Appendix 4.3 Market failure

A 4.3.1 Public goods

In the two person, two commodity, two resource economy considered in the preceding Appendix, now let X be a public good and Y a private good. Given the results established there regarding the conditions for efficiency in relation to firms in the same industry, we can simplify here without loss by assuming that each commodity is produced in an industry which has just one firm. Given that we are taking the defining characteristic of a public good to be that it is consumed in the same quantity by all, we can state the problem from which the necessary conditions for efficiency are to be derived as:

Max $U^A(X, Y^A)$

subject to

$U^B(X, Y^B) = Z$

$X(K^X, L^X) = X$

$Y(K^Y, L^Y) = Y^A + Y^B$

$K^T = K^X + K^Y$

$L^T = L^X + L^Y$

The Lagrangian for this problem is

$$\begin{aligned} L = U^A(X, Y^A) &+ \lambda_1[U^B(X, Y^B) - Z] \\ &+ \lambda_2[X(K^X, L^X) - X] \\ &+ \lambda_3[Y(K^Y, L^Y) - Y^A - Y^B] \\ &+ \lambda_4[K^T - K^X - K^Y] \\ &+ \lambda_5[L^T - L^X - L^Y] \end{aligned}$$

from which the necessary conditions for maximisation are:

$$\frac{\partial L}{\partial X} = U_X^A + \lambda_1 U_X^B - \lambda_2 = 0 \qquad (4.44.a)$$

$$\frac{\partial L}{\partial Y^A} = U_Y^A - \lambda_3 = 0 \qquad (4.44.b)$$

$$\frac{\partial L}{\partial Y^B} = \lambda_1 U_Y^B - \lambda_3 = 0 \qquad (4.44.c)$$

$$\frac{\partial L}{\partial K^X} = \lambda_2 X_K - \lambda_4 = 0 \qquad (4.44.d)$$

$$\frac{\partial L}{\partial L^X} = \lambda_2 X_L - \lambda_5 = 0 \qquad (4.44.e)$$

$$\frac{\partial L}{\partial K^Y} = \lambda_3 Y_K - \lambda_4 = 0 \qquad (4.44.f)$$

$$\frac{\partial L}{\partial L^Y} = \lambda_3 Y_L - \lambda_5 = 0 \qquad (4.44.g)$$

Consider first equations 4.44.d to 4.44.g, which relate to production. They imply

$$\frac{X_L}{X_K} = \frac{\lambda_5}{\lambda_4} = \frac{Y_L}{Y_K}$$

which is

$MRTS_X = MRTS_Y$

so that production efficiency is required. They also imply

$$\frac{Y_K}{X_K} = \frac{\lambda_2}{\lambda_3} = \frac{Y_L}{X_L}$$

which is

$$MRT_K = MRT_L = \frac{\lambda_2}{\lambda_3} \qquad (4.45)$$

so that as regards production activities, the conditions in the presence of a public good are the same as in the standard case, see Appendix 4.1, where there are no public goods.

Now consider equations 4.44.a to 4.44.c, which relate to consumption. From equations a and b there

$$\frac{U_X^A}{U_Y^A} = \frac{\lambda_2}{\lambda_3} - \frac{\lambda_1 U_X^B}{\lambda_3} \qquad (4.46.a)$$

and using equation 4.44.c we can write

$$\frac{U_X^B}{U_Y^B} = \frac{U_X^B}{\lambda_3/\lambda_1} = \frac{\lambda_1 U_X^B}{\lambda_3} \qquad (4.46.b)$$

and adding 4.46.a and 4.46.b gives:

$$\frac{U_X^A}{U_Y^A} + \frac{U_X^B}{U_Y^B} = \frac{\lambda_2}{\lambda_3} - \frac{\lambda_1 U_X^B}{\lambda_3} + \frac{\lambda_1 U_X^B}{\lambda_3} = \frac{\lambda_2}{\lambda_3} \qquad (4.47)$$

Using the definition for MRUS, equation 4.47 is

$$MRUS^A + MRUS^B = \frac{\lambda_2}{\lambda_3}$$

so that from equation 4.45 we have the condition

$$MRUS^A + MRUS^B = MRT \qquad (4.48)$$

stated as equation 4.15 in the chapter.

