Appendix 19.1 National income, the return on wealth, Hartwick's rule and sustainable income

Written by: Jack Pezzey, Environment Department, University of York

In this appendix and the next we use the dot notation for derivatives with respect to time so as to reduce clutter in the exposition. For the same reason we omit the *t* subscript when referring to derivatives such as marginal utilities and marginal products, writing, for example, U_C rather than U_{Ct} for $\partial U_t/\partial C_t$.

National income and the return on wealth

We begin here with the simplest optimal growth model where there is a single produced good which may be consumed or added to the stock of reproducible capital, which does not depreciate, and where the environment affects neither utility nor production. The problem to be considered is

$$\text{Max}\int_0^\infty U\left(C_t\right)e^{-\rho t}dt \text{ subject to } \dot{K}_t = Q\left(K_t\right) - C_t \tag{19.44a}$$

for which the current-value Hamiltonian is

$$H_t = U\left(C_t\right) + w_t\left(Q\{K_t\} - C_t\right)$$

where the necessary conditions are

$$\partial H_t / \partial C_t = U_C - w_t = 0 \qquad (19.44b)$$

$$\dot{w}_t - \rho w_t = -\partial H_t / \partial K_t = -w_t Q_K \qquad (19.44c)$$

Replacing $Q(K_t) - C_t$ by D_t, we can write the maximised value of the Hamiltonian as

$$H_t^* = U\left(C_t\right) + w_t \dot{K}_t$$

which by equation 19.44b can also be written as

$$H_t^* = U\left(C_t\right) + U_C \dot{K}_t \qquad (19.44d)$$

where C_t and D_t are the *optimal* values for the maximisation problem 19.44a. We can interpret H_t^* as instantaneous national income measured in units which are utils. Observe that the right-hand side of 19.44d is the current flow of utility plus the value of the change in the capital stock measured in units which reflect its contribution to future, maximised, utility.

This interpretation of H_t* can be further supported by noting that if we linearise the utility function so that $U(C_t) = U_C C_t$, we can write equation 19.44d as

$$H_t^* = U_C C_t + U_C \dot{K}_t$$

so that

$$H_t^* / U_C = C_t + I_t$$

where I_t for investment is D_t.[i] Given the assumption that K does not depreciate, the right-hand side here is just the usual expression for net national income and if we use NDP for this, we have

$$\text{NDP}_t = H_t^* / U_C = C_t + I_t \qquad (19.44e)$$

Now introduce the use of a non-renewable natural resource into production, as in the simple exhaustible resource depletion problem considered in Chapter 14, and see Appendix 14.2 there. We saw there that for

$$\text{Max} \int_0^\infty U(C_t) e^{-\rho t} dt \text{ subject to } \dot{S}_t = -R_t$$

and

$$\dot{K}_t = Q(K_t R_t) - C_t$$

the current-value Hamiltonian is

$$H_t = U(C_t) + P_t(-R_t) + w_t(Q\{K_t R_t\} - C_t)$$

so that the necessary conditions are

$$\partial H_t/\partial C_t = U_C - w_t = 0 \quad (19.45a)$$

$$\partial H_t/\partial R_t = -P_t + w_t Q_R = 0 \quad (19.45b)$$

$$\dot{w}_t - \rho w_t = -\partial H_t/\partial K_t = -w_t Q_K \quad (19.45c)$$

$$\dot{P}_t - \rho P_t = -\partial H_t/\partial S_t = 0 \quad (19.45d)$$

These can respectively be written as

$$w_t = U_C \quad (19.45e)$$

$$\dot{P}_t/P_t = \dot{w}_t/w_t + \dot{Q}_R/Q_R \quad (19.45f)$$

$$Q_K = \rho - \dot{w}_t/w_t \quad (19.45g)$$

$$\dot{P}_t/P_t = \rho \quad (19.45h)$$

Note that equation 19.45f comes from differentiating equation 19.45b with respect to time, dividing both sides by *P*, and then substituting for *P* on the right-hand side from equation 19.45b. For constant consumption, equations 19.45g and 19.45a then give, since U_C constant means $G_C = 0$,

$$Q_K = \rho - \dot{U}_C/U_C = \rho \quad (19.45i)$$

Finally, equations 19.45g, 19.45f and 19.45h together give

$$\dot{Q}_R/Q_R = Q_K \quad (19.45j)$$

as an alternative statement of the Hotelling rule which is used later.

