Appendix 5.1 Matrix algebra

A5.1.1 Introduction

In this chapter, and in a few of the later ones (particularly Chapter 8 and the appendix to Chapter 14), some use is made of matrix algebra notation and elementary matrix operations. This appendix provides, for the reader who is unfamiliar with matrix algebra, a brief explanation of the notation and an exposition of a few of its fundamental operations. We deal here only with those parts of matrix algebra that are necessary to understand the use made of it in this text. The reader who would like a more extensive account should go to any good first-year university-level mathematics text. For example, chapter 4 of Chiang (1984) provides a relatively full account of introductory-level matrix algebra in an accessible form.

A5.1.2 Matrices and vectors

A *matrix* is a set of elements laid out in the form of an array occupying a number of rows and columns. Consider an example where the elements are numbers. Thus, the array of numbers

0.7	0.1
0.9	0.2
0.3	0.2
0.1	0.0

can be called a matrix. In such an array, the relative positions of the elements *do* matter. Two matrices are identical if the elements are not only the same but also occupy the same positions in each matrix. If the positions of two or more elements were interchanged, then a different matrix would result (unless the interchanged elements were themselves identical).

It is conventional, for presentational purposes, to place such an array within square brackets, and to label the matrix by a single bold letter (usually upper-case).[20] So in the following expression, **A** is the name we have given to this particular matrix of eight numbers.

$$A = \begin{bmatrix} 0.7 & 0.1 \\ 0.9 & 0.2 \\ 0.3 & 0.2 \\ 0.1 & 0.0 \end{bmatrix}$$

It is also conventional to define the *dimension* of a matrix by the notation $m \times n$ where m is the number of rows occupied by the elements of the matrix and n is the number of columns occupied by elements of the matrix. So, for our example, **A** is of dimension 4 × 2 as its elements span four rows and two columns. Notice that because elements of matrices span rows and columns, they can be handled very conveniently within spreadsheet programs.

Sometimes we want to define a matrix in a more general way, such that its elements are numbers, but those numbers are as yet unspecified. To do this we could write **A** in the more general form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

Notice the way in which each of the elements of this matrix has been labelled. Any one of them is a_{ij} where i denotes the row in which it is found and j denotes its column. With this convention, the bottom right element of the matrix – here a_{42} – will necessarily have a subscript identical to the dimension of the matrix, here 4 ×2.

[20] The use of square brackets is not universal; some authors prefer round brackets or braces.

It is convenient to have another shorthand notation for the matrix array. This is given by

$$A = [a_{ij}]_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$$

The bracketed term here lets the reader know that what is being referred to is a matrix with $m \times n$ elements a_{ij}.

A5.1.2.1 A special form of matrix: the identity matrix

A matrix is said to be *square* if its row and column dimensions are equal (it has the same number of rows and columns). Thus, the matrix

$$B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

is a 2×2 square matrix. Furthermore, if the coefficients of a square matrix satisfy the restrictions that each element along the leading (top left to bottom right) diagonal is 1 and every other coefficient is zero, then that matrix is called an *identity matrix*. Thus the matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a 2 × 2 identity matrix. An identity matrix is often denoted by the symbol **I**, or sometimes by $\mathbf{I}_n$ where the n serves to indicate the row (and column) dimension of the identity matrix in question. In our example, it would be $\mathbf{I}_2$.

A5.1.2.2 Vectors

A vector is a special case of a matrix in which all elements are located in a single row (in which case it is known as a row vector) or in a single column (known as a column vector). Looking at the various rows and columns in the 4×2 matrix **A** above, it is evident that we could make up six such vectors from that matrix. We could construct four row vectors from the elements in each of the four rows of the matrix. And we could make up two column vectors from the elements in each of the two columns.[21] The four row vectors constructed in this way are

$$[a_{11} \quad a_{2}] \quad [a_{21} \quad a_{12}] \quad [a_{31} \quad a_{32}] \quad \text{and}$$
$$[a_{41} \quad a_{42}]$$

each of which is of dimension 1 X 2, while the two column vectors, each of dimension 4 X 1, are given by

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} \text{ and } \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix}$$

A5.1.2.3 The transpose of a matrix or a vector

Various 'operations' can be performed on matrices.[22] One of the most important – and commonly used – is the operation of forming the 'transpose' of a matrix. The transpose of a matrix is obtained by interchanging its rows and columns, so that the first column of the original matrix becomes the first row of the transpose matrix, and so on. Doing this implies that if the original matrix **A** were of dimension $m \times n$, its transpose will be of dimension $n \times m$. The transpose of **A** is denoted as $\mathbf{A}'$, or sometimes as $\mathbf{A}^{\mathrm{T}}$.

