Appendix 17.3 The dynamics of renewable resource harvesting

In this appendix we derive the optimal solution to the net present value maximising fishery described in Section 17.8.3 in the text. We begin the analysis with general functional forms for the biological and economic equations underlying the model. Two cases are considered. First the case in which the market price of fish is an exogenously given constant. In the second case, the fishery faces a downward sloping demand curve for fish, and so fish prices are endogenous.

In both cases, the derivation involves specifying the private optimising problem, obtaining the current-value Hamiltonian, deriving necessary first order conditions, and solving those conditions to obtain a set of analytical conditions that constitute the main elements of the model solution. We obtain the steady state solution to the model from these conditions. In addition, an elementary account of system dynamics is given.

This appendix also derives parametric solutions for the particular functional forms used in this chapter. Finally, it solves these parametric solutions for their numerical values for the 'baseline' parameters assumed in Box 17.2 and for i = 0.

To assist comparison with the main text, equations in this appendix that appeared previously in the text are given their earlier textbook number, but marked with an asterisk. Other equations are numbered sequentially and are not asterisked. Where notation has been used before in the text, it will not be redefined here.

A.17.3.1 The Optimising Problem

The problem to be solved is

$$\mathrm{Max} \int_0^\infty \{R(H_t) - C(S_t, H_t)\} e^{-it}\, dt$$

$$\text{subject to}$$

$$\frac{dS}{dt} = G(S_t) - H_t$$

and initial stock level $S(0) = S_0$., where R(H) is the total revenue from resource harvesting.

A.17.3.2 Private present value-maximising harvesting in a competitive market with enforceable property rights and constant prices.

With constant price R(H) = PH

and so the current-value Hamiltonian is

$$L(H,S)_t = PH_t - C(S_t, H_t) + p_t(G(S_t) - H_t)$$

with necessary first order conditions

$$\text{(i)}\ \frac{\partial L_t}{\partial H_t} = 0 = P - \frac{\partial C}{\partial H_t} - p_t \qquad (13)$$

$$\text{(ii)}\ \frac{d p_t}{dt} = ip_t - p_t\frac{dG}{dS_t} + \frac{\partial C}{\partial S_t} \qquad \text{(17.30* in main text)}$$

$$\text{(iii)}\ \frac{dS}{dt} = G(S_t) - H_t$$

Equation 17.30* is the Hotelling efficiency condition for a renewable resource in which costs depend upon the stock level. It applies whether or not the fishery is in steady state equilibrium.

As described in the text, a decision about whether to defer some harvesting until the next period is made by comparing the marginal costs and benefits of adding additional units to the resource stock. The marginal cost is the foregone current return, the value of which is the net price of the resource, p. However, since we are considering a decision to *defer* this revenue by one period, the present value of this sacrificed return is ip.

Out of steady state the benefits obtained by the resource investment are threefold:

- The unit of stock may appreciate in value by the amount dp/dt.
- With an additional unit of stock, *total* harvesting costs will be reduced by the quantity $\partial C/\partial S$ (note that $\partial C/\partial S < 0$).
- The additional unit of stock will grow by the amount dG/dS, the value of which is this quantity multiplied by the net price of the resource, p.

A present value-maximising owner will add to (subtract from) the resource stock provided the marginal cost of doing so is less (greater) than the marginal benefit. These imply the asset equilibrium condition 17.30*.

Dividing both sides of Equation 17.30* by the net price p (and ignoring time subscripts from now on for simplicity) gives

$$i = \frac{\left(\frac{dp}{dt}\right)}{p} - \frac{\left(\frac{\partial C}{\partial S}\right)}{p} + \frac{dG}{dS} \qquad (14)$$

This reformulation shows that it is Hotelling's rule of efficient resource use, albeit in a modified form. (It is instructive to compare this version of Hotelling's rule with other versions that we have derived previously. See, for example, Equation 7.20c in Chapter 7.)

The left-hand side of Equation 14 is the rate of return that can be obtained by investing in assets elsewhere in the economy. The right-hand side is the rate of return that is obtained from the renewable resource. This is made up of three elements:

- the proportionate growth in net price
- the proportionate reduction in harvesting costs that arises from a marginal increase in the resource stock
- the natural rate of growth in the stock from a marginal change in the stock size.

