Appendix 4.1 Conditions for efficiency and optimality

A 4.1.1 Marginal Rates of substitution and transformation

For an individual consumer the marginal rate of utility substitution, MRUS, between two commodities is defined as the rate at which one commodity can be substituted for the other holding utility constant. For marginal changes in consumption levels, for U=U(X,Y)

$$dU = U_X dX + U_Y dY$$

where dU, dX and dY are differentials, and we are using U_X for $\partial U/\partial X$ and U_Y for $\partial U/\partial Y$, the marginal utilities. Setting dU = 0

$$0 = U_X dX + U_Y dY$$

so that

$$-U_Y dY = U_X dX$$

and

$$-dY/dX = Ux/U_Y$$

gives the MRUS as the ratio of the marginal utilities:

$$MRUS = U_X/U_Y \qquad (4.20)$$

The MRUS is the slope of the indifference curve at the relevant (X,Y) combination times -1. Since the slope is negative, the MRUS itself is positive, as it must be here given positive marginal utilities.

The marginal rate of technical substitution, MRTS, between two inputs to production is the rate at which one can be substituted for the other holding output constant. For marginal changes in input levels, for X = X(K,L)

$$dX = X_K dK + X_L dL$$

where dX, dK and dL are differentials, and where $X_K = \partial X/\partial K$ and $X_L = \partial X/\partial L$ are the marginal products of capital and labour. Setting

$$dX = 0$$

$$0 = X_K dK + X_L dL$$

$$-X_K dK = X_L dL$$

and

$-dK/dL = X_L/X_K$

gives the MRTS as the ratio of the marginal products of the labour and capital inputs:

$$MRTS = X_L/X_K \qquad (4.21)$$

The MRTS is the slope of the isoquant at the relevant (K, L) combination times -1. Since the slope is negative, the MRTS itself is positive, as it must be here given positive marginal products.

The marginal rate of transformation, MRT, refers to the rate at which one commodity can be transformed into the other by means of marginal re-allocations of one of the inputs to production. Thus MRT_K refers the effect on the output of Y when capital is, at the margin, shifted from use in the production of X to the production of Y, and MRT_L refers the effect on the output of Y when labour is, at the margin, shifted from use in the production of X to the production of Y. Consider shifting capital at the margin. For $X = X(K^X, L^X)$ and $Y = Y(K^Y, L^Y)$

$$dX = X_K dK^X + X_L dL^X \quad \text{and} \quad dY = Y_K dK^Y + Y_L dL^Y$$

where dK^X, for example, is a marginal increase/decrease in the use of capital in the production of X. The definition of the marginal rate of transformation for capital is

$$MRT_K \equiv -dY/dX$$

when there is no re-allocation of labour. Note the use of the three bar identity sign here to indicate a matter of definition. Then

$$MRT_K = -\left[\frac{Y_K dK^Y + Y_L dL^Y}{X_K dK^X + X_L dL^X}\right]$$

which for $dL^Y = dL^X = 0$ is

$$MRT_K = -\left[\frac{Y_K dK^Y}{X_K dK^X}\right]$$

and $dK^Y = - dK^X$, so

$$MRT_K = -\left[\frac{Y_K(-dK^X)}{X_K dK^X}\right]$$

where the dK^X's cancel, and taking account of the two minus signs we have

$$MRT_K = Y_K/X_K \qquad (4.22.a)$$

so that the marginal rate of transformation for capital is the ratio of the marginal products of capital in each line of production. A similar derivation, for $dK^Y = dK^X = 0$ and $dL^Y = -dL^X$, establishes that

$$MRT_L = Y_L/X_L \qquad (4.22.b)$$

A 4.1.2 Efficiency conditions

Allocative efficiency exists when it is impossible to make one individual better off without making some other individual(s) worse off. We consider an economy with two individuals each consuming two commodities, where each commodity is produced by an industry comprising two firms, each of which uses two inputs - capital and labour.[1] For such an economy, the conditions characterising allocative efficiency can be derived by considering the following constrained maximisation problem:

