_{C0,C1} = \text{MRUS}^{\text{B}}_{C0,C1} = \text{MRT}^{\text{C}}_{0,1}$$

which can also be stated as

$$r^{\text{A}} = r^{\text{B}} = \delta \tag{11.51}$$

i.e. the equality of the (common) consumption discount rate with the rate of return to investment.

Given this aggregation, the optimality problem is

$$\text{Max } W\{U^{A}(C_0^A, C_1^A), U^{B}(C_0^B, C_1^B)\}$$

subject to

$$Q_0(K_0) - (K_1 - K_0) = C_0^A + C_0^B$$

$$Q_1(K_1) - (K_2 - K_1) = C_1^A + C_1^B$$

It is left to the reader to confirm that the necessary conditions for a welfare optimum here are 11.51 plus

$$\frac{W_A}{W_B} = \frac{U_{C0}^B}{U_{C0}^A} = \frac{U_{C1}^B}{U_{C1}^A} \tag{11.52}$$

where $W_A = \partial W/\partial U^A$ and $W_B = \partial W/\partial U^B$.

With this specification of the problem there is no condition requiring the equality of rates of return to investment. This condition can be recovered by modifying the specification so that there is a single commodity produced by many firms across which production functions differ. The outputs of the various firms are, that is, perfect substitutes in consumption. In this case, with $i = 1,..., N$ firms, the efficiency problem is

$$\text{Max } U^{A}(C_0^A, C_1^A)$$

subject to

$$U^{B}(C_0^B, C_1^B) = Z$$

$$\sum_1^N \{Q_0^i(K_0^i) - (K_1^i - K_0^i)\} = C_0^A + C_0^B$$

$$\sum_1^N \{Q_1^i(K_1^i) - (K_2^i - K_1^i)\} = C_1^A + C_1^B$$

for which the necessary conditions are

$$U^{A}_{C0} - \lambda_2 = 0$$

$$U^{A}_{C1} - \lambda_3 = 0$$

$$\lambda_1 U^{B}_{C0} - \lambda_2 = 0$$

$$\lambda_1 U^{B}_{C1} - \lambda_3 = 0$$

$$-\lambda_2 + \lambda_3 Q^i_{K1} + \lambda_3 = 0,\ i = 1, 2, \ldots, N$$

from which

$$r^{A} = r^{B} = \delta_i$$

for all *i*.

A11.1.4.2 Aggregation over individuals

In order to focus solely on matters intertemporal, we could further specialise the problem specification by explicitly considering just one ‘representative’ individual. In that case we consider

$$\text{Max } U(C_0, C_1)$$

subject to

$$Q_0(K_0) - (K_1 - K_0) = C_0$$

$$Q_1(K_1) - (K_2 - K_1) = C_1$$

with Lagrangian

$$L = U(C_0, C_1) + \lambda_2[Q_0(K_0) - K_1 + K_0 - C_0] \\ + \lambda_3[Q_1(K_1) - K_2 + K_1 - C_1]$$

for necessary conditions

$$U_{C0} - \lambda_2 = 0$$

$$U_{C1} - \lambda_3 = 0$$

$$-\lambda_2 + \lambda_3 Q_{K1} + \lambda_3 = 0$$

from which we get

MRUS = MRT

or

$$r = \delta \tag{11.53}$$

A widely used variant of $U(C_0, C_1)$ is

$$W\{U(C_0), U(C_1)\} = U(C_0) + \left(\frac{1}{1+\rho}\right)U(C_1)$$

which has overall utility as the sum of current utility and discounted future utility; the parameter ρ is the utility discount rate. Some observations on terminology and notation here appear in the text of the chapter.

