Appendix 5.2 Spatially differentiated stock pollution: a numerical example

This appendix provides a numerical example of a spatially differentiated ambient pollution problem. We obtain the efficient level of *M* for each source and *A* for each receptor. Some of the material below is copied from the output of a Maple file *ambient.mw*. The interested reader can find the Maple file itself in the *Additional Materials* for Chapter 5.

The problem is one in which in the relevant spatial area ('airshed') there are two emissions sources, and two pollution receptors. The **D** matrix of transition coefficients is, therefore of the following form:

$$D_{ij} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

for which we use below the specific values

$$\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$$

Assumptions used:

1. The marginal damage of pollution function is MD(A) = A (a very simple special case), and is identical everywhere.
2. The marginal benefit of emissions function, MB(M), is identical for each firm, and is given by

   $\text{MB}(M_i) = a - bM_i$

   where we assume a = 344 and b = 7.

As shown in the text, an efficient solution requires that for each i, i = 1,2

$$\mathrm{MB}(M_i) = \sum_{j=l}^{N} \left(\frac{\partial}{\partial A_j} D(A_j) \right) d_{ji}$$

which under Assumption (1) is

$$\mathrm{MB}(M_i) = \sum_{j=l}^{N} A_j d_{ji}$$

This is here a two-equation linear system:

$$a - bM_1 = d_{11}A_1 + d_{21}A_2$$

$$a - bM_2 = d_{12}A_1 + d_{22}A_2$$

which gives:

$$a - bM_1 = d_{11}(d_{11}M_1 + d_{12}M_2) + d_{21}(d_{21}M_1 + d_{22}M_2)$$

$$a - bM_2 = d_{12}(d_{11}M_1 + d_{12}M_2) + d_{22}(d_{21}M_1 + d_{22}M_2)$$

We next define an expression (called sys1) that consists of these two equations:

$$sys1 := \{a - bM_2 = d_{12}(d_{11}M_1 + d_{12}M_2)$$

$$+d_{22}(d_{21}M_1 + d_{22}M_2),$$

$$a - bM_1 = d_{11}(d_{11}M_1 + d_{12}M_2)$$

$$+d_{21}(d_{21}M_1 + d_{22}M_2)\}$$

This can be solved (using the 'solve' command in Maple) to obtain solutions for M_1 and M_2 in terms of the parameters, *a*, *b* and the components of the **D** matrix. The solutions are given by

$$M_1 = \frac{\left(b - d_{11}d_{12} - d_{21}d_{22} + d_{12}^2 + d_{22}^2\right)a}{\left(\begin{array}{c} b^2 + bd_{12}^2 + bd_{22}^2 + d_{11}^2 b + d_{11}^2 d_{22}^2 + \\ d_{21}^2 b + d_{21}^2 d_{12}^2 - 2d_{11}d_{12}d_{21}d_{22} \end{array}\right)}$$

$$M_2 = \frac{a\left(b - d_{11}d_{12} + d_{21}^2 + d_{11}^2 + d_{21}d_{22}\right)}{\left(\begin{array}{c} b^2 + bd_{12}^2 + bd_{22}^2 + d_{11}^2 b + d_{11}^2 d_{22}^2 + \\ d_{21}^2 b + d_{21}^2 d_{12}^2 - 2d_{11}d_{12}d_{21}d_{22} \end{array}\right)}$$

To obtain specific values for the solutions, we now substitute the particular values a = 344, b = 7, d_{11} = 2, d_{12} = 4, d_{21} = 3 and d_{22} = 2 for the parameters, giving the solution:

$\{M_1 = 13, M_2 = 6\}$

We next find the efficient ambient pollution levels in the two receptor areas. First define a new system of equations:

sys11 $:=\{A_1 = d_{11}M_1 + d_{12}M_2, A_2 = d_{21}M_1 + d_{22}M_2\}$

This can be solved (using the ‘solve’ command) to obtain solutions for A_1 and A_2 in terms of the components of the **D** matrix and the emission levels, M_1 and M_2:

sols22 $:=\{A_1 = d_{11}M_1 + d_{12}M_2, A_2 = d_{21}M_1 + d_{22}M_2\}$

To obtain specific values for the solutions, we now substitute our assumed particular values for the parameters, giving

$\{A_1 = 50, A_2 = 51\}$