A 4.3.2 Externalities: consumer to consumer

As in the text, we ignore production in looking at this case. Given that we have not previously looked at a pure exchange economy, it will be convenient first to look at such an economy where there is no external effect.

To identify the necessary conditions for efficiency, we look at

$\text{Max } U^A(X^A, Y^A)$

subject to

$U^B(X^B, Y^B) = Z$

$X^T = X^A + X^B$

$Y^T = Y^A + Y^B$

where X^T and Y^T are the total amounts of the two commodities to be allocated as between A and B. The Lagrangian for this problem is

$$L = U^A(X, Y^A) + \lambda_1[U^B(X, Y^B) - Z]$$

$$+ \lambda_2[X^T - X^A - X^B]$$

$$+ \lambda_3[Y^T - Y^A - Y_B]$$

and the necessary conditions are

$$\frac{\partial L}{\partial X^A} = U_X^A - \lambda_2 = 0$$

$$\frac{\partial L}{\partial Y^A} = U_Y^A - \lambda_3 = 0$$

$$\frac{\partial L}{\partial X^B} = \lambda_1 U_X^B - \lambda_2 = 0$$

$$\frac{\partial L}{\partial Y^B} = \lambda_1 U_Y^B - \lambda_3 = 0$$

from which we get

$$\frac{U_X^A}{U_Y^A} = \frac{U_X^B}{U_Y^B} = \frac{\lambda_2}{\lambda_3}$$

which is the same consumption efficiency condition as for the economy with production, i.e. $MRUS^A = MRUS^B$. We already know, from Appendix 4.2, that consumers facing given and fixed prices P_X and P_Y and maximising utility subject to a budget constraint will satisfy this condition.

Now, suppose that B's consumption of Y is an argument in A's utility function. We are assuming that Y^B is a source of dis-utility to A. Then the maximisation problem to be considered is

$$\text{Max } U^A(X^A, Y^A, Y^B)$$

subject to

$$U^B(X^B, Y^B) = Z$$

$$X^T = X^A + X^B$$

$$Y^T = Y^A + Y^B$$

for which the Lagrangian is

$$L = U^A(X^A, Y^A, Y^B) + \lambda_1[U^B(X^B, Y^B) - Z]$$

$$+ \lambda_2[X^T - X^A - X^B]$$

$$+ \lambda_3[Y^T - Y^A - Y^B]$$

with necessary conditions

$$\frac{\partial L}{\partial X^A} = U_X^A - \lambda_2 = 0 \qquad (4.49.a)$$

$$\frac{\partial L}{\partial Y^A} = U_Y^A - \lambda_3 = 0 \qquad (4.49.b)$$

$$\frac{\partial L}{\partial X^B} = \lambda_1 U_X^B - \lambda_2 = 0 \qquad (4.49.c)$$

$$\frac{\partial L}{\partial Y^B} = U_{YB}^A + \lambda_1 U_Y^B - \lambda_3 = 0 \qquad (4.49.d)$$

where $U_{YB}^A = \partial U^A/\partial Y^B$. Note that Y^B is a source of dis-utility to A so that $U_{YB}^A < 0$. From 4.49.a and 4.49.b we get

$$\frac{U_X^A}{U_Y^A} = \frac{\lambda_2}{\lambda_3} \qquad (4.50.a)$$

from 4.49.c

$$U_X^B = \frac{\lambda_2}{\lambda_1} \tag{4.50.b}$$

and from 4.49.d

$$U_Y^B = \frac{\lambda_3}{\lambda_1} - \frac{U_{YB}^A}{\lambda_1} \tag{4.50.c}$$

so that using 4.50.b and 4.50.c

$$\frac{U_X^B}{U_Y^B} = \frac{\lambda_2}{\lambda_3 - U_{YB}^A} \tag{4.50.d}$$

Looking at 4.50.a and 4.50.d we see that with the externality, efficiency does not require the condition $MRUS^A = MRUS^B$. But we have just seen that, facing just the prices P_X and P_Y, market trading between A and B will give $MRUS^A = MRUS^B$. So given the existence of this externality, market exchange will not satisfy the conditions, 4.50.a and 4.50.d, for efficiency.