Consider two examples. First, let **a** be the 4×1 column vector

$$\text{a}=\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}$$

[21] One could also, of course, make up other vectors as mixtures of elements from different rows or columns.

then its transpose, **a′** is given by the row vector **a′** = $[a_{11}\ a_{21}\ a_{31}\ a_{41}]$.

As a second example, consider the first array that we introduced in this appendix. That matrix and its transpose are given by

$$A=\begin{bmatrix}0.7 & 0.1\\ 0.9 & 0.2\\ 0.3 & 0.2\\ 0.1 & 0.0\end{bmatrix} \quad A'=\begin{bmatrix}0.7 & 0.9 & 0.3 & 0.1\\ 0.1 & 0.2 & 0.2 & 0.0\end{bmatrix}$$

A5.1.2.4 Bold notation for vectors and matrices

As we mentioned earlier, it is conventional to use the **bold** font to denote vectors or matrices, and to use an ordinary (non-bold) font to denote a scalar (single number) term. Hence, in the following expression, we can deduce from the context and the notation employed that each of $\mathbf{a_1}$ and $\mathbf{a_2}$ is a column vector consisting respectively of the first column of scalars and the second column of scalars. We know that the element a_{21}, for example, is a scalar because it is not written in bold font.

$$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\\ a_{41} & a_{42}\end{bmatrix}=\begin{bmatrix}a_1 & a_2\end{bmatrix}$$

[22] From this point on in this appendix, we shall use the term matrix to include both vectors and matrices, unless the context requires that we distinguish between the two.

A5.1.3 Other operations on matrices

As with algebra more generally, several operations such as addition and multiplication can, under some conditions, be performed on matrices.

A5.1.3.1 Addition and subtraction

Two matrices can be added (or subtracted) if they have the same dimension. Essentially, these operations involve adding (or subtracting) comparably positioned elements in the two individual matrices. Suppose that we wish to add the two ($m \times n$) matrices **A** = [a_{ij}] and **B** = [b_{ij}]. The sum, **C** = [c_{ij}] is defined by

$$\mathbf{C} = [c_{ij}] = [a_{ij}] + [b_{ij}] + \text{ where } c_{ij} = a_{ij} + b_{ij}$$

Example:

$$\begin{bmatrix} 7 & 1 \\ 9 & 2 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 9 & 1 \\ 0 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7+3 & 1+0 \\ 9+9 & 2+1 \\ 3+0 & 2+4 \\ 1+2 & 0+3 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 18 & 3 \\ 3 & 6 \\ 3 & 3 \end{bmatrix}$$

Matrix subtraction is equivalent, but with the addition operation replaced by the subtraction operation in the previous expression.

A5.1.3.2 Scalar multiplication

Scalar multiplication involves the multiplication of a matrix by a single number (a scalar). To implement this, one merely multiplies every element of the matrix by that scalar.

Example:

$$\text{If A}=\begin{bmatrix}0.7 & 0.1\\ 0.9 & 0.2\\ 0.3 & 0.2\\ 0.1 & 0.0\end{bmatrix}\text{ then 2A}=\begin{bmatrix}1.4 & 0.2\\ 1.8 & 0.4\\ 0.6 & 0.4\\ 0.2 & 0.0\end{bmatrix}$$

A5.1.3.3 Multiplication of matrices

Suppose that we have two matrices, **A** and **B**. Can these be multiplied by one another? The first thing to note is that here (unlike with ordinary algebra) the order of multiplication matters. Call **A** the lead matrix and **B** the lag matrix. For the matrix multiplication to be possible (or even meaningful) the following condition on the dimensions of the two matrices must be satisfied:

Number of columns in **A** = Number of rows in **B**

If this condition is satisfied, then the matrices are said to be 'conformable' and a new matrix **C** can be obtained which is the matrix product **AB**. The matrix **C** will have the same number of rows as **A** and the same number of columns as **B**.

How are the elements of **C** obtained? The following rule is used.

$$c_{ij} = \sum_{k=l}^{n} a_{ik} b_{kj} \text{ for } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n$$

Example:

$$\text{A}=\begin{bmatrix}2\ 1\\ 0\ 3\\ 1\ 2\end{bmatrix}\text{B}=\begin{bmatrix}3\ 2\\ 4\ 1\end{bmatrix}=\text{AB}=\begin{bmatrix}(2\times 3)+(1\times 4) & (2\times 2)+(1\times 1)\\ (0\times 3)+(3\times 4) & (0\times 2)+(3\times 1)\\ (1\times 3)+(2\times 4) & (1\times 2)+(2\times 1)\end{bmatrix}=\begin{bmatrix}10 & 5\\ 12 & 3\\ 11 & 4\end{bmatrix}$$

An intuitive way of thinking about this is as follows. Suppose we want to find element c_{ij} of the product matrix **C** (the element in the cell corresponding to row *i* and column *j*). To obtain this, we do the following:

- multiply the first element in row *i* by the first element in column *j*
- multiply the second element in row *i* by the second element in column *j*

  .

