Appendix 14.2 The optimal solution to the simple exhaustible resource depletion problem

In this appendix, we derive the optimal solution to the simple exhaustible resource depletion problem discussed in Part 1 of this chapter. In doing this, we will make extensive reference to the solution method outlined in Appendix 14.1.

The objective function to be maximised is:

$$W = \int_{t=0}^{t=\infty} U(C_t)\mathrm{e}^{-\rho t}\mathrm{d}t$$

Comparing this with the form and notation used for an objective function in Tables 14.2 and 14.3 it is evident that:

- we are here using W (rather than J) to label the objective function;
- the initial time period (t_0) is here written as $t = 0$;
- the terminal time (t_T) is infinity: therefore we describe the terminal point as free;
- there is a discounting factor present in the objective function: Table 14.3 is therefore appropriate;
- the integral function which in general takes the form $L(\mathbf{x}, \mathbf{u}, t)$ (ignoring the discounting term) here has the form $U(C_t)$. It is a function of one variable only, consumption, which is a control variable (u). Note that we have written this variable as C_t rather than C to make it explicit that

the value of the control variable changes over time. No state variable enters the objective function in this problem, nor does time, *t*, enter the integral function directly (it enters only through the discounting factor).

Be careful not to confuse *U* and *u*. The term *U* in Appendix 14.2 denotes utility; it is what is being maximised in the objective function; **u** in Tables 14.2 and 14.3 is the notation used for control variables.

There are two state variables (the **x** variables in Table 14.3) in this problem: S_t and K_t, the resource stock at time *t* and the capital stock at time *t*, respectively. Corresponding to these two state variables are two state equations of motion (the equations $\dot{x} = f(x, u, t)$ in Table 14.3). These are

$$\dot{S}_t = -R_t$$

and

$$\dot{K}_t = Q_t - C_t$$

There are two control variables in this problem: C_t and R_t (the rate of resource extraction). These are the two variables whose values are chosen by the decision maker to form a time path that will maximise the objective function. Note that in neither of the state equations of motion does a state variable (**x**) or time (*t*) appear as an argument of the function.

The economic system consists of:

- the two state equations;
- initial values for the state variables: the initial resource stock (S_0, see equation 14.9) and the initial capital stock (K_0, see footnote 5 in the main text);
- a production function, linking output Q (which is neither a state nor a control variable) to the capital stock and rate of resource extraction at each point in time:

$$Q_t = Q(K_t, R_t)$$

One final thing remains to be specified: the terminal state conditions. We do not state these explicitly in the text. However, by implication, the problem is one in which both the capital stock and the resource stock become zero at the end of the (infinite) planning horizon, so we have $K_{t=\infty} = 0$ and $S_{t=\infty} = 0$ (i.e. $\mathbf{x}(t_T) = \mathbf{x}_T = 0$, in the notation of Table 14.3). As a result of $\mathbf{x}(t_T) = 0$ and t_T free (with an infinite horizon), it is the third column from the right in Table 14.3 which is relevant for obtaining the solution to this problem.

The current-value Hamiltonian for this problem is

$$H_{C_t} = U(C_t) + P_t(-R_t) + \omega_t(Q_t - C_t)$$

in which P_t and ω_t are the co-state variables (shadow prices) expressed in units of current-value utility associated with the resource stock and the capital stock at time *t* respectively. After substituting for Q_t from the production function, the Hamiltonian is

$$H_{C_t} = U(C_t) + P_t(-R_t) + \omega_t(Q\{K_t, R_t\} - C_t)$$

The necessary conditions for a maximum include:

$$\frac{\partial H_{C_t}}{\partial C_t} = U_{C,t} - \omega_t = 0 \tag{14.25a}$$

$$\frac{\partial H_{C_t}}{\partial R_t} = -P_t + \omega_t Q_{R,t} = 0 \tag{14.25b}$$

$$\dot{P}_t = -\frac{\partial H_{C_t}}{\partial S_t} + \rho P_t \Leftrightarrow \dot{P}_t = \rho P_t \tag{14.25c}$$

$$\dot{\omega}_t = -\frac{\partial H_{C_t}}{\partial K_t} + \rho\omega_t \Leftrightarrow \dot{\omega}_t = \rho\omega_t - Q_{K,t}\omega_t \tag{14.25d}$$

The pair of equations 14.25a and 14.25b correspond to the 'Max *H*' condition $\partial H_C/\partial \mathbf{u} = 0$ in Table 14.3, for the two control (**u**) variables *R* and *C*. The second pair, 14.25c and 14.25d, are the equations of motion for the two co-state variables [$\dot{\mu} = \rho\boldsymbol{\mu} - \partial H_C/\partial \mathbf{x}$] that are associated with the two state variables S and K. Note that in 14.25c the term $-\partial H_C/\partial S = 0$ as S does not enter the Hamiltonian function. The four equations 14.14a to 14.14d given in the main text of this chapter are identical to equations 14.25a to 14.25d above (except that the equations in the text, rather loosely, use H rather than H_C).

Obtaining an expression for the growth rate of consumption

An expression for the growth rate of consumption along the optimal time path can be obtained by combining equations 14.25a and 14.25d as follows (dropping the time subscripts for simplicity). First, differentiate equation 14.25a with respect to time, yielding:

$$\dot{\omega} = U''(C)\dot{C} \tag{14.26}$$

Next, combine equations 14.26 and 14.25d to obtain:

$$U''(C)\dot{C} = \rho\omega - Q_K\omega$$

Hence

$$\dot{C}U''(C) = \omega(\rho - Q_K)$$

But since from equation 14.25a we know that $U'(C_t) = \omega_t$ the previous equation can be re-expressed as

$$\dot{C}U''(C) = U'(C)(\rho - Q_K)$$

Therefore

$$\frac{\dot{C}U''(C)}{C} = \frac{U'(C)(\rho - Q_K)}{C}$$

and so

$$\frac{\dot{C}}{C} = \frac{1}{\left[\dfrac{U''(C)C}{U'(C)}\right]}(\rho - Q_K) \qquad (14.27)$$

Now by definition the elasticity of marginal utility with respect to consumption, η, is

$$\eta = -\frac{\partial MU / MU}{\partial C / C}$$

Noting that MU = $U'(C)$, then the expression for η can be rearranged to give

$$\eta = -\frac{U''(C)C}{U'(C)}$$

Then 14.27 can be rewritten as

$$\frac{\dot{C}}{C} = -\frac{1}{\eta}(\rho - Q_K) = \frac{Q_K - \rho}{\eta}$$

which is the expression we gave for the growth rate of consumption in the text.