# Appendix 13.1 Irreversibility and development: future known

In the absence of irreversibility, the efficient levels of development would be chosen, so as to

$$\text{Max } F_1(A_1) + F_2(A_2)$$

where $F_1(A_1)$ and $F_2(A_2)$ are the net benefit functions. The necessary conditions are:

$$\partial F_1/\partial A_1 = 0 \qquad (13.34a)$$

$$\partial F_2/\partial A_2 = 0 \qquad (13.34b)$$

With the linear MNB functions shown in Figure 13.4, these conditions are

$$\alpha - \beta A_1 = 0 \qquad (13.35a)$$

$$\mathrm{k}\alpha - \beta A_2 = 0 \qquad (13.35b)$$

where $k > 1$. Using the notation of Figure 13.4, solving equations 13.35 gives:

$$A_1^{\mathrm{NI}} = \alpha/\beta \qquad (13.36a)$$

$$A_2^{\mathrm{NI}} = k(\alpha/\beta) = kA_1^{\mathrm{NI}} \qquad (13.36b)$$

If there is irreversibility but it is not taken account of in decision making, the result will be:

$$A_1' = A_1^{\mathrm{NI}} = \alpha/\beta \qquad (13.37a)$$

$$A_2' = A_1' = \alpha/\beta \qquad (13.37b)$$

With irreversibility taken into account in decision making, the problem is to

$$\text{Max } F_1(A_1) + F_2(A_2)$$

subject to

$$A_1 = A_2$$

for which the Lagrangian

$$L = F_1(A_1) + F_2(A_2) + \lambda[A_1 - A_2]$$

gives the necessary conditions:

$$\partial F_1/\partial A_1 + \lambda = 0 \tag{13.38a}$$

$$\partial F_2/\partial A_2 - \lambda = 0 \tag{13.38b}$$

$$A_1 - A_2 = 0 \tag{13.38c}$$

Substituting $\alpha - \beta A_1$ for $\partial F_1/\partial A_1$ and $k\alpha - \beta A_2$ for $\partial F_2/\partial A_2$ in equations 13.38a and 13.38b and solving leads to:

$$A^I_1 = A^I_2 = (\alpha/\beta)\{(1 + k)/2\} \tag{13.39}$$

Comparing equation 13.39 with equations 13.36 and 13.37, for $k > 1$ it is seen that

$$A^I_1 > A_1^{NI} = A'_1 \tag{13.40}$$

and

$$A'_2 < A^I_2 < A_2^{NI} \tag{13.41}$$

as shown in Figure 13.4.

Now consider the cost of irreversibility, when it is taken into account in decision making. As discussed in the chapter, this cost is the sum of the triangles **abc** and **def** in Figure 13.4. The area of **abc** is given by 0.5 × **ac** × **ab** where

$$\mathbf{ac} = A^{I}_{1} - A_{1}^{NI} = (\alpha/\beta)\{(1 + k)/2\} - (\alpha/\beta)$$

$$= (\alpha/\beta)\{(k - 1)/2\} \qquad (13.42)$$

and

$$\mathbf{ab} = \alpha - \beta A^{I}_{1} = \alpha - \beta\, (\alpha/\beta)\{(1 + k)/2\}$$

$$= \{\alpha\, (1 - k)/2\} \qquad (13.43)$$

so that

$$\mathbf{abc} = \{\alpha^{2}(k - 1)(1 - k)\}/8\beta \qquad (13.44)$$

Proceeding in the same way, we get

$$\mathbf{ef} = A_{2}^{NI} - A^{I}_{2} = k(\alpha/\beta) - \{(1 + k)/2\}(\alpha/\beta)$$

$$= (\alpha/\beta)\{(k - 1)/2\} \qquad (13.45)$$

and

$$\mathbf{de} = k\alpha - \beta A^{I}_{2} = k\alpha - \beta(\alpha/\beta)\{(1 + k)/2\}$$

$$= \alpha\{(k - 1)/2\} \qquad (13.46)$$

so that

$$\mathbf{def} = \{\alpha^{2}(k - 1)(k - 1)\}/8\beta \qquad (13.47)$$

Comparing equations 13.43 and 13.46 we see that **ab** and **de** are equal, as stated in the text discussion of Figure 13.4, but of opposite sign. Equation 13.44 shows **abc** as negative, so to get the cost of irreversibility we use the absolute value for **abc** (the right-hand side of 13.44 multiplied by –1) plus **def**. This gives

$$\begin{aligned} |\mathbf{abc}| + \mathbf{def} &= 2 \times \{\alpha^2(k-1)^2\}/8\beta \\ &= \{\alpha^2(k-1)^2\}/4\beta \end{aligned} \qquad (13.48)$$

for the cost of irreversibility.

Now consider the cost of ignoring irreversibility in decision making. This leads to $A_1^{NI}$ instead of $A^I_1$, and to $A'_2$ instead of $A^I_2$. As shown in Figure 13.4, in the first period there is a gain equal to the area of triangle **abc**, and in the second a loss equal to the area **edhi**. If **edhi** > **abc**, there is a net loss. Since **abc** and **def** have the same areas, this condition is **edhi** > **def**. Clearly, **edhi** is greater than **def** if **ie** = **ef**, which it does as **ie** = **ac** and by equations 13.42 and 13.45 **ac** and **ef** are equal.

So there is a cost to ignoring irreversibility when it exists. For the linear MNB functions used in Figure 13.4, we can show that the cost of ignoring irreversibility is greater than the cost of irreversibility. The cost of ignoring irreversibility is area **edhi**, which is area **hgd** plus area **gdei**. Consider the latter first. We have

$$\mathbf{gdei} = \mathbf{de} \times \mathbf{ie} = \alpha\{(k-1)/2\} \times \mathbf{ie}$$

using equation 13.46 for **de**. The distance **ie** is $A^I_2 - A'_2$, so that, using equations 13.39 and 13.37,

$$\textbf{gdei} = \alpha\{(k - 1)/2\} \times [(\alpha/\beta)\{(1 + k)/2\} - (\alpha/\beta)]$$

which, on simplifying, is

$$\textbf{gdei} = \{\alpha^2(k - 1)^2\}/4\beta \qquad (13.49)$$

Comparing equations 13.48 and 13.49 gives **gdei** equal to the cost of irreversibility. But the cost of ignoring irreversibility is **gdei** plus **hdg**, so the cost of ignoring irreversibility is greater than the cost of irreversibility.