$$\text{Max} \quad U^A(X^A, Y^A)$$

subject to

$$U^B(X^B, Y^B) = Z$$

$$X_1(K_1^X, L_1^X) + X_2(K_2^X, L_2^X) = X^A + X^B$$

$$Y_1(K_1^Y, L_1^Y) + Y_2(K_2^Y, L_2^Y) = Y^A + Y^B$$

$$K^T = K_1^X + K_2^X + K_1^Y + K_2^Y$$

$$L^T = L_1^X + L_2^X + L_1^Y + L_2^Y$$

We are looking for the conditions under which A's utility will be maximised, given that B's is held at some arbitrary level Z. The other constraints are that the total consumption of each commodity is equal to the amount produced, and that the sum of the capital and labour inputs across all firms is equal to the economy's respective endowments, K^T and L^T.

[1] Using two individuals, two commodities and two firms in each industry does not really involve any loss of generality. Exactly the same qualitative conditions in terms of marginal rates of substitution and transformation would emerge if we used h individuals, n commodities and m firms in each industry. Our analysis could be generalised by having individual utility depend also on labour supplied, so that the total amount of labour available to the economy would be a variable rather than a constraint. This would introduce additional conditions, but would not alter those derived here. Another direction of generalisation would be over time so that the availability of capital is a matter of choice rather than a constraint - Chapter 11 looks at intertemporal efficiency and optimality.

This problem can be dealt with using the Lagrangian method reviewed in Appendix 3.1. Here the Lagrangian is

$$
\begin{aligned}
L = {} & U^A(X^A, Y^A) + \lambda_1[U^B(X^B, Y^B) - Z] \\
& + \lambda_2[X_1(K_1^X, L_1^X) + X_2(K_2^X, L_2^X) - X^A - X^B] \\
& + \lambda_3[Y_1(K_1^Y, L_1^Y) + Y_2(K_2^Y, L_2^Y) - Y^A - Y^B] \\
& + \lambda_4[K^T - K_1^X - K_2^X - K_1^Y - K_2^Y] \\
& + \lambda_5[L^T - L_1^X - L_2^X - L_1^Y - L_2^Y]
\end{aligned}
$$

We now need a way of indicating the marginal product of an input to the production of a commodity in a particular firm. A straightforward extension of the notation already introduced here is to use, for example, X_K^1 for $\partial X_1 / \partial K_1^X$, the marginal product of capital in the production of commodity X in firm 1 in the industry producing X.

In this notation, the first order conditions are:

$$\frac{\partial L}{\partial X^{A}} = U_{X}^{A} - \lambda_{2} = 0 \qquad (4.23.a)$$

$$\frac{\partial L}{\partial Y^{A}} = U_{Y}^{A} - \lambda_{3} = 0 \qquad (4.23.b)$$

$$\frac{\partial L}{\partial X^{B}} = \lambda_{1} U_{X}^{B} - \lambda_{2} = 0 \qquad (4.23.c)$$

$$\frac{\partial L}{\partial Y^{B}} = \lambda_{1} U_{Y}^{B} - \lambda_{3} = 0 \qquad (4.23.d)$$

$$\frac{\partial L}{\partial K_{1}^{X}} = \lambda_{2} X_{K}^{1} - \lambda_{4} = 0 \qquad (4.23.e)$$

$$\frac{\partial L}{\partial K_{2}^{X}} = \lambda_{2} X_{K}^{2} - \lambda_{4} = 0 \qquad (4.23.f)$$

$$\frac{\partial L}{\partial L_{1}^{X}} = \lambda_{2} X_{L}^{1} - \lambda_{5} = 0 \qquad (4.23.g)$$

$$\frac{\partial L}{\partial L_{2}^{X}} = \lambda_{2} X_{L}^{2} - \lambda_{5} = 0 \qquad (4.23.h)$$

$$\frac{\partial L}{\partial K_{1}^{Y}} = \lambda_{3} Y_{K}^{1} - \lambda_{4} = 0 \qquad (4.23.i)$$

$$\frac{\partial L}{\partial K_{2}^{Y}} = \lambda_{3} Y_{K}^{2} - \lambda_{4} = 0 \qquad (4.23.j)$$

$$\frac{\partial L}{\partial L_{1}^{Y}} = \lambda_{3} Y_{L}^{1} - \lambda_{5} = 0 \qquad (4.23.k)$$

$$\frac{\partial L}{\partial L_{2}^{Y}} = \lambda_{3} Y_{L}^{2} - \lambda_{5} = 0 \qquad (4.23.l)$$

