Appendix 3.1 The Lagrange multiplier method of solving constrained optimisation problems

Suppose we have the following problem in which a function of three variables is to be maximised subject to two constraints:

$$\max \quad f(x_1, x_2, x_3)$$

subject to

$$g(x_1, x_2, x_3) = 0$$
$$h(x_1, x_2, x_3) = 0$$

To obtain a solution to this problem, we begin by writing the Lagrangian (L) for the problem. The Lagrangian consists of two components. The first of these is the function to be maximised. The second contains the constraint functions (but without being set equal to zero), with each constraint being preceded by a separate Lagrange multiplier variable. The Lagrangian is the sum of all these terms.

So in this case the Lagrangian, L, is

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = f(x_1, x_2, x_3) + \lambda_1 g(x_1, x_2, x_3) + \lambda_2 h(x_1, x_2, x_3) \qquad (3.28)$$

in which λ_1 and λ_2 are two Lagrange multipliers (one for each constraint) and the term $L(x_1, x_2, x_3, \lambda_1, \lambda_2)$ signifies that we are now to regard the Lagrangian as a function of the original choice variables of the problem and of the two Lagrange multiplier variables.

We now proceed by using the standard method of unconstrained optimisation to find a maximum of the Lagrangian with respect to x_1, x_2, x_3, λ_1 and λ_2. The necessary first-order conditions for a maximum are

$$\frac{\partial L}{\partial x_1} = f_1 + \lambda_1 g_1 + \lambda_2 h_1 = 0$$

$$\frac{\partial L}{\partial x_2} = f_2 + \lambda_1 g_2 + \lambda_2 h_2 = 0$$

$$\frac{\partial L}{\partial x_3} = f_3 + \lambda_1 g_3 + \lambda_2 h_3 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = g(x_1, x_2, x_3) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = h(x_1, x_2, x_3) = 0$$

where

$$f_i = \frac{\partial f}{\partial x_i},\ g_i = \frac{\partial g}{\partial x_i},\ h_i = \frac{\partial h}{\partial x_i} \text{ for } i = 1,2,3$$

These are solved simultaneously to obtain solution values for the choice variables.

The second-order conditions for a maximum require that the following determinant be positive:

$$\begin{vmatrix} L_{11} & L_{12} & L_{13} & g_1 & h_1 \\ L_{21} & L_{22} & L_{23} & g_2 & h_2 \\ L_{31} & L_{32} & L_{33} & g_3 & h_3 \\ g_1 & g_2 & g_3 & 0 & 0 \\ h_1 & h_2 & h_3 & 0 & 0 \end{vmatrix}$$

where

$$L_{ij} = \frac{\partial^2 L}{\partial x_i \partial x_j}$$

For a constrained maximum, a sufficient second-order condition can be stated in terms of the signs of the bordered principal minors of the Hessian matrix. Details of this condition are beyond the scope of this appendix, but can be found on page 386 of Chiang (1984).

The Lagrange multiplier method is widely used in economic analysis generally, and in resource and environmental economics particularly. This is because the Lagrange multipliers have a very useful interpretation in analysis. They are 'shadow prices' on the constraints. In the case of a constrained maximisation problem as considered above, this means that the value of a Lagrange multiplier tells us what the effect on the maximised value of the objective function would be for a small - strictly an infinitesimal (or vanishingly small) - relaxation of the corresponding constraint would be. The same interpretation arises in constrained minimisation problems. Clearly, this is very useful information. We now illustrate this interpretation using a simple example from an environmental economics context. We consider the problem of the least-cost allocation across sources of a reduction in total emissions, which problem will be discussed at length in Chapter 6.

Suppose that there are two firms, 1 and 2, where production gives rise to emissions M_1 and M_2. In the absence of any regulation of their activities, the firms' profit maximising emissions levels are 1000 and 7500 tonnes respectively. The firms can cut back, or abate, emissions, but so doing reduces profits and is costly. Further, abatement costs as a function of the level of abatement vary as between the two firms. The abatement cost functions are

$$C_1 = 10A_1 + 0.01A_1^2 = 10(1000 - M_1) + 0.01(1000 - M_1)^2 \qquad (3.29.a)$$

$$C_2 = 5A_2 + 0.001A_2^2 = 5(7500 - M_2) + 0.001(7500 - M_2)^2 \qquad (3.29.b)$$

where A_1 and A_2 are the levels of abatement, the amount by which emissions in some regulated situation are less than they would be in the absence of regulation.

The regulatory authority's problem is to determine how a reduction in total emissions from 8500 = (1000 + 7500) to 750 tonnes should be allocated as between the two firms. Its criterion is the minimisation of the total cost of abatement. The problem is, that is, to find the levels of A_1 and A_2, or equivalently of M_1 and M_2, which minimise C_1 plus C_2 given that M_1 plus M_2 is to equal 750. Formally, using M_1 and M_2 as the control or choice variables, the problem is

