Appendix 18.1 Mathematical derivations

(1) The present-value-maximising first-order condition derived from equation 18.7

$$\Pi = \frac{(pS_T \mathrm{e}^{-iT} - k}{1 - \mathrm{e}^{-iT}} \tag{18.7}$$

$$\Pi = \frac{(pS_T - k\mathrm{e}^{iT}}{\mathrm{e}^{-iT} - 1}$$

$$\Pi = \frac{(pS_T - k}{\mathrm{e}^{-iT} - 1} - k \tag{18.7b}$$

Differentiating 18.7b with respect to T and setting the result equal to zero gives

$$\frac{\mathrm{d}\Pi}{\mathrm{d}T} = -1(pS_T - k)(e^{iT} - 1)^2 i\mathrm{e}^{iT} + (e^{iT} - 1)^{-1} p\frac{dS_T}{dT} = 0$$

$$\frac{\mathrm{d}\Pi}{\mathrm{d}T} = -\frac{(pS_T - k)i\mathrm{e}^{iT}}{(\mathrm{e}^{iT} - 1)^2} + \frac{p\frac{\mathrm{d}S_T}{\mathrm{d}T}}{(e^{iT} - 1)} = 0$$

$$\frac{(pS_T - k)i\mathrm{e}^{iT}}{(\mathrm{e}^{iT} - 1)^2} = \frac{p\frac{\mathrm{d}S_T}{\mathrm{d}T}}{(e^{iT} - 1)}$$

$$\frac{(pS_T - k)i\mathrm{e}^{iT}}{(\mathrm{e}^{iT} - 1)} = p\frac{\mathrm{d}S_T}{\mathrm{d}T}$$

$$\frac{p\frac{\mathrm{d}S_T}{\mathrm{d}T}}{(pS_T - k)} = \frac{i\mathrm{e}^{iT}}{(e^{iT} - 1)} = \frac{i}{1 - e^{-iT}}$$

equation 18.8a, as required

(2) Obtaining the alternative version of the first-order condition

$$\frac{p\dfrac{\mathrm{d}S_T}{\mathrm{d}T}}{(pS_T-k)}=\frac{i}{1-e^{-iT}} \tag{18.8a}$$

$$p\frac{\mathrm{d}S_T}{\mathrm{d}T}=\frac{i(pS_T-k)}{1-e^{-iT}}$$

$$p\frac{\mathrm{d}S_T}{\mathrm{d}T}=\frac{ipS_T-ik}{1-e^{-iT}}$$

$$p\frac{\mathrm{d}S_T}{\mathrm{d}T}=\frac{ipS_T-ipS_T\mathrm{e}^{-iT}+ipS_T\mathrm{e}^{-iT}-ik)}{1-e^{-iT}}$$

$$p\frac{\mathrm{d}S_T}{\mathrm{d}T}=\frac{ipS_T(1-\mathrm{e}^{-iT})+ipS_T\mathrm{e}^{-iT}-ik}{1-e^{-iT}}$$

$$p\frac{\mathrm{d}S_T}{\mathrm{d}T}=ipS_T+i\left(\frac{pS_T\mathrm{e}^{-iT}-k}{1-e^{-iT}}\right)$$

$$p\frac{\mathrm{d}S_T}{\mathrm{d}T}=ipS_T+i\Pi \tag{18.8b}$$

(3) The optimal rotation at i = 0

$$\frac{p\dfrac{\mathrm{d}S_T}{\mathrm{d}T}}{(pS_T-k)}=\frac{i}{1-e^{-iT}} \tag{18.8a}$$

By l'Hopital's rule:

$$\lim_{i\to 0}\frac{i}{1-e^{-iT}}=\frac{1}{T}$$

L'Hopital's rule:

Suppose $f(a)=g(a)=0$, $f'(a)$ and $g'(a)$ exist, and $g'(a)\neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}$$

Hence, as i goes to zero in the limit we have

$$\frac{p\dfrac{\mathrm{d}S_T}{\mathrm{d}T}}{(pS_T-k)}=\frac{1}{T} \text{ or}$$

$$p\frac{dS_T}{dT}=\frac{(pS_T-k)}{T}$$

This implies that cutting will be done at an age T which maximises the average economic yield, $(pS_T - k)/T$. This point is illustrated in Figure 18.7 (using a diagram that is an adaptation of one used in Clark, 1990, p. 273). At the tangency point vertically above T = 99, the average economic yield (given by the slope of the ray from the origin) is at its maximum.

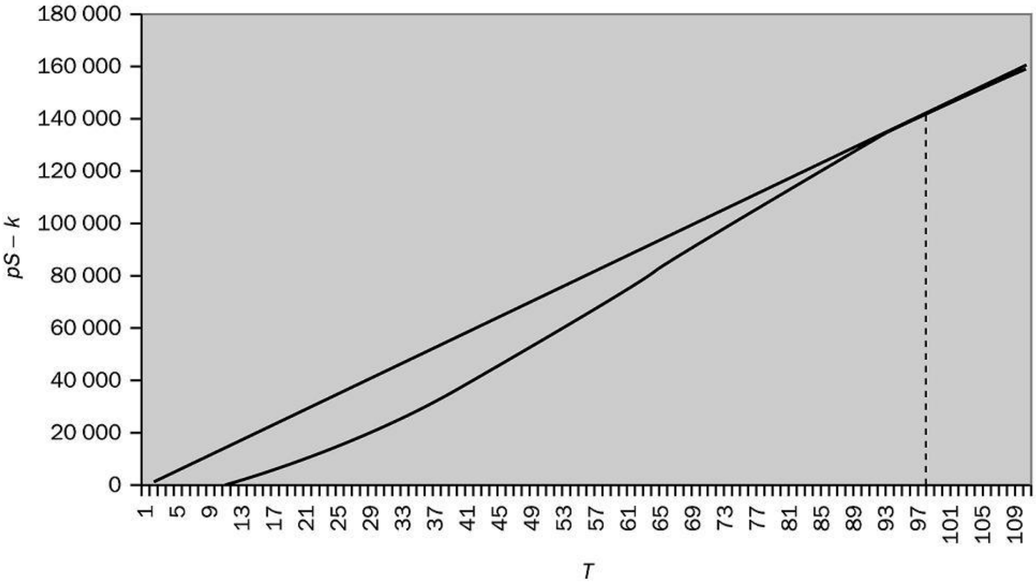

Figure 18.7 The optimising rotation at i = 0: maximising average economic yield