From equations a and b here

$$\frac{U_{X}^{A}}{U_{Y}^{A}} = \frac{\lambda_{2}}{\lambda_{3}} \qquad (4.23.m)$$

and from c and d

$$\frac{U_X^B}{U_Y^B} = \frac{\lambda_2 / \lambda_1}{\lambda_3 / \lambda_1} = \frac{\lambda_2}{\lambda_3} \qquad (4.23.n)$$

so that

$$\frac{U_X^A}{U_Y^A} = \frac{U_X^B}{U_Y^B}$$

which from equation 4.20 in section A 4.1.1 above is

$$MRUS^A = MRUS^B \qquad (4.24)$$

which is the consumption efficiency condition stated as equation 4.3 in the text of the chapter.

Now, from equations 4.23.e and 4.23.f we have

$$X_K^1 = X_K^2 = \lambda_4/\lambda_2 \qquad (4.23.o)$$

from equations 4.23.g and 4.23.h

$$X_L^1 = X_L^2 = \lambda_5/\lambda_2 \qquad (4.23.p)$$

from equations 4.23.i and 4.23.j

$$Y_K^1 = Y_K^2 = \lambda_4/\lambda_3 \qquad (4.23.q)$$

and from equations 4.23.k and 4.23.l

$$Y_L^1 = Y_L^2 = \lambda_5/\lambda_3 \qquad (4.23.r)$$

From equations 4.23.o and 4.23.p

$$\frac{X_L^1}{X_K^1} = \frac{X_L^2}{X_K^2} = \frac{\lambda_5/\lambda_2}{\lambda_4/\lambda_2} = \frac{\lambda_5}{\lambda_4}$$

and from equations 4.23.q and 4.23.r

$$\frac{Y_L^1}{Y_K^1} = \frac{Y_L^2}{Y_K^2} = \frac{\lambda_5/\lambda_3}{\lambda_4/\lambda_3} = \frac{\lambda_5}{\lambda_4}$$

so that

$$\frac{X_L^1}{X_K^1} = \frac{X_L^2}{X_K^2} = \frac{Y_L^1}{Y_K^1} = \frac{Y_L^2}{Y_K^2} \qquad (4.23.s)$$

Recall from equation 4.21 in section A.4.1.1 above that for X = X(K, L), MRTS = X_L/X_K. Hence, equation 4.23.s here can be written as

$$MRTS_X^1 = MRTS_X^2 = MRTS_Y^1 = MRTS_Y^2 \qquad (4.25)$$

where $MRTS_X^1$, for example, is the marginal rate of technical substitution for capital and labour in the production of commodity X by firm 1 in the X industry. What equation 4.25 says is a) that all firms in an industry must have the same MRTS and b) that the MRTS must be the same in all industries. As b), equation 4.25 is the production efficiency condition as stated, equation 4.4, and explained intuitively in the text of the chapter. It is a) here that makes it legitimate to consider, as we did in the text, each industry as comprising a single firm.

Given that firms in the same industry operate with the same marginal products, we can write equations 4.23.o to 4.23.r as

$$X_K = \lambda_4/\lambda_2 \qquad (4.23.t)$$

$$X_L = \lambda_5/\lambda_2 \qquad (4.23.u)$$

$$Y_K = \lambda_4/\lambda_3 \qquad (4.23.v)$$

and

$$Y_L = \lambda_5/\lambda_3 \qquad (4.23.w)$$

Then from equations 4.23.v and 4.23.t

$$\frac{Y_K}{X_K} = \frac{\lambda_4/\lambda_3}{\lambda_4/\lambda_2} = \frac{\lambda_2}{\lambda_3}$$

