Appendix 15.1 Solution of the multi-period resource depletion model

We wish to maximize

$$W = \int_{t=0}^{t=T} U\left(R_t\right) e^{-\rho t} dt$$

subject to

$$\dot{S} \;=\; -R_t$$

The current-valued Hamiltonian for this problem is

$$H = U(R_t) + P_t(-R_t)$$

The necessary conditions for maximum social welfare are

$$\dot{P}_t \;=\; \rho P_t \tag{15.14}$$

$$\frac{\partial H}{\partial R} = -P_t + \frac{dU}{dR} = 0 \tag{15.15}$$

Rearranging equation 15.15 we obtain

$$P_t \;=\; \frac{dU}{dR}$$

so that the resource shadow price, P_t, is equal to the marginal utility of the non-renewable resource, an equality used in the main text. Equation 15.14 is, of course, the Hotelling efficiency condition, given as equation 15.7b in the chapter.

As we noted in the chapter, an optimal solution must have the property that the stock goes to zero at exactly the point that demand goes to zero. In order for demand to be zero at time T (which we determine in a moment) the net price must reach the choke price at time T. That is,

$$P_T = K$$

This, together with equation 15.7a in the main text, implies

$$K = P_0 \mathrm{e}^{\rho T} \tag{15.16}$$

To solve for R_t, it can be seen from equations 15.7a and 15.8 that

$$P_0 \mathrm{e}^{\rho t} = K\mathrm{e}^{-aR}$$

Substituting for K from equation 15.16 we obtain

$$\begin{aligned} P_0\mathrm{e}^{\rho t} &= P_0\mathrm{e}^{-(aR-\rho T)} \\ \Rightarrow \rho t &= -aR + \rho T \qquad (15.17) \\ \Rightarrow R_t &= \frac{\rho}{a}(T-t) \end{aligned}$$

This gives an expression for the rate at which the resource should be extracted along the optimal path.

To find the optimal time period, T, over which extraction should take place, recall that the fixed stock constraint is:

$$\int_0^T R_t \mathrm{d}t = \overline{S}$$

and so by substitution for R_t from equation 15.17 we obtain

$$\int_0^T \left[\frac{\rho}{a}(T-t)\right]\mathrm{d}t = \overline{S}$$

Therefore

$$\frac{\rho}{a}\left[Tt - \frac{t^2}{2}\right]_0^T = \overline{S}$$

$$\frac{1}{2}\frac{\rho}{a}r^2 = \overline{S}$$

or

$$T = \sqrt{\frac{2\bar{S}a}{\rho}}$$

Next we solve, using equation 15.16, for the initial royalty level, P_0:

$$P_0 = K\mathrm{e}^{-\rho T} = K\mathrm{e}^{-\sqrt{2\rho\bar{S}a}}$$

To obtain an expression for the resource royalty at time t, we substitute equation 15.7a into the expression just derived for the initial royalty level to obtain the required condition:

$$P_t = K\mathrm{e}^{\rho(t-T)}$$

The optimal initial extraction level is, from equation 15.17,

$$R_0 = \frac{\rho}{a}(T-0) = \frac{\rho T}{a} = \sqrt{\frac{2\rho\bar{S}}{a}}$$