However, our main interest here is in the Hamiltonian itself. Using the equations of motion, the maximised Hamiltonian can be written

$$H_t^* = U(C_t) + w_t \dot{K}_t + P_t \dot{S}_t \tag{19.45k}$$

and proceeding to linearise the utility function as for the simple model above, this becomes

$$H_t^*/U_C = C_t + I_t + (P_t/w_t)\ \dot{S}_t$$

which by equation 19.45b, and substituting for F_t from the equation of motion, can be written

$$H_t^*/U_C = C_t + I_t - Q_R R_t = \text{NDP}_t - Q_R R_t = \text{EDP}_t \tag{19.45l}$$

where we still use NDP_t for national income as conventionally measured, and now introduce EDP_t to refer to national income as properly measured given the use of the natural resource in production. According to equation 19.45l, EDP_t is NDP_t minus the rent $Q_R R_t$ arising in the extraction of the resource, where that rent is the measure of the depreciation of the asset which is the resource stock. Depreciation is the amount extracted valued at the marginal product of the resource, which in this model with costless extraction is the unit rent.

We could write equation 19.45l as

$$\text{EDP}_t = C_t + I_t - (P_t/w_t)R_t \tag{19.45m}$$

where P_t/w_t is the relative (to the price of the numeraire commodity which is the consumption/capital good) price of the extracted resource, which in a model with costless extraction is the same as the price of the resource *in situ*. As we saw in Chapter 14, in a fully competitive economy the relative price of the resource would move over time as required by the necessary conditions for the maximisation of discounted utility. This is the basis for taking equation 19.45m as a guide to how the conventional measure of national income should be adjusted to account for non-renewable-resource depletion in an actual economy. The assumption is, that is, that actual economies should be treated as if they were fully competitive economies. Recall from Chapter 14 that the conditions characterising a fully competitive economy are strong.

Now, note that we have $H_t^* = H_t^*(K, S, w, P)$ and consider the differentiation of H_t^* with respect to time. We have

$$\dot{H}_t^* = (\partial H^*/\partial K)\,\dot{K}_t ++ (\partial H^*/\partial S)\,\dot{S}_t + (\partial H^*/\partial w)\,\dot{w} + (\partial H^*/\partial P)\text{E}_t$$

Using equation 19.45c for $(\partial H^*/\text{d}K)$, equation 19.45d for $(\partial H^*/\text{d}S)$, and $\partial H^*/\partial w = \text{D}_t$ and $\partial H^*/\partial P = \text{F}_t$ from equation 19.45k, we get

$$\dot{H}_t^* = \rho w_t \dot{K}_t + \rho P_t \dot{S}_t = \rho\left(w_t \dot{K}_t + P_t \dot{S}_t\right) \tag{19.45n}$$

From equation 19.45l, using $G_C = 0$ and equation 19.45a,

$$(\mathrm{d}/\mathrm{d}t)\,\mathrm{EDP}_t = \dot{H}_t^*/U_C = \dot{H}_t^*/w_t \qquad (19.45o)$$

Combining equations 19.45n, 19.45o and 19.45l then gives

$$(\mathrm{d}/\mathrm{d}t)\mathrm{EDP}_t = \rho[\dot{K}_t + (P_t/w_t)\dot{S}_t] = \rho(I_t - Q_R R_t)$$

$$= \rho(\mathrm{EDP}_t - C_t) \qquad (19.45p)$$

Using equation 19.45i the solution of this differential equation in EDP_t can be shown to be

$$\mathrm{EDP}_t = \rho W_t = Q_K W_t \qquad (19.45q)$$

where W_t is the economy's wealth at time *t*, as defined by the present discounted value of consumption from time *t* onwards:

$$W_t \int_t^\infty C_t e^{\rho(\tau - t)} d\tau \qquad (19.45r)$$

This is a rewritten version of a famous result due to Weitzman (1976). Since the marginal product of capital Q_K is the interest rate in a competitive economy, this is the basis for the interpretation of EDP_t, which is properly measured national income, as the 'return' (at the going rate of interest) on the economy's total stock of wealth. If, moreover, there are constant returns to scale in the economy's production function $Q(K_t, R_t)$, then wealth could be interpreted not just as the present discounted value of future consumption, but also as the value today in consumption units of the economy's productive assets:

$$W_t = K_t + Q_R S_t \qquad (19.45s)$$

Hartwick's rule and sustainable income

There is a powerful appeal in the idea that income is the interest earned on wealth, and that consuming exactly one's income – no more and no less – should be sustainable for ever. Is that what equation 19.45q above is saying for an economy? That is, are there circumstances in which EDP is *sustainable* national income? The answer is 'yes', but only in a case of severely restricted practical value, which unfortunately is often misunderstood in the literature, creating much confusion on this topic. What can be said is the following. *If* optimal consumption happens to equal EDP *always*, and *if* constant consumption is physically feasible – note the 'if' and 'always' caveats – then both consumption and EDP will be constant for ever; or in other words, sustainable. The proof of this is as follows. We start from the first 'if', by assuming

