# Appendix 18.2 The length of a forest rotation in the infinite-rotation model: some comparative statics

In this appendix, we deal with the same infinite rotation model (and all the associated assumptions) as given in Section 18.3.

## The optimum rotation length, T

Our analysis has found that the optimal rotation length, $T^*$, is the value of $T$ which maximises $\Pi$ (the present value of an infinite cycle of rotations) in the equation

$$\Pi = [pS_T \mathrm{e}^{-iT} - k] + \mathrm{e}^{-iT}\Pi \tag{18.6}$$

or equivalently, in

$$\Pi = \frac{pS_T \mathrm{e}^{-iT} - k}{1 - e^{-iT}} \tag{18.7}$$

It is clear from inspection of either equation that the present value of profits for any rotation length, and implicitly $T^*$, will depend on the values of the parameters $p$, $k$ and $i$, and on the timber growth function $S(t)$.

We know from standard optimisation theory that maximisation of $\Pi$ requires that $T$ be chosen so that the first derivative of $\Pi$ with respect to $T$ is equal to zero. Using this first-order condition, we obtained, after some rearrangement, the equation:

$$p\frac{\mathrm{d}S_T}{\mathrm{d}T} = ipS_T + i\Pi \tag{18.8b}$$

It will be useful for what follows to put this into the alternative form

$$p\frac{\mathrm{d}S_T}{\mathrm{d}T} - ipS_T - i\Pi = 0 \qquad (18.8^*)$$

## Comparative statics

We are now ready to do the comparative statics. First, it will be useful to give some general perspective on how this is to be done. As you will see, the key thing we shall need to use is the total differential of a function.

Looking again at equation 18.7, you see will that the value function being maximised is of the general form

$$\Pi = \Pi\ (i, p, S, T, k)$$

Optimisation requires that the first derivative of Π with respect to $T$ is zero at $T = T^*$. That is,

$$\frac{\partial\Pi(i, p, S, T, k)}{\partial T^*} = f(i, p, S, T, k) = 0$$

Now imagine asking what must happen to the *optimal* rotation period, $T^*$, if one of the other determinants of Π changes.[8] Suppose, for example, that the other determinant is the discount rate, $i$. To get the answer, we obtain the *total differential* of the first-order condition:

$$\frac{\partial f}{\partial T} dT + \frac{\partial f}{\partial i} di$$

[8] In the rest of this note, as in the text of the book itself, for notational simplicity we omit the * symbol being used on any variable to denote its optimal value. Whether the optimal value or any other value is being referred to should be clear from the context.

Note that this total differential is examining how the function changes as the variables in which we are interested change. There are two variables we are allowing to change here: the interest rate, $i$, and the forest rotation period, $T$. So our total differential contains two 'change' terms, d$i$ and d$T$.

However, the value of the function will *not* change here. Although $\Pi$ itself will probably change, the *slope of* $\Pi$ *with respect to T* (which is the function we are examining here) must remain at zero at any optimising value of $T$. So the value of this differential must be zero, given that $T$ and $i$ are being chosen simultaneously to maximise present value. Therefore

$$\frac{\partial f}{\partial T}\mathrm{d}T + \frac{\partial f}{\partial i}\mathrm{d}i = 0$$

or

$$\frac{\partial f}{\partial T}\mathrm{d}T = -\frac{\partial f}{\partial i}\mathrm{d}i$$

Inspection of this shows that if the terms $\partial f/\partial T$ and $-\partial f/\partial i$ are each positive (or are each negative), then d$T$/d$i$ must then be positive. *Be careful to take account of the negative sign here*. Similarly, if the terms $\partial f/\partial T$ and $-\partial f/\partial i$ are of opposite sign, then d$T$/d$i$ must then be negative. This gives us a method for establishing our comparative static results, as we show below. And if we want to find the effect of changes in other variables (such as planting costs) on the optimal rotation, it is simply a matter of having the appropriate variables in the total differential.