and from equations 4.23.w and 4.23.u

$$\frac{Y_L}{X_L} = \frac{\lambda_5/\lambda_3}{\lambda_5/\lambda_2} = \frac{\lambda_2}{\lambda_3}$$

so that

$$\frac{Y_K}{X_K} = \frac{Y_L}{X_L} = \frac{\lambda_2}{\lambda_3}$$

which from equations 4.22.a and 4.22.b in section A 4.1.1 above can be written as

$$\mathrm{MRT_L} = \mathrm{MRT_K} = \lambda_2/\lambda_3 \qquad (4.23.x)$$

At equations 4.23.m and 4.23.n we obtained

$$\frac{U_X^A}{U_Y^A} = \frac{U_X^B}{U_Y^B} = \frac{\lambda_2}{\lambda_3}$$

which by equation 4.20 from section A 4.1.1 is

$$\mathrm{MRUS}^A = \mathrm{MRUS}^B = \lambda_2/\lambda_3 \qquad (4.23.y)$$

From equations 4.23.x and 4.23.y we get

$$\mathrm{MRUS}^A = \mathrm{MRUS}^B = \mathrm{MRT_L} = \mathrm{MRT_K} \qquad (4.26)$$

which is the product mix efficiency condition stated as equation 4.5 in the chapter.

A 4.1.3 Optimality conditions

We now introduce a social welfare function, so as to derive the conditions that characterise an optimal allocation. Using the same assumptions about utility and production as in section 4.1.2, the problem to be considered here is:

Max $W\{U^A(X^A, Y^A), U^B(X^B, Y^B)\}$

subject to

$$X_1(K_1^X, L_1^X) + X_2(K_2^X, L_2^X) = X^A + X^B$$

$$Y_1(K_1^Y, L_1^Y) + Y_2(K_2^Y, L_2^Y) = Y^A + Y^B$$

$$K^T = K_1^X + K_2^X + K_1^Y + K_2^Y$$

$$L^T = L_1^X + L_2^X + L_1^Y + L_2^Y$$

Here the Lagrangian is

$$\begin{aligned} L = W\{U^A(X^A, Y^A), U^B(X^B, Y^B)\} \\ + \lambda_2[X_1(K_1^X, L_1^X) + X_2(K_2^X, L_2^X) - X^A - X^B] \\ + \lambda_3[Y_1(K_1^Y, L_1^Y) + Y_2(K_2^Y, L_2^Y) - Y^A - Y^B] \\ + \lambda_4[K^T - K_1^X - K_2^X - K_1^Y - K_2^Y] \\ + \lambda_5[L^T - L_1^X - L_2^X - L_1^Y - L_2^Y] \end{aligned}$$

where we have started numbering the multipliers at 2 so as to bring out more transparently the correspondences between the necessary conditions for efficiency and optimality - the fact that we use the same symbols and numbers in both cases does not, of course, mean that the multipliers take the same values in both cases.

The first order conditions for this welfare maximisation problem are:

$$\frac{\partial L}{\partial X^A} = W_A U_X^A - \lambda_2 = 0 \qquad (4.27.a)$$

$$\frac{\partial L}{\partial Y^A} = W_A U_Y^A - \lambda_3 = 0 \qquad (4.27.b)$$

$$\frac{\partial L}{\partial X^B} = W_B U_X^B - \lambda_2 = 0 \qquad (4.27.c)$$

$$\frac{\partial L}{\partial Y^B} = W_B U_Y^B - \lambda_3 = 0 \qquad (4.27.d)$$

$$\frac{\partial L}{\partial K_1^X} = \lambda_2 X_K^1 - \lambda_4 = 0 \qquad (4.27.e)$$

$$\frac{\partial L}{\partial K_2^X} = \lambda_2 X_K^2 - \lambda_4 = 0 \qquad (4.27.f)$$

$$\frac{\partial L}{\partial L_1^X} = \lambda_2 X_L^1 - \lambda_5 = 0 \qquad (4.27.g)$$

$$\frac{\partial L}{\partial L_2^X} = \lambda_2 X_L^2 - \lambda_5 = 0 \qquad (4.27.h)$$