Suppose now that there exists a central planner who knows the two agents' utility functions and the quantities of X and Y available. The planner's objective is an efficient allocation, to be realised by the two agents individually maximising utility on terms set by the planner, rather by the planner telling the agents at what levels to consume. The planner declares prices P_X and P_Y, and also requires B to compensate A for her Y^B suffering at the rate c per unit of Y^B. In that case, A's utility maximisation problem is

Max $U^A(X^A, Y^A, Y^B)$

subject to

$P_X X^A + P_Y Y^A = M^A + cY^B$

where M^A is A's income before the receipt of any compensation from B. The Lagrangian for this problem is:

$L = U^A(X^A, Y^A, Y^B) + \lambda_A[P_X X^A + P_Y Y^A - M^A - cY^B]$

Note that Y^B is not a choice variable for A. The level of Y^B is chosen by B. The necessary conditions for A's maximisation problem are

$$\frac{\partial L}{\partial X^A} = U_X^A + \lambda_A P_X = 0$$

$$\frac{\partial L}{\partial Y^A} = U_Y^A + \lambda_A P_Y = 0$$

from which

$$\frac{U_X^A}{U_Y^A} = \frac{P_X}{P_Y} \tag{4.51.a}$$

B's utility maximisation problem is

Max $U^B(X^B, Y^B)$

subject to

$P_X X^B + P_Y Y^B = M^B - cY^B$

the Lagrangian for which is

$L = U^B(X^B, Y^B) + \lambda_B[P_X X^B + P_Y Y^B - M^B + cY^B]$

with necessary conditions

$$\frac{\partial L}{\partial X^B} = U_X^B + \lambda_B P_X = 0$$

$$\frac{\partial L}{\partial Y^B} = U_Y^B + \lambda_B P_Y + \lambda_B c = 0$$

from which

$$\frac{U_X^B}{U_Y^B} = \frac{P_X}{P_Y + c} \tag{4.51.b}$$

So, we have 4.50.a and 4.50.d as the efficiency conditions and 4.51.a and 4.51.b as the individual utility maximising conditions. Comparing 4.50.a and 4.50.d with 4.51.and 4.51.b, it will be seen that they are the same for:

$\lambda_2 = P_X$, $\lambda_3 = P_Y$ and $c = -U_{YB}^A$

If, that is, the planner solves the appropriate maximisation problem and sets P_X and P_Y at the shadow prices of the commodities, and requires B to compensate A at a rate which is equal to, but of opposite sign to, A's marginal dis-utility in respect of the external effect, then A and B individually maximising utility given those prices and that compensation rate will bring about an efficient allocation. The planner is putting a price on the external effect, and the required price is A's marginal dis-utility.

However, as shown in the discussion of the Coase theorem in the body of the chapter, it is not actually necessary to have this kind of intervention by the planner. If A had the legal right to extract full compensation from B, had a property right in an unpolluted environment, then the right price for efficiency would emerge as the result of bargaining between A and B.

In considering the consumption to consumption case in the chapter we argued that the liability/property right could be assigned the other way round and still bring about an efficient outcome. The corresponding procedure with a planner setting the terms on which the two agents maximised utility would be to have the planner work out what Y^B would be with the externality un-corrected, say Y^B*, and then require A to compensate B for reducing Y^B below that level. In that case, A's maximisation problem would be

$$\text{Max } U^A(X^A, Y^A, Y^B)$$

subject to

$$P_X X^A + P_Y Y_A = M^A - b(Y^B* - Y^B)$$

and B's would be

$$\text{Max } U^B(X^B, Y^B)$$

subject to

$$P_X X^B + P_Y Y^B = M^B + b(Y^B* - Y^B)$$

where we use b for 'bribe'. It is left as an exercise to confirm that this arrangement would, given suitable P_X, P_Y and b, produce an efficient outcome.

The situation considered in the chapter actually differed from that considered here in a couple of respects. First, in that example the external effect involved A doing something - playing a musical instrument - which did not have a price attached to it, and which B did not do. In the un-corrected externality situation there, A pursued the 'polluting' activity up to the level where its marginal utility was zero. In the chapter, we considered things in terms of monetary costs and benefits in a partial equilibrium context, rather than utility maximisation in a general equilibrium context. Thinking about that noise pollution example in the following way may help to make the connections, and make a further point.