Before obtaining our results, it will be convenient to write equation 10.8* in yet one more form:

$$\frac{\mathrm{d}S_T}{\mathrm{d}T} - iS_T = \frac{i}{p}\Pi \tag{18.8.1}$$

remembering that

$$\Pi = \frac{pS_T\mathrm{e}^{-iT} - k}{1 - e^{-iT}} \qquad \text{from equation 18.7}$$

### A change in the discount rate, i

Totally differentiate equation 18.8.1 with respect to *T* and *i*, to obtain the total differential

$$\frac{\partial\left[\frac{\mathrm{d}S_T}{\mathrm{d}T}-iS_T\right]}{\partial T}\mathrm{d}T-S_T\mathrm{d}i=\frac{\Pi}{p}\mathrm{d}i+\frac{i}{p}\left[\frac{\partial\Pi}{\partial i}\right]\mathrm{d}i \qquad (18.8.2)$$

Noting that

$$\frac{\partial\Pi}{\partial i}=-\frac{pST\mathrm{e}^{-iT}}{1-e^{-iT}}-\frac{(pS\mathrm{e}^{-iT}-k)T\mathrm{e}^{-iT}}{(1-e^{-iT})^2} \qquad (18.8.3)$$

we can substitute 18.8.3 into 18.8.2 and simplify to give

$$\frac{\partial\left[\frac{\mathrm{d}S_T}{\mathrm{d}T}-iS_T\right]}{\partial T}\mathrm{d}T=\left(\frac{\Pi}{p}+S_T\right)\left[1-\frac{iT}{\mathrm{e}^{iT}-1}\right]\mathrm{d}i \qquad (18.8.4)$$

## Hint

Doing differentiation by hand can be difficult. Try learning and using Maple or Mathematica to do calculus for you. See the Maple file *Timber.mws* to see how easy this is. One way of learning these packages is to read Ron Shone's *Economic Dynamics* (1997). The web site associated with that book is full of Maple and Mathematica example files to take you through the package. The book is also a superb account of economic dynamics.

Under reasonable assumptions about the form of the timber biological growth function, the coefficient associated with d*T* is negative. (You could check this in the Excel spreadsheet.) It should also be clear that the coefficient associated with the term d*i* is positive. It follows from the reasoning given above that d*T*/d*i* must be negative.

## A change in planting costs, *k*

As we are now investigating the effect of a change in $k$ on the optimal value of $T$, the total differential required now contains terms in d$T$ and d$k$. Totally differentiating equation 18.8.1, we obtain:

$$\frac{\partial\left[\frac{\mathrm{d}S_T}{\mathrm{d}T} - iS_T\right]}{\partial T}\mathrm{d}T = \frac{i}{p}\frac{\partial\Pi}{\partial k}\mathrm{d}k = \frac{i}{p}\left[\frac{-\mathrm{e}^{iT}}{\mathrm{e}^{iT}-1}\right]\mathrm{d}k \tag{18.8.5}$$

Inspection of equation 18.8.5 shows that the coefficients associated with d$T$ and d$k$ are negative, and therefore d$T$/d$k$ is positive. A *fall* in planting/replanting costs would then result in a shortening of the optimal rotation length.

## A change in timber net price, *p = P – c*

Totally differentiate equation 18.8.1 with respect to $T$ and $p$:

$$\frac{\partial\left[\frac{\mathrm{d}S_T}{\mathrm{d}T} - iS_T\right]}{\partial T}\mathrm{d}T = \frac{-i\Pi}{p^2}\mathrm{d}p + \frac{i}{p}\left[\frac{\partial\Pi}{\partial p}\right]\mathrm{d}p \tag{18.8.6}$$

and so

$$\frac{\partial\left[\frac{\mathrm{d}S_T}{\mathrm{d}T} - iS_T\right]}{\partial T}\mathrm{d}T = \frac{i}{p}\left[\frac{-pS_T + \mathrm{e}^{iT}k}{(\mathrm{e}^{iT}-1)p} + \frac{S_T}{\mathrm{e}^{iT}-1}\right]\mathrm{d}p$$

which simplifies to

$$\frac{\partial\left[\frac{\mathrm{d}S_T}{\mathrm{d}T}-iS_T\right]}{\partial T}\mathrm{d}T=\frac{ik}{p^2}\left[\frac{\mathrm{e}^{iT}}{\mathrm{e}^{iT}-1}\right]\mathrm{d}p \tag{18.8.7}$$

As the coefficient on d$T$ is negative and the coefficient with d$p$ is positive, it follows that d$T$/d$p$ is negative. This implies that d$T$/d$P$ < 0 and d$T$/d$c$ > 0.