$$\frac{\partial L}{\partial K_1^Y} = \lambda_3 Y_K^1 - \lambda_4 = 0 \qquad (4.27.i)$$

$$\frac{\partial L}{\partial K_2^Y} = \lambda_3 Y_K^2 - \lambda_4 = 0 \qquad (4.27.j)$$

$$\frac{\partial L}{\partial L_1^Y} = \lambda_3 Y_L^1 - \lambda_5 = 0 \qquad (4.27.k)$$

$$\frac{\partial L}{\partial L_2^Y} = \lambda_3 Y_L^2 - \lambda_5 = 0 \qquad (4.27.l)$$

where $W_A = \partial W/\partial U^A$ and $W_B = \partial W/\partial U^B$

Note that equations e through to l in the set 4.27 are the same as e through to l in the set 4.23. It follows that optimality requires the efficiency in production condition, equation 4.25, re-written here as

$$MRTS_X^1 = MRTS_X^2 = MRTS_Y^1 = MRTS_Y^2 \qquad (4.28)$$

From a and b in set 4.27

$$\frac{U_X^A}{U_Y^A} = \frac{\lambda_2}{\lambda_3}$$

as W_A cancels. Similarly, from c and d in set A 4.8

$$\frac{U_X^B}{U_Y^B} = \frac{\lambda_2}{\lambda_3}$$

so that optimality requires

$$\frac{U_X^A}{U_Y^A} = \frac{U_X^B}{U_Y^B}$$

or

$$MRUS^A = MRUS^B = \lambda_2 / \lambda_3 \qquad (4.29)$$

which is the same as the consumption efficiency condition, 4.24, in the previous section.

From equations 4.27.e through to 4.27.l we can, as in the previous section, derive

$$MRT_L = MRT_K = \lambda_2 / \lambda_3 \qquad (4.30)$$

and from 4.29 and 4.30 we have

$$MRUS^A = MRUS^B = MRT_L = MRT_K \qquad (4.31)$$

which is the same product mix condition as is required for efficiency.

Optimality requires the fulfilment of all of the efficiency conditions. In deriving the efficiency conditions, the utility of B is set at some arbitrary level. The maximisation problem considered there, as well as producing the conditions that any efficient allocation must satisfy, identifies the maximum level for A's utility conditional on the selected level of B's utility. In the welfare maximisation problem the function $W\{U^A, U^B\}$ selects the utility levels for A and B. As discussed in the text, only combinations of U^A and U^B that lie along the utility possibility frontier are relevant for welfare maximisation. All such combinations satisfy the efficiency conditions, and hence welfare maximisation entails satisfying the efficiency conditions as shown above. It also entails the condition stated as equation 4.7 in the chapter, which condition fixes the utility levels for A and B using the social welfare function.

From equations 4.27.a through to 4.27.d we have

$$W_A = \frac{\lambda_2}{U_X^A} \qquad (4.32.a)$$

$$W_A = \frac{\lambda_3}{U_Y^A} \qquad (4.32.b)$$

$$W_B = \frac{\lambda_2}{U_X^B} \qquad (4.32.c)$$

$$W_B = \frac{\lambda_3}{U_Y^B} \qquad (4.32.d)$$

From a and c here we get

$$\frac{W_A}{W_B} = \frac{U_X^B}{U_X^A}$$

and from b and d we get

$$\frac{W_A}{W_B} = \frac{U_Y^B}{U_Y^A}$$

so that

$$\frac{W_A}{W_B} = \frac{U_X^B}{U_X^A} = \frac{U_Y^B}{U_Y^A} \qquad (4.33)$$

which is equation 4.7 in the chapter.

The SWF is $W = W(U^A, U^B)$ so that

$dW = W_A dU^A + W_B dU^B$

Setting the left-hand side here equal to zero so as to consider small movements along a social welfare indifference curve, and re-arranging gives

$$-\frac{dU^B}{dU^A} = \frac{W_A}{W_B}$$

for the slope of a social welfare indifference curve. The slope of the utility possibility frontier is $-dU^B/dU^A$ which is equal to U_X^B / U_X^A and to U_Y^B / U_Y^A .