Let Y^A be the number of hours that A plays her instrument. Consider each individual's utility to depend on income and Y^A, so that $U^A = U^A(M^A, Y^A)$ and $U^B = U^B(M^B, Y^A)$, where $\partial U^A/\partial Y^A > 0$ and $\partial U^B/\partial Y^A < 0$. Consider welfare maximisation for given M^A and M^B. The problem is

$$\text{Max } W\{U^A(M^A, Y^A), U^B(M^B, Y^A)\}$$

where the only choice variable is Y^A, so that the necessary condition is:

$$W_A U^A_{YA} = -W_B U^B_{YA}$$

For equal welfare weights, this is

$$U^A_{YA} = - U^B_{YA}$$

or

Marginal cost of music to A = Marginal benefit of music to B

which is the condition as stated in the chapter. The further point that the derivation of this condition here makes is that in the standard simple story about the Coase Theorem implicitly assigns equal welfare weights to the two individuals.

A 4.3.3 Externalities: producer to producer

To begin here, we suppose that the production function for Y is

$$Y = Y(K^Y, L^Y, S) \text{ with } Y_S = \partial Y/\partial S > 0$$

and for X is

$$X = X(K^X, L^X, S) \text{ with } X_S = \partial X/\partial S < 0$$

where S is pollutant emissions arising in the production of Y and adversely affecting the production of X. The Lagrangian from which the conditions for efficiency are to be derived is:

$$\begin{aligned} L = U^A(X^A, Y^A) &+ \lambda_1[U^B(X^B, Y^B) - Z] \\ &+ \lambda_2[X(K^X, L^X, S) - X^A - X^B] \\ &+ \lambda_3[Y(K^Y, L^Y, S) - Y^A - Y^B] \\ &+ \lambda_4[K^T - K^X - K^Y] \\ &+ \lambda_5[L^T - L^X - L^Y] \end{aligned}$$

The reader can readily check that in this case, taking derivates of L with respect to X^A, Y^A, X_B, Y^B, K^X, L^X, K^Y and L^Y gives, allowing for the fact that there is just one firm in each industry, the consumption, production and product mix conditions derived in Appendix 4.1.2 and stated in the chapter. Taking the derivative of L with respect to S gives the additional condition

$$\frac{\partial L}{\partial S} = \lambda_2 X_S + \lambda_3 Y_S = 0$$

or

$$\frac{\lambda_2}{\lambda_3} = -\frac{Y_S}{X_S} \qquad (4.52)$$

Now, suppose that a central planner declares prices $P_X = \lambda_2$, $P_Y = \lambda_3$, $P_K = \lambda_4$, $P_L = \lambda_5$, and requires that that the firm producing Y pay compensation to the firm affected by its emissions at the rate c per unit S. Then, the Y firm's problem is

$$\text{Max } P_Y Y(K^Y, L^Y, S) - P_K K^Y - P_L L^Y - cS$$

with the usual necessary conditions

$$P_Y Y_K - P_K = 0$$

$$P_Y Y_L - P_L = 0$$

plus

$$P_Y Y_S - c = 0 \tag{4.53}$$

Compare equation 4.52 with 4.53. If we set $c = -P_X X_S$ then the latter becomes

$$P_Y Y_S = -P_X X_S$$

or

$$\frac{P_X}{P_Y} = -\frac{Y_S}{X_S} \tag{4.54}$$

which, for $P_X = \lambda_2$ and $P_Y = \lambda_3$, is the same as equation 4.52. With this compensation requirement in place, the profit maximising behaviour of the Y firm will be as required for efficiency. Note that the rate of compensation makes sense. $P_X X_S$ is the reduction in X's profit for a given level of output when Y increases S. Note also that while we have called this charge on emissions of S by the Y firm 'compensation', we have not shown that efficiency requires that the X firm actually receives such compensation. The charge c, that is, might equally well be collected by the planner, in which case we would call it a tax on emissions.[1]

In the chapter we noted that one way of internalising a producer to producer externality could be for the firms to merge, or to enter into an agreement to maximise joint profits. A proof of this claim is as follows. The problem then is

$$\text{Max } P_X X(K^X, L^X, S) + P_Y(K^Y, L^Y, S) - P_K(K^X + K^Y) - P_L(L^X + L^Y)$$

for which the necessary conditions are

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[1] However, if c takes the form of a tax rather than compensation paid to the X firm, the question arises as to what happens to the tax revenue. It cannot remain with the planner, otherwise the government, as she does not count as an agent. If the planner/government has unspent revenues, it would be possible to make some agent better off without making any other agent(s) worse off. Given the simple model specification here, where, for example, there is no tax/welfare system and no public goods supply, we cannot explore this question further. It is considered, for example, in chapter 4 of Baumol and Oates (1988), and the 'double dividend' literature reviewed in chapter 10 here is also relevant.