A.17.3.2.1 Steady state equilibrium.

In a steady state, all variables (including shadow prices) are unchanging with respect to time, so dp/dt = 0. So the three first order conditions collapse to

$$ip = p\frac{dG}{dS} - \frac{\partial C}{\partial S} \qquad (17.32^*)$$

$$p = P(H) - \frac{\partial C}{\partial H} \qquad (17.29^*)$$

$$G(S) = H \qquad (17.14^*)$$

This constitutes a system of three equations in three unknowns S, H and p. Given functional forms for G(S) and C(H,S), we may then be able to obtain the steady state solution for those three unknowns in terms of the model parameters. These are derived later in this section for our assumed functional forms.

If i = 0 the steady-state equation 17.32* simplifies to

$$\frac{\partial C}{\partial S} = \frac{dG}{dS}p \qquad (15)$$

This was the condition 7 that we found in Appendix 17.2 for a private property fishery, and demonstrates that the static private property fishery is a special case of the PV maximising fishery when the discount rate is zero

We can also express Equation 15 in terms of market price rather than net price. Noting that $p = P - C_H$ (where $C_H = \partial C/\partial H$) then

$$\frac{\partial C}{\partial S} = \frac{dG}{dS}(P - C_H)$$
$$\frac{\partial C}{\partial S} = \frac{dG}{dS}P - \frac{dG}{dS}C_H$$

or

$$\frac{\partial G}{\partial S}P = \frac{dG}{dS}C_H + \frac{\partial C}{\partial S} \qquad (16)$$

This is the steady-state condition for maximising the net present value of the renewable resource in the special case where the discount rate is zero. The left-hand side can be interpreted as the steady-state marginal revenue of harvesting, the right-hand side the steady-state marginal cost of harvesting, taking account of the dependence of costs on effort, the cost per unit effort and the dependence of cost on the size of the resource stock.

A.17.3.2.2 Initial and terminal conditions, and adjustment dynamics.

The problem has a single initial condition, namely that there exists a fixed, positive initial stock level, S_0. Let $\{H_T, S_T\}$ denote the terminal, steady-state values of the harvest rate and resource stock, such that

$$\frac{dH}{dt} = 0 \quad \text{and} \quad \frac{dS}{dt} = 0$$

The nature of the adjustment dynamics depends on whether or not the Hamiltonian is linear in the control variable (here taken to be H). Adjustment may be rapid and discontinuous (along a most rapid adjustment path, MRAP) or gradual and continuous, depending on the underlying functions. We return to this below.

A.17.3.2.3 Parametric solution for the assumed functional forms, where P is constant

For our assumed functions, the Hamiltonian and first-order conditions are:

$$L(H,S) = PH - \frac{wH}{eS} + p(gS - \frac{gS^2}{S_{MAX}} - H) \qquad (17)$$

$$\text{(i)}\ \frac{\partial L}{\partial H} = 0 = P - \frac{w}{eS} - p$$

$$\text{(ii)}\ \dot{p} = ip - p(g - \frac{2gS}{S_{MAX}}) - \frac{wH}{eS^2} \qquad (16)$$

$$\text{(iii)}\ \dot{S} = gS - \frac{gS^2}{S_{MAX}} - H$$

In steady state, Equation (16) can be rearranged to the form

$$ip = p(g - \frac{2gS}{S_{MAX}}) + \frac{wH}{eS^2}$$

Solving this first order condition equation to obtain an expression for H as a function of p (net price) and S gives:

$$H = \frac{p(iS_{MAX} - gS_{MAX} + 2gS)eS^2}{wS_{MAX}}$$

and then substituting for p from the first order condition p = P - ∂C/∂H yields:

$$H = \frac{\left(P - \frac{w}{eS}\right)(iS_{MAX} - gS_{MAX} + 2gS)eS^2}{wS_{MAX}}$$

Given that G(S) = H in steady state, we next substitute the biological growth function, G(S) into the previous equation:

$$g\left(1 - \frac{S}{S_{MAX}}\right)S = \frac{(PeS - w)(iS_{MAX} - gS_{MAX} + 2gS)S}{wS_{MAX}}$$