$$C_t = Q(K_t, R_t) - \dot{K}_t = \text{EDP}_t \tag{19.46a}$$

always, which from equation 19.45l means that

$$\dot{K}_t = Q_R R_t \tag{19.46b}$$

always. The rule in equation 19.46b, which says 'investment in reproducible capital is always equal to resource rents', is *Hartwick's rule* (after John Hartwick who discovered it in 1977, see Hartwick, 1977). Taking the time derivative of consumption from equation 19.46a then gives

$$\begin{aligned}\dot{C}_t &= Q_K \dot{K}_t + Q_R \dot{R} - \ddot{K}_t \\ &= Q_K \dot{K}_t + Q_R \dot{R}_t - \left(\dot{R}_R R_t + Q_R \dot{R}_t \right)\end{aligned}$$

using Hartwick's rule. Note that without the 'always', we could not have taken and used the time derivative of $\dot{K} = Q_R R_t$ to substitute for $\ddot{K}$. Using Hotelling's rule written as equation 19.45j then gives

$$\dot{C} = Q_K \dot{K}_t - Q_R Q_K R_t$$

which using Hartwick's rule again is

$$\dot{C} = 0$$

that is, constant consumption.

However, *there is no reason why optimal consumption should equal EDP*, and hence why Hartwick's rule (and hence constant consumption) should hold on an optimal path. Indeed, constant consumption may not even be feasible. In general, optimal consumption will rise or fall over time, and will be more or less than EDP at any point in time; and hence capital investment will be more or less than resource rents at any time.

However again, what happens if the economy is constrained to follow Hartwick's rule? (We will not show how this constraint might be achieved, but one way would be to introduce a macroeconomically significant policy of tax incentives to invest more.) It turns out that consumption will indeed be constant, but the constraint policy will force the economy off the optimal path: equation 19.44a will no longer be maximised. As a result, both prices and quantities on a constant-consumption path will generally be different from their optimal values. Nevertheless, some sort of present-value function, using a different utility discount factor (say $\lambda(t)$ instead of e −

ρt in equation 19.44a) will still be maximised on the highest possible constant consumption path. ρ will be replaced throughout by $-\dot{\lambda}/\lambda$, but (as the reader can readily check) the form of Hotelling's rule as equation 19.45j, used above in the proof of constant consumption, will be unchanged.

At any point in time, aggregate investment, defined as $\dot{K}_t - Q_R R_t$, is therefore an unreliable indicator of an economy's sustainability. The optimal path of an economy may be unsustainable at time *t*, and yet aggregate investment may be positive then. Or, if there is technical progress in production (which we have ignored above), it turns out that the economy can be sustainable at *t* even though aggregate investment is negative then. And the problem remains even if one tries to use the 'right' price, Q_R, which would apply on the constant-consumption path, because the quantities of investment $\dot{K}_t$ and resource depletion R_t will still be wrong. Trying to use Hartwick's rule ('invest resource rents') as either a policy prescription to achieve sustainability, or as the basis for 'sustainability accounting', therefore faces a fundamental chicken-and-egg problem. The rule works only if sustainability, in the form of constant consumption, and hence both sustainability prices and quantities, have already been achieved! Moreover, achieving constant consumption when it is not the optimal path raises an awkward political question: which matters more, sustainability or optimality?

Another frequent misunderstanding in the literature is that keeping consumption constant means keeping wealth constant. The trouble is that wealth can be defined in different ways. If wealth is defined as the time integral of aggregate investment, then obviously it remains constant on a constant-consumption path, thanks to Hartwick's rule. But if wealth is defined as earlier, as the present value of future consumption or the aggregate value of current assets, then wealth need not be constant. Indeed, in the best-known example of constant consumption with non-renewable resources, discovered in 1974 by Robert Solow (Solow, 1974a), wealth must be *rising* for ever to keep consumption constant. Intuitively, what is happening is that in the Cobb–Douglas production function Solow uses, $Q\left(K_{t,}R_t\right) = K_t^{\alpha} R_t^{\beta}$, the marginal product of capital investment Q_K is falling because an ever-rising stock of capital K_t has to be combined with an ever-shrinking resource flow R_t. So, by equation 19.45q, wealth W_t must be rising if the product $Q_K W_t$ = EDP$_t$ = C_t is to be constant.

---

[i] Strictly speaking, linearising the utility function makes the Hamiltonian linear in consumption, and so gives rise to what are known as ‘corner’ or ‘non-interior’ solutions to the optimal control problem, for which equations like 19.41a and 19.41b do not hold. However, in common with much of the relevant literature, we will overlook this technicality.