$P_X X_K - P_K = 0$

$P_X X_L - P_L = 0$

$P_Y Y_K - P_K = 0$

$P_Y Y_L - P_L = 0$

which, given $P_X = \lambda_2$, $P_K = \lambda_4$ etc satisfy the standard (no externality)efficiency conditions, plus

$P_X X_S + P_Y Y_S = 0$

This last condition for joint profit maximisation can be written as

$$\frac{P_X}{P_Y} = -\frac{Y_S}{X_S}$$

which is just equation 4.54, previously shown to be necessary, in addition to the standard conditions, for efficiency in the presence of this kind of externality.

In Chapter 2 we noted that the fact that matter can neither be created nor destroyed is sometimes overlooked in the specification of economic models. We have just been guilty in that way ourselves - writing

$Y = Y(K^Y, L^Y, S)$

with S as some kind of pollutant emission, has matter, S, appearing from nowhere when, in fact, it must have a material origin in some input to the production process. A more satisfactory production function for the polluting firm would be

$Y = Y(K^Y, L^Y, R^Y, S\{R^Y\})$

where R^Y is the input of some material, say tonnes of coal, and $S\{R^Y\}$ maps coal burned into emissions, of say smoke, and $\partial Y/\partial R^Y = Y_R > 0$, $\partial Y/\partial S = Y_S > 0$ and $\partial S/\partial R^Y = S_{RY} > 0$ We shall now show that while this more plausible model specification complicates the story a little, it does not alter the essential message.

To maintain consistency with the producer to producer case as analysed above, and in the chapter, we will assume that in the production of X the use of R does not give rise to emissions of smoke. Then, the Lagrangian for deriving the efficiency conditions is:

$$
\begin{aligned}
L = U^A(X^A, Y^A) &+ \lambda_1[U^B(X^B, Y^B) - Z] \\
&+ \lambda_2[X(K^X, L^X, R^X, S\{R^Y\}) - X^A - X^B] \\
&+ \lambda_3[Y(K^Y, L^Y, R^Y, S\{R^Y\}) - Y^A - Y^B] \\
&+ \lambda_4[K^T - K^X - K^Y] \\
&+ \lambda_5[L^T - L^X - L^Y] \\
&+ \lambda_6[R^T - R^X - R^Y]
\end{aligned}
$$

In the production function for X, $\partial X/\partial R^X = X_R > 0$ and $\partial X/\partial S = X_S < 0$. The reader can confirm that taking derivatives here with respect to all the choice variables except R^X and R^Y gives all of the standard conditions. Then with respect to R^X and R^Y, we get

$$\frac{\partial L}{\partial R^X} = \lambda_2 X_R - \lambda_6 = 0 \tag{4.55.a}$$

$$\frac{\partial L}{\partial R^Y} = \lambda_2 X_S S_{RY} + \lambda_3 Y_R + \lambda_3 Y_S S_{RY} - \lambda_6 = 0 \tag{4.55.b}$$

As before, suppose a planner who sets $P_X = \lambda_2$,....$P_L = \lambda_5$ plus $P_R = \lambda_6$ and a tax on the use of R in the production of Y at the rate t. Then the profit maximisation problem for the firm producing Y is

$$\text{Max } P_Y Y(K^Y, L^Y, R^Y, S\{R^Y\}) - P_K K^Y - P_L L^Y - P_R R^Y - tR^Y$$

and for the firm producing X it is

$$\text{Max } P_X X(K^X, L^X, R^X, S\{R^Y\}) - P_K K^X - P_L L^X - P_R R^X$$

If the reader derives the necessary conditions here, which include

$$P_Y Y_R + P_Y Y_S S_{RY} - P_R - t = 0 \tag{4.56}$$

she can verify that for $P_X = \lambda_2$,....$P_L = \lambda_5$ and $P_R = \lambda_6$ with

$$t = - P_X X_S S_{RY} \tag{4.57}$$

independent profit maximisation by both firms satisfies the standard efficiency conditions plus the externality correction conditions stated above as equations 4.55.a and 4.55.b. The rationale for this rate of tax should also be apparent - S_{RY} is the increase in smoke for an increase in Y's use of R, X_S gives the effect of more smoke on the output of X for given K^X and L^X, and P_X is the price of X.