After further manipulation and simplification, we can obtain from this last equation an explicit solution for S* in terms of model parameters alone:

$$S^* = \frac{1}{4}\frac{\left(-PeiS_{MAX} + PegS_{MAX} + wg + \sqrt{\begin{array}{l}P^2e^2i^2S_{MAX}{}^2 - 2P^2e^2iS_{MAX}{}^2g + 6PegwiS_{MAX}\\ + P^2e^2g^2S_{MAX}{}^2 + +2Peg^2S_{MAX}w + w^2g^2\end{array}}\right)}{Peg}$$

which is the rather cumbersome expression that has been used to calculate S^* in the associated Excel spreadsheets. A simpler form emerging after further manipulation is:

$$S^* = \frac{1}{4}S_{MAX}\left(\frac{w}{PeS_{MAX}} + 1 - \frac{i}{g} + \sqrt{\left(\frac{w}{PeS_{MAX}} + 1 - \frac{i}{g}\right)^2 + \frac{8wi}{PegS_{MAX}}}\right) \quad (18)$$

We may now substitute S* into G(S) to get H*:

$$H^* = \frac{1}{4}g\left(\frac{3}{4} - \frac{1}{4}\frac{w}{PeS_{MAX}} + \frac{\frac{1}{4}i}{g} - \frac{1}{4}\sqrt{\left(\frac{w}{PeS_{MAX}} + 1 - \frac{i}{g}\right)^2 + \left(\frac{8wi}{PegS_{MAX}}\right)}\right)S_{MAX} \times$$

$$\left(\frac{w}{PeS_{MAX}} + 1 - \frac{i}{g} + \sqrt{\left(\frac{w}{PeS_{MAX}} + 1 - \frac{i}{g}\right)^2 + \frac{8wi}{PegS_{MAX}}}\right) \quad (19)$$

and then finally obtain an expression for steady-state effort, E*, by using E* = H*/(eS*):

$$E^* = \frac{g}{e}\left(\frac{3}{4} - \frac{1}{4}\frac{w}{PeS_{MAX}} + \frac{\frac{1}{4}i}{g} - \frac{1}{4}\sqrt{\left(\frac{w}{PeS_{MAX}} + 1 - \frac{i}{g}\right)^2 + \left(\frac{8wi}{PegS_{MAX}}\right)}\right) \quad (20)$$

Equations 19 and 20 are used to obtain the PV maximising steady state values of H and E in the associated Excel spreadsheets. If i = 0 these expressions collapse to those for a static profit maximising model.

A.17.3.2.4 Dynamics

The Hamiltonian (Equation 17) is linear in H, the instrument variable. In this case, adjustment is of the bang-bang (most rapid approach path or MRAP) variety. That is, if $S < S^*$ then harvesting is zero until $S = S^*$, after which $H = H^*$. If $S > S^*$ then harvesting is undertaken at its maximum possible rate until $S = S^*$, after which $H = H^*$.

A.17.3.3 Private present value-maximising harvesting in a competitive market with enforceable property rights and a downward sloping demand curve for fish.

The Hamiltonian is now

$$L(H,S)_t = R(H_t) - C(S_t, H_t) + p_t(G(S_t) - H_t) \qquad (21)$$

The necessary conditions for a maximum are now

$$\frac{\partial L_t}{\partial H_t} = 0 = \frac{dR}{dH_t} - \frac{\partial C}{\partial H_t} - p_t \qquad (22)$$

$$\frac{d p_t}{dt} = ip_t - p_t \frac{dG}{d S_t} + \frac{\partial C}{\partial S_t} \qquad (17.30^*)$$

Differentiating 22 with respect to t (and omitting time subscripts for simplicity) yields

$$\frac{dp}{dt} = \frac{d^2R}{dH^2}\frac{dH}{dt} - \frac{\partial^2 C}{\partial H^2}\frac{dH}{dt}$$

Equating this with 17.30* gives

$$ip - p\frac{dG}{dS} + \frac{\partial C}{\partial S} = \frac{d^2R}{dH^2}\frac{dH}{dt} - \frac{\partial^2 C}{\partial H^2}\frac{dH}{dt} \qquad (23)$$

which can be solved to give dH/dt as a function of H and S.

We also have

$$\frac{dS}{dt} = G(S_t) - H_t \qquad (24)$$

Equations 23 and 24 constitute a system of two differential equations in two variables H and S which can be solved for the paths and steady state solutions of those two variables.