$\min C_1 + C_2$

subject to

$M_1 + M_2 = 750$

Substituting for C_1 and C_2 from equations 3.29.a and 3.29.b, and writing the Lagrangian, we have

$$L = 113750 - 30M_1 - 20M_2 + 0.01M_1^2 + 0.001M_2^2 + \lambda[750 - M_1 - M_2] \qquad (3.30)$$

where the necessary conditions are

$$\frac{\partial L}{\partial M_1} = -30 + 0.02M_1 - \lambda = 0 \qquad (3.31.a)$$

$$\frac{\partial L}{\partial M_2} = -20 + 0.002M_2 - \lambda = 0 \qquad (3.31.b)$$

$$\frac{\partial L}{\partial \lambda} = 750 - M_1 - M_2 = 0 \qquad (3.31.c)$$

Eliminating λ from equations 3.31.a and 3.31.b gives

$$-30 + 0.02M_1 = -20 + 0.002M_2 \qquad (3.32)$$

and solving equation 3.31.c for M_1 gives

$$M_1 = 750 - M_2 \qquad (3.33)$$

so that substituting equation 3.33 into equation 3.32 and solving leads to M_2 equal to 227.2727, and then using equation 3.33 leads to M_1 equal to 522.7272. The corresponding abatement levels are A_1 equal to 477.2728 and A_2 equal to 7272.7273. Note that firm 2, where abatement cost are much lower than in firm 1, does proportionately more abatement.

Now, in order to get the allocation of abatement across the firms we eliminated λ from equations 3.31.a and 3.31.b. Now that we know M_1 and M_2 we can use one of these equations to calculate the value of λ as -19.5455. This is the shadow price of pollution, in the units of the objective function, which are here £s, when it is constrained to be a total emissions level of 750 tonnes. This shadow price gives what the impact on the minimised total cost of abatement would be for a small relaxation of the constraint that is the target regulated level of total emissions. To see this, we can compare the minimised total cost for 750 tonnes and 751 tonnes. To get the former, simply substitute M_1 = 522.7272 and M_2 = 227.2727 into

$$C = 113750 - 30M_1 - 20M_2 + 0.01M_1^2 + 0.001M_2^2 \qquad (3.34)$$

to get 96306.819. To get the latter, replace 750 by 751 in equation 3.31.c, and then solve equations 3.31.a, 3.31.b and 3.31.c as before to get M_1 = 522.8181 and M_2 = 228.1818, which on substitution into equation 3.34 for C gives the total cost of abatement to 751 tonnes as 96287.272. The difference between 96287.272 and 96306.819 is - 19.547, to be compared with the value for λ calculated above as -19.5455. The two results do not agree exactly because strictly the value for λ is for an infinitesimally small relaxation of the constraint, whereas we actually relaxed it by one tonne. Note that the shadow price is £s per tonne, so that the Lagrangian is in the same units as the objective function, £s.

It is not always necessary to use the method of Lagrange multipliers to solve constrained optimisation problems. Sometimes the problem can be solved by substituting the constraint(s) into the objective function. This is the case in our example here. We want to find the values for M_1 and M_2 which minimise C as given by equation 3.34, given that $M_1 + M_2 = 750$. That means that $M_1 = 750 - M_2$, and if we use this to eliminate M_1 from equation 3.34, after collecting terms we get

$$C = 96875 - 5M_2 + 0.011M_2^2 \qquad (3.35)$$

where the necessary condition for a minimum is

$$\frac{\partial C}{\partial M_2} = -5 + 0.022M_2$$

which solves for $M_2 = 227.2727$, and from $M_1 = 750 - M_2$ we then get M_1 as 522.7273.

Even where solution by the substitution method is possible, using the method of Lagrange multipliers is generally preferable in that it provides extra information on shadow prices, with the interpretation set out above. In fact, these shadow prices often are useful in a further way, in that they have a natural interpretation as the prices that could be used to actually achieve a solution to the problem under consideration. Again, this can be illustrated with the emissions control example. If the regulatory authority had the information on the abatement cost functions for the two firms, it could do the calculations as above to find that for the least cost attainment of a reduction to 750 tonnes firm 1 should be emitting 522.7272 tonnes and firm 2 227.2727 tonnes. It could then simply instruct the two firms that these were their permissible levels of emissions.

Given that it can also calculate the shadow price of pollution at its desired level, it can achieve the same outcome by imposing on each firm a tax per unit emission at a rate which is the shadow price. A cost minimising firm facing a tax on emissions will abate to the point where its marginal abatement cost is equal to the tax rate. With t for the tax rate, and M* for the emissions level in the absence of any regulation or taxation, total costs are

$$C(A) + tM = C(A) + t(M^* - A)$$

so that total cost minimisation implies

$$\frac{\partial C}{\partial A} - t = 0$$

or

$$\frac{\partial C}{\partial A} = t \qquad (3.36)$$

For firm 1, the abatement cost function written with A_1 as argument is

$$C_1 = 10A_1 + 0.01A_1^2 \quad (3.37)$$

so that marginal abatement costs are given by

$$\frac{\partial C_1}{\partial A_1} = 10 + 0.02A_1 \quad (3.38)$$

Using the general condition which is equation 3.36 with equation 3.38, we get

$10 + 0.02A_1 = t$

and substituting for t equal to the shadow price of pollution, 19.5455, and solving yields A_1 equal to 477.275, which is, rounding errors apart, the result that we got when considering what level of emissions the authority should regulate for in firm 1. Proceeding in the same way for firm 2, it will be found that it will do as required for the least cost allocation of total abatement if it also faces a tax of £19.5455 pounds per tonne of emissions.

When we return to the analysis of instruments for pollution control in Chapter 6 we shall see that the regulatory authority could reduce emissions to 750 by issuing tradable permits in that amount. Given the foregoing, it should be intuitive that the equilibrium price of those permits would be £19.5455.