Now consider joint profit maximisation. From

$$\text{Max } P_X X(K^X, L^X, R^X, S\{R^Y\}) + P_Y(K^Y, L^Y, R^Y, S\{R^Y\})$$

$$- P_K(K^X + K^Y) - P_L(L^X + L^Y) - P_R(R^X + R^Y)$$

the necessary conditions are

$$P_X X_K - P_K = 0$$

$$P_X X_L - P_L = 0$$

$$P_Y Y_K - P_K = 0$$

$$P_Y Y_L - P_L = 0$$

$$P_X X_R - P_R = 0$$

$$P_Y Y_R + P_Y Y_S S_{RY} + P_X X_S S_{RY} - P_R = 0$$

Substituting from equation 4.57 into 4.56 for t gives the last of these equations, showing that the outcome under joint profit maximisation is the same as with the tax on the use of R in the production of Y.

A 4.3.4 Externalities: producer to consumers

The main point to be made for this case concerns the implications of non-rivalry and non-excludability These are not peculiar to the producer to consumers case, but are conveniently demonstrated using it. To simplify the notation, we revert to having emissions in production occur without any explicit representation of their material origin. As noted in the analysis of the producer to producer case, this simplifies without, for present purposes, missing anything essential. We assume that the production of Y involves pollutant emissions which affect both A and B equally, though, of course, A and B might have different preferences over pollution and commodities. Pollution is, that is, in the nature of a public bad - A/B's consumption is non-rival with respect to B/A's consumption, and neither can escape, be excluded from, consumption.

The Lagrangian for the derivation of the efficiency conditions is

$$L = U^A(X^A, Y^A, S) + \lambda_1[U^B(X^B, Y^B, S) - Z]$$

$$+ \lambda_2[X(K^X, L^X) - X^A - X^B]$$

$$+ \lambda_3[Y(K^Y, L^Y, S) - Y^A - Y^B]$$

$$+ \lambda_4[K^T - K^X - K^Y]$$

$$+ \lambda_5[L^T - L^X - L^Y]$$

where $\partial U^A/\partial S = U_S^A < 0$, $\partial U^B/\partial S = U_S^B < 0$ and $\partial Y/\partial S = Y_S > 0$. The necessary conditions are:

$$\frac{\partial L}{\partial X^A} = U_X^A - \lambda_2 = 0 \qquad (4.58.a)$$

$$\frac{\partial L}{\partial Y^A} = U_Y^A - \lambda_3 = 0 \qquad (4.58.b)$$

$$\frac{\partial L}{\partial X^B} = \lambda_1 U_X^B - \lambda_2 = 0 \qquad (4.58.c)$$

$$\frac{\partial L}{\partial Y^B} = \lambda_1 U_Y^B - \lambda_3 = 0 \qquad (4.58.d)$$

$$\frac{\partial L}{\partial S} = U_S^A + \lambda_1 U_S^B + \lambda_3 Y_S = 0 \qquad (4.58.e)$$

$$\frac{\partial L}{\partial K^X} = \lambda_2 X_K - \lambda_4 = 0 \qquad (4.58.f)$$

$$\frac{\partial L}{\partial L^X} = \lambda_2 X_L - \lambda_5 = 0 \qquad (4.58.g)$$

$$\frac{\partial L}{\partial K^Y} = \lambda_3 Y_K - \lambda_4 = 0 \qquad (4.58.h)$$

$$\frac{\partial L}{\partial L^Y} = \lambda_3 Y_L - \lambda_5 = 0 \qquad (4.58.i)$$

The reader can check that these can be expressed as the standard consumption, production and product mix conditions plus

$$U_S^A + \lambda_1 U_S^B = -\lambda_3 Y_S \qquad (4.59)$$

from equation 4.58.e.