A.17.3.3.1 Worked example for assumed functions and parameter values

We assume that dB/dH = P(H) = a – bH, with a = 250 and b = 2000. As before, the biological growth equation, the fishery production function, and fishing costs are as specified in the right-hand column of Table 17. Therefore C(H,S) = wH/eS. Parameter values are those given in Table 17.2.

The current-value Hamiltonian and the first order conditions (omitting time subscripts for simplicity except where necessary) are

$$L(H,S) = \int_0^h (a - bH)dh - \frac{wH}{eS} + p(gS - \frac{gS^2}{S_{MAX}} - H) \qquad (25)$$

$$\text{(i)}\ \frac{\partial L}{\partial H} = 0 = a - bH - \frac{w}{eS} - p \qquad (26)$$

$$\text{(ii)}\ \dot{p} = ip - p(g - \frac{2gS}{S_{MAX}}) - \frac{wH}{eS^2} \qquad (27)$$

$$\text{(iii)}\ \dot{S} = gS - \frac{gS^2}{S_{MAX}} - H \qquad (28)$$

Differentiating 26 with respect to time gives

$$\dot{p} = -b\dot{H} + \frac{w}{eS^2}\dot{S} \qquad (29)$$

Substituting 29 into 27 gives

$$-b\dot{H} + \frac{w}{eS^2}\dot{S} = ip - p(g - \frac{2gS}{S_{MAX}}) - \frac{wH}{eS^2}$$

which solving for $\dot{H}$ gives

$$\dot{H} = \frac{1}{-b}\left((i - g + \frac{2gS}{S_{MAX}})p - \frac{wH}{eS^2} - \frac{w}{eS^2}\dot{S} \right)$$

which after substituting for $\dot{S}$ from (28), p from (26) and simplifying gives

$$\dot{H} = \frac{1}{-b}\left((i - g + \frac{2gS}{S_{MAX}})[a - bH - \frac{w}{eS}] - \frac{wg}{eS}(1 - \frac{S}{S_{MAX}}) \right) \quad (30)$$

Equations (18) and (30) constitute the required system of 2 differential equations in the variables S and H. From these, we can (after parameter values have been substituted in) obtain solution values for S and H. The dynamic trajectories may be shown graphically by means of a phase plane diagram (to be explained below).

The steady-state solutions to the system are found where $\dot{S} = 0$ and $\dot{H} = 0$. That is, we solve simultaneously

$$0 = gS - \frac{gS^2}{S_{MAX}} - H$$

$$0 = \frac{1}{-b}\left((i - g + \frac{2gS}{S_{MAX}})[a - bH - \frac{w}{eS}] - \frac{wg}{eS}(1 - \frac{S}{S_{MAX}}) \right)$$

which, with the baseline parameter values, yield S* = 0.449 and H* = 0.0371. Then P* = a – bH* = 250 – 2000(0.0371) = 175.8. Computations can be found in the Maple file *Chapter 17.mws*, in which the phase plane diagram showing the system dynamics is generated.

It is of interest to note that if the intercept of the inverse demand function has been a = 273.26 rather than a = 250, the solution yields S* = 0.4239, H* = 0.0366 and P* = 200, identical results to those obtained for the exogenously fixed price model variant with P = 200.

A.17.3.4 Private present value-maximising harvesting in a monopolistic market with enforceable property rights and a downward sloping demand curve for fish.

For a monopolistic industry, R = P(H)H and so

$$\frac{dR}{dH} = \frac{d}{dH}\left(P(H)H\right) = \frac{dP}{dH}H + P(H)$$

and so dR/dH < P, as dP/dH <0. Hence the monopolistic profit-maximising solution is different from the competitive market (and the socially efficient) solution. To see precisely how it is different, look at the net price first-order condition for the two alternative market structures.

Monopoly:

$$P(H) = \frac{\partial C}{\partial H} + p - \frac{dP}{dH}H$$

Competition:

$$P(H) = \frac{\partial C}{\partial H} + p$$

As dP/dH is negative, this implies that the market price is higher in monopolistic than in competitive markets, and so less is harvested and sold. One would expect the stock size to be larger in monopolistic than in competitive markets.