Appendix 3.2 Social welfare maximisation

For two persons and a fixed amount of the consumption good, the problem is to choose X^A and X^B so as to maximise

$$W = W\{U^A(X^A), U^B(X^B)\}$$

subject to the constraint

$$X^A + X^B = \overline{X}$$

The Lagrangian for this problem is

$$L = W\{U^A(X^A), U^B(X^B)\} + \lambda[\overline{X} - X^A - X^B]$$

and the necessary conditions include

$$\frac{\partial L}{\partial X^A} = W_A U_X^A - \lambda = 0 \qquad (3.39.a)$$

$$\frac{\partial L}{\partial X^B} = W_B U_X^B - \lambda = 0 \qquad (3.39.b)$$

where we are using the notation for derivatives introduced in the chapter - W_A for $\partial W / \partial U^A$ and U_X^A for $\partial U^A / \partial X^A$ etc - and making the same assumptions - $W_A > 0$, $U_X^A > 0$, $U_{XX}^A < 0$ etc. From equations 3.39 here we get the condition stated as equation 3.3 in the chapter:

$$W_A U_X^A = W_B U_X^B \qquad (3.40)$$

For $W = W\{U^A, U^B\} = w_A U^A + w_B U^B$, $W_A = w_A$ and $W_B = w_B$ so that the necessary condition (3.40) here becomes

$$w_A U_X^A = w_B U_X^B \qquad (3.41)$$

and for $w_A = w_B$ this is

$$U_X^A = U_X^B \qquad (3.42)$$

which is equation 3.6 in the chapter text.

Now consider a case where the social welfare function is

$W = U^A + U^B$

and where the two individuals have identical utility functions. Specifically, suppose that

$$U^A(X^A) = 2(X^A)^{1/2} \text{ and } U^B(X^B) = 2(X^B)^{1/2}$$

so that

$$U_X^A = (X^A)^{-1/2} \text{ and } U_X^B = (X^B)^{-1/2}$$

and

$$U_{XX}^A = -\frac{1}{2}(X^A)^{-3/2} \text{ and } U_{XX}^B = -\frac{1}{2}(X^B)^{-3/2}$$

Then equation 3.42 becomes

$$(X^A)^{-1/2} = (X^B)^{-1/2}$$

so that

$$X^A = X^B = 0.5\overline{X}$$

and each individual gets half of the available X.

Now consider a case where the social welfare function is again

$W = U^A + U^B$

but the two individuals have different utility functions. Specifically, suppose that

$$U^A(X^A) = 2(X^A)^{1/2} \text{ and } U^B(X^B) = (X^B)^{1/2}$$

so that

$$U_X^A = (X^A)^{-1/2} \text{ and } U_X^B = 0.5(X^B)^{-1/2}$$

In this case, the condition which is equation 3.42 still applies, but now it gives

$$X^B = \frac{X^A}{4}$$

In section 3.4.1.2 we considered the iso-elastic utility function

$$U = \frac{X^{1-\eta}}{1-\eta} \qquad (3.43)$$

which was used when discussing utilitarian formulations of Rawlsian differentiation in favour of the worst-off. It was stated there that the relative weight accorded to increases in consumption for the worse-off individual increases as the degree of inequality between the individuals increases, and increases as the elasticity of marginal utility increases, ie as $\eta \rightarrow \infty$. To see this, we proceed as follows. For

$$W = U^A + U^B$$

with the U's given by equation 3.43 we have

$$W = \frac{\left(X^A\right)^{1-\eta}}{1-\eta} + \frac{\left(X^B\right)^{1-\eta}}{1-\eta}$$

so that

$$W_A = \frac{\partial W}{\partial X^A} = (X^A)^{-\eta} \text{ and } W_B = \frac{\partial W}{\partial X^B} = (X^B)^{-\eta}$$

and

$$\frac{W_A}{W_B} = \frac{(X^A)^{-\eta}}{(X^B)^{-\eta}} = \left(\frac{X^A}{X^B}\right)^{-\eta} = r^{-\eta} \qquad (3.44)$$

where r is the ratio of X_A to X_B and r > 1 for B the worse-off person.

From equation 3.44

$$\frac{\partial\left(\frac{W_A}{W_B}\right)}{\partial r} = -r^{-\eta-1} < 0$$

and

$$\frac{\partial\left(\frac{W_A}{W_B}\right)}{\partial \eta} = r^{-\eta-1}\ln r > 0$$

so that the relative weight assigned to A's utility decreases with r and decreases as η increases.