  .

  and so on up to
- multiply the final element in row *i* by the final element in column *j*

The sum of all these multiplications gives us the number required for c_{ij}. (Note that this process requires the dimension condition that we stated earlier to be satisfied.) This process is then repeated for all combinations of *i* and *j*.

Doing this kind of exercise by hand for even quite small matrices can be very time-consuming, and prone to error. It is better to use a spreadsheet for this purpose. To see how this is done – and to try it out for yourself with an Excel spreadsheet, *Matrix.xls* – read the file *Matrix.doc* in the *Additional Materials* for Chapter 5.

However, we suggest you calculate the products **AB** and **BA** of the following two 2 × 2 matrices **A** and **B** to convince yourself that **AB** does not equal **BA**.

$$A=\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} B=\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

A5.1.3.4 Division

Whereas obtaining the product of two matrices is a meaningful operation in matrix algebra, and can be done providing the two matrices are ‘conformable’, the same cannot be said of matrix division. Indeed, the division of one matrix by another is not a meaningful operation.

A5.1.3.5 The inverse matrix

However, a related concept – matrix inversion – does exist and is fundamental to much that is done in matrix algebra. To motivate this concept, think of ordinary algebra. If *a* and *b* are two numbers then the division of *a* by *b* (i.e. *a*/*b*) can be done, provided that *b* is non-zero. But notice that *a*/*b* can also be written as ab^{-1}, where b^{-1} is the inverse (or reciprocal of *b*).

Where **B** is a matrix, we can under some conditions obtain its inverse matrix, $\mathbf{B}^{-1}$. And if we have a second matrix, say **A**, which has the same number of rows as $\mathbf{B}^{-1}$ has columns, then the product $\mathbf{B}^{-1}\mathbf{A}$ can be obtained.

How is the inverse of B defined? The matrix inverse must satisfy the following equality:

$$\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$$

That is, the product of a matrix and its inverse matrix is the identity matrix. Inspecting the dimension conditions implied by this definition shows that a matrix can only have an inverse if it is a square matrix.

Let us look at an example. The inverse of the matrix

$$A=\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

is given by

$$A^{-1}=\begin{bmatrix} 0 & 1 \\ 0.5 & -1.5 \end{bmatrix}$$

as

$$\begin{bmatrix} 0 & 1 \\ 0.5 & -1.5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0.5 & -1.5 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We will not give any methods here by which an inverse can be obtained. There are many such rules, all of which are tedious or difficult to implement once the matrix has more than 3 rows. Instead, we just report that a modern spreadsheet package can obtain inverse matrices by one simple operation, even for matrices of up to about 70 rows in size. There is clearly no need to bother about deriving an inverse by hand! And, of course, it is always possible to verify that the inverse is correct by checking that its product with the original matrix is **I**.

Once again, to see how this is done, see *Matrix.doc* and *Matrix.xls*.

A5.1.4 The uses of matrix algebra

The two main uses we make of matrix algebra in this text are

- to describe a system of linear equations in a compact way;
- to solve systems of equations or to carry out related computations.

Each of these is used in this chapter (in Section 5.8, where we discuss ambient pollution standards) and in Chapter 8. As an example of the first use, it is evident that the system of equations used in our ambient pollution example,

$$A_1 = d_{11}M_1 + d_{12}M_2$$

$$A_2 = d_{21}M_1 + d_{22}M_2$$

$$A_3 = d_{31}M_1 + d_{32}M_2$$

$$A_4 = d_{41}M_1 + d_{42}M_2$$

can be more compactly written as **A = DM**

where

$$D=\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \\ d_{41} & d_{42} \end{bmatrix} \quad M=\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad A=\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

Check for yourself that, after the matrix multiplication **DM**, this reproduces the original system of four equations.

The potential power of matrix algebra as a computational or solution device is illustrated in our analysis of input–output analysis in Chapter 8. We will leave you to follow the exposition there. As you will see, it is in this context that the inverse of a matrix is useful.

References:

Chiang (1984): Fundamental Methods of Mathematical Economics. 3rd edition. McGraw Hill.