Now suppose that a central planner declares prices $P_X = \lambda_2$, $P_Y = \lambda_3$, $P_K = \lambda_4$ and $P_Y = \lambda_5$. Proceeding as done previously in this Appendix, the reader can check that utility and profit maximisation at these prices will satisfy all of the standard conditions, but not equation 4.59. Suppose then that the planner also requires the producer of Y to pay a tax at the rate t on emissions of S. Considering

$$\text{Max } P_Y Y(K^Y, L^Y, S) - P_K K^Y - P_L L^Y - tS$$

gives the standard conditions

$$P_Y Y_K - P_K = 0$$

$$P_L L_Y - P_L = 0$$

plus

$$P_Y Y_S - t = 0$$

which can be written as

$$t = \lambda_3 Y_S \tag{4.60}$$

Comparing equations 4.59 and 4.60, we have the result that, in this case, achieving efficiency as the result of individual utility and profit maximisation requires, in addition to the usual 'ideal' institutional arrangements that the producer of Y faces an emissions tax at the rate:

$$t = -[U_S^A + \lambda_1 U_S^B] \tag{4.61}$$

Note that since U_S^A and U_S^B are both negative, the tax rate required is positive.

In the chapter, we stated that the correction of this kind of externality required that the tax rate be set equal to the marginal external cost at the efficient allocation. We will now show that this is exactly what the result 4.61 requires. From equation 4.58.c

$$\lambda_1 = \frac{\lambda_2}{U_X^B}$$

and from equation 4.58.a

$$1 = \frac{\lambda_2}{U_X^A}$$

so that equation 4.61 can be written

$$t = -\left[\frac{\lambda_2}{U_X^A} U_S^A + \frac{\lambda_2}{U_X^B} U_S^B\right]$$

which, using $P_X = \lambda_2$, is

$$t = -P_X\left[\frac{U_S^A}{U_X^A} + \frac{U_S^B}{U_X^B}\right]$$

or

$$t = P_X\left[MRUS_{XS}^A + MRUS_{XS}^B\right] \qquad (4.62)$$

as stated at equation 4.17 in the chapter.[2] The tax rate is the monetary value of the increases in X consumption that would be required to hold each individual's utility constant in the face of a marginal increase in S. We could, of course, have derived the marginal external cost in terms of Y, rather than X, compensation.

In this case, the joint profit maximisation solution is clearly not, even in principle, available for the correction of the market failure problem. Nor, given the public good characteristic of the suffering of A and B, is property rights/legal liability solution. The way to correct this kind of market failure is to tax the emissions at a rate which is equal to the marginal external cost arising at the efficient allocation. It can be shown that where there is more than one source of the emissions, all sources are to be taxed at the same rate. The checking of this statement by considering

$$\begin{aligned} L = U^A(X^A, Y^A, S) &+ \lambda_1[U^B(X^B, Y^B, S) - Z] \\ &+ \lambda_2[X(K^X, L^X, S^X) - X^A - X^B] \\ &+ \lambda_3[Y(K^Y, L^Y, S^Y) - Y^A - Y^B] \\ &+ \lambda_4[K^T - K^X - K^Y] \\ &+ \lambda_5[L^T - L^X - L^Y] \\ &+ \lambda_6[S - S^X - S^Y] \end{aligned}$$

[2] To recapitulate, the marginal rate of substitution here is derived as follows. For U(X, Y, S)

$dU = U_X dX + U_Y dY + U_S dS$

so for dU and dY = 0

$0 = U_X dX + U_S dS$

and

$$\frac{U_S}{U_X} = -\frac{dX}{dS}$$

is left to the reader as an exercise. The result also applies where total emissions adversely affect production as well as having utility impacts - consider

$$L = U^A(X^A, Y^A, S) + \lambda_1[U^B(X^B, Y^B, S) - Z]$$
$$+ \lambda_2[X(K^X, L^X, S^X, S) - X^A - X^B]$$
$$+ \lambda_3[Y(K^Y, L^Y, S^Y, S) - Y^A - Y^B]$$
$$+ \lambda_4[K^T - K^X - K^Y]$$
$$+ \lambda_5[L^T - L^X - L^Y]$$
$$+ \lambda_6[S - S^X - S^Y]$$

where $\partial X/\partial S < 0$ and $\partial Y/\partial S < 0$.

A 4.3.5 The polluting monopolist

As in section 5.11, it is convenient here to refer to a monopolist, but the analysis applies to any firm that faces a downward sloping demand function. The analysis here is based on Barnett (1980): see also Baumol and Oates (1988).