With this form of maximand, the two-period optimisation problem becomes

$$\text{Max } W = U(C_0) + \left(\frac{1}{1+\rho}\right)U(C_1)$$

subject to

$$Q_0(K_0) - (K_1 - K_0) = C_0$$

$$Q_1(K_1) - (K_2 - K_1) = C_1$$

with Lagrangian

$$\begin{aligned} L = {} & U(C_0) + \left(\frac{1}{1+\rho}\right)U(C_1) \\ & + \lambda_2\left[Q_0(K_0) - K_1 + K_0 - C_0\right] \\ & + \lambda_3\left[Q_1(K_1) - K_2 + K_1 - C_1\right] \end{aligned}$$

for necessary conditions

$$U_{Co} - \lambda_2 = 0$$
$$\left(\frac{1}{1+\rho}\right)U_{C1} - \lambda_3 = 0$$
$$-\lambda_2 + \lambda_3 Q_{K1} + \lambda_3 = 0$$

from which we get

$$\frac{U_{Co}}{[1/(1+\rho)]U_{C1}} = 1 + Q_{K1}$$

or

$$\frac{U_{C1}}{U_{C0}} = \frac{1+\rho}{1+Q_{K1}} \tag{11.54}$$

Note that 11.54 implies, given decreasing marginal utility, $U_{CC} < 0$, that:

For $\rho > Q_{K1}$, $C_1 < C_0$

For $\rho = Q_{K1}$, $C_1 = C_0$

For $\rho < Q_{K1}$, $C_1 > C_0$

The model just considered is the simple optimal growth model considered in the chapter here, and in Chapter 3. For given utility and production functions, a given utility discount rate and given initial and terminal stocks of capital, it determines period 0 saving/investment, and hence consumption levels in the two periods. Such a model takes it as given that the intratemporal and intertemporal efficiency conditions are satisfied, and aggregates over commodities and individuals.

A11.1.4.3 Consumption and utility discount rates

Given that in

$$W\{U(C_0),U(C_1)\} = U(C_0) + \left(\frac{1}{1+\rho}\right)U(C_1)$$

contemporaneous utility is a function only of current consumption, $W\{.\}$ can be expressed with consumption levels as arguments. Utility discounting then implies consumption discounting. The consumption discount rate is defined as

$$r \equiv -\frac{dC_1}{dC_0} - 1$$

For

$$W = U(C_0) + \left(\frac{1}{1+\rho}\right)U(C_1)$$

we have

$$dW = U_{C0}dC_0 + \left(\frac{1}{1+\rho}\right)U_{C1}dC_1$$

so that with d*W* = 0, the MRUS is

$$\frac{dC_1}{dC_0} = \frac{U_{Co}}{[1/(1+\rho)]U_{C1}}$$

and

$$r = \frac{U_{Co}}{[1/(1+\rho)]U_{C1}} - 1\frac{(1+\rho)U_{C0}}{U_{C1}} - 1 \qquad (11.55)$$

This shows that the consumption discount rate depends on the utility discount rate, and on the levels of marginal utility, and hence, given the utility function, on the consumption levels, in each period. In fact, with diminishing marginal utility, we can see that for given consumption levels *r* increases as ρ increases, and that for given ρ *r* increases as the ratio of C_1 to C_0 increases. Working in continuous time it is possible to derive the expression for *r* in terms of ρ and the growth of *C* that was used in Chapter 3 in the discussion of the ethics of discounting. First write equation 11.55 in terms of any two adjacent periods *t* and *t* + 1 as

$$r = \frac{U_{C,t} - [1/(1+\rho)]U_{C,t+1}}{[1/(1+\rho)]U_{C,t+1}}$$

which in continuous time is

$$r = \frac{\frac{d}{dt}[e^{-\rho t}U'(C_t)]}{e^{-\rho t}U'(C_t)}$$

which gives

$$r = \rho - \frac{U''(C_t)\dot{C}_t}{U'(C_t)} \tag{11.56}$$

Define as the elasticity of marginal utility

$$\eta \equiv -\frac{U''(C_t)C_t}{U'(C_t)} \tag{11.57}$$

and 11.56 can be written as

$$r = \rho + \eta g \tag{11.58}$$

where g is the growth rate for consumption, $\dot{C}_t / C_t$.