Appendix 15.2 The monopolist's profit-maximising extraction programme

To solve for the monopolist's profit-maximising extraction programme, we need to do some additional calculation. First, let us derive an expression for the firm's marginal profit function, $M\Pi$:

$$M\Pi_t = \frac{\partial \Pi_t}{\partial R_t} = \frac{\partial\left(P(R)R_t\right)}{\partial R_t} = \frac{\partial P_t}{\partial R_t} R_t + P(R) \tag{15.18}$$

Now, substituting for $P(R)$ from the resource demand function (equation 15.8) we can express this equation as

$$\begin{aligned} M\Pi_t &= -aR_t K\mathrm{e}^{-aR_t} + K\mathrm{e}^{-aR_t} \\ &= K(-aR_t + 1)\mathrm{e}^{-aR_t} \approx K\mathrm{e}^{-ahRt} \end{aligned} \tag{15.19}$$

where $h = 2.5$. Notice the approximation here. We use this because otherwise it is not possible to obtain an analytical solution, given the double appearance of R_t.

Since resource extraction at the end of the planning horizon must be zero ($R_T = 0$) we have

$$M\Pi_t = K\mathrm{e}^{-ahR(T)} = K \tag{15.20}$$

To obtain $M\Pi_0$, using equation 15.9 we obtain

$$M\Pi_0 = M\Pi_T\, \mathrm{e}^{-iT} = K\mathrm{e}^{-iT} \tag{15.21}$$

To obtain an expression for $M\Pi_t$, using equations 15.9 and 15.21, we have

$$M\Pi_t = M\Pi_0\, \mathrm{e}^{it} = K\mathrm{e}^{i(t-T)} \tag{15.22}$$

Now we may obtain a solution equation for R_t, using equations 15.9 and 15.22:

$$Ke^{-ahR_t} = Ke^{i(t-T)}$$

implying that

$$i(t-T) = -ahR_t$$

$$\Rightarrow R_t = \frac{i}{ha}(T-t) \qquad (15.23)$$

In order to obtain the optimal depletion time period T we use the fixed-stock constraint together with equation 15.23, the result we have just obtained:

$$\int_0^T R_t \mathrm{d}t = \overline{S}$$

$$\Rightarrow \int_0^T \frac{i}{ha}(T-t)\mathrm{d}t = \overline{S}$$

$$\Rightarrow \frac{i}{ha}\left[Tt - \frac{t^2}{2}\right]_0^r = \overline{S}$$

$$\frac{1}{2}\frac{i}{ha}r^2 = \overline{S}$$

Therefore

$$T = \sqrt{\frac{2\overline{S}ha}{i}}$$

To solve the initial extraction R_0, from equation 15.22:

$$R_0 = \frac{i}{ha}\left(T - 0\right) = \frac{iT}{ha} = \sqrt{\frac{2i\bar{S}}{ha}}$$

Finally, to solve the initial net price P_0, from equation 15.8, (the demand curve)

$$P_0 = K\mathrm{e}^{-aR_0} = K\ \exp\left(-\sqrt{\frac{2i\bar{S}a}{h}}\right)$$