The monopolist's demand function is P(Y) with $\partial P/\partial Y < 0$ and the cost function is C = C(Y, S) with $\partial C/\partial Y > 0$ and $\partial C/\partial S < 0$ - reducing emissions for output constant is costly. If emissions are taxed at the rate t, the monopolist selects Y and S so as to maximise net profits

$$\pi = P(Y)Y - C(Y, S) - tS$$

with necessary conditions

$$\frac{\partial \pi}{\partial Y} = \frac{\partial P}{\partial Y}Y + P - \frac{\partial C}{\partial Y} = 0 \qquad (4.63a)$$

and

$$\frac{\partial \pi}{\partial S} = -\frac{\partial C}{\partial S} - t = 0 \qquad (4.63b)$$

The EPA wants to set the tax rate on emissions so as to maximise the net, of pollution damage costs, benefits from the production of Y. In terms of the partial analysis of section 5.6, this is the area under the demand function for Y less the monopolist's cost of production less the cost of pollution damage. It chooses t, that is, so as to maximise

$$\int_0^Y P(Y)dY - C(Y,S) - D(S)$$

where D stands for damage costs related to emissions according to the function D(S). To find the rate required for t, totally differentiate this maximand with respect to t to get the condition:

$$P\frac{dY}{dt} - \frac{\partial C}{\partial Y}\frac{dY}{dt} - \frac{\partial C}{\partial S}\frac{dS}{dt} - \frac{\partial D}{\partial S}\frac{dS}{dt} = 0 \qquad (4.64)$$

The rate for t has to be calculated to account for profit maximisation by the monopolist, whose profit maximising conditions 4.63a and 4.63b can be written as:

$$P = \frac{\partial C}{\partial Y} - \frac{\partial P}{\partial Y}Y$$

and

$$t = -\frac{\partial C}{\partial S}$$

Substituting from here into the condition 4.64 gives

$$\left(\frac{\partial C}{\partial Y} - \frac{\partial P}{\partial Y}Y\right)\frac{dY}{dt} - \frac{\partial C}{\partial Y}\frac{dY}{dt} + t\frac{dS}{dt} - \frac{\partial D}{\partial S}\frac{dS}{dt} = 0$$

where multiplying both sides by dt/dS, cancelling and re-arranging gives:

$$t = \frac{\partial P}{\partial Y}Y\frac{dY}{dS} + \frac{\partial D}{\partial S} \qquad (4.65)$$

Now, the price elasticity of demand for Y is

$$\eta = -\frac{\partial Y}{\partial P}\frac{P}{Y}$$

from which

$$\frac{\partial P}{\partial Y}Y = -\frac{P}{\eta}$$

so that equation 4.65 can be re-written as:

$$t = \frac{\partial D}{\partial S} - \frac{P}{\eta}\frac{dY}{dS} \qquad (4.66)$$

The first term here is the marginal cost of emissions damage, ie marginal external cost. The emissions tax rate required for the price taker firm is equal to marginal external cost. According to 4.66, the tax rate required for the monopolist is lower than that, by an amount that depends on the price elasticity of demand, η, price, P, and dY/dS, which is the marginal response of output to decreased emissions. Note that as η gets larger, demand becomes more elastic, so the second term in 4.66 gets smaller, and that in the limit for perfectly elastic demand - the price taker case where η is infinity - the tax rate required is equal to marginal external cost.

There is another way of writing this result. From the definition for η

$$\frac{P}{\eta} = -\frac{P}{\frac{\partial Y}{\partial P}\frac{P}{Y}} = -\frac{\partial P}{\partial Y}Y$$

while

$$P - MR = P - \left(P + \frac{\partial P}{\partial Y}Y\right) = -\frac{\partial P}{\partial Y}Y$$

where MR stands for marginal revenue, so that 4.66 can be written as

$$t = \frac{\partial D}{\partial S} - (P - MR)\frac{dY}{dS} \qquad (4.67a)$$

which by virtue of MC = MR for the profit maximising monopolist can also be written as

$$t = \frac{\partial D}{\partial S} - (P - MC)\frac{dY}{dS} \qquad (4.67b)$$

where MC stands for marginal cost. Given that the price taking firm maximises profit with MC = P, we see again that for the price taking firm this collapses to tax rate equal to marginal external cost.