Appendix 8.2 The algebra of the two-sector CGE model

The utility maximisation problem for the household is

$$\text{Max } C_1^{\alpha} C_2^{\beta} \text{ subject to } Y = P_1 C_1 + P_2 C_2$$

for which the Lagrangian is

$$\psi = C_1^{\alpha} C_2^{\beta} + \lambda [Y - P_1 C_1 - P_2 C_2]$$

giving the first-order conditions:

$$\partial \psi / \partial C_1 = \alpha C_1^{\alpha - 1} C_2^{\beta} - \lambda P_1 = 0 \quad (8.41)$$
$$\partial \psi / \partial C_2 = \beta C_1^{\alpha} C_2^{\beta - 1} - \lambda P_2 = 0 \quad (8.42)$$
$$\partial \psi / \partial \lambda = Y - P_1 C_1 - P_2 C_2 = 0 \quad (8.43)$$

Moving the terms in λ to the right-hand side in equations 8.41 and 8.42 and then dividing the first equation by the second gives

$$(\alpha C_1^{\alpha - 1} C_2^{\beta}) / (\beta C_1^{\alpha} C_2^{\beta - 1}) = P_1 / P_2$$

which can be solved for C_2 as

$$C_2 = (\beta P_1 / \alpha P_2) C_1$$

which on substitution for C_2 in equation 8.43 solves for

$$C_1 = [\alpha / (\alpha + \beta) P_1] Y \quad (8.44)$$

Using equation 8.44 to eliminate C_1 from the budget constraint and solving for C_2 yields

$$C_2 = [\beta / (\alpha + \beta) P_2] Y \quad (8.45)$$

Equations 8.44 and 8.45 are equations 8.25 in the text of the chapter.

Now consider the derivation of factor demands and the supply function for a profit-maximising firm, where the production function is Cobb–Douglas in labour, *L*, and oil, *R*:

$$X = L^a R^b \tag{8.46}$$

With *W* and *P* for the prices of labour and oil respectively, total cost is given by:

$$TC = WL + PR \tag{8.47}$$

For cost minimisation, the Lagrangian is

$$\psi = WL + PR + \lambda[X - L^a R^b]$$

and the necessary conditions are

$$\partial\psi/\partial L = W - \lambda a L^{a-1} R^b = 0 \tag{8.48}$$
$$\partial\psi/\partial R = P - \lambda b L^a R^{b-1} = 0 \tag{8.49}$$
$$\partial\psi/\partial\lambda = X - L^a R^b = 0 \tag{8.50}$$

Moving the terms in λ to the right-hand side in equations 8.48 and 8.49 and dividing the first of the resulting equations by the second so as to eliminate λ, we get

$$W / P = (a / b) L^{-1} R$$

or

$$L / R = (a / b) P W^{-1} = (a / b)(P / W) \tag{8.51}$$

which gives the ratio of factor use levels for cost minimisation as depending on the factor price ratio, and the parameters of the production function.

From equation 8.50 we can write

$$L = (X / R^b)^{1/a}$$

and using this in equation 8.51 to eliminate *L* yields, after rearrangement,

$$R = [(b / a)(W / P)]^{a/(a+b)} X^{1/(a+b)} \tag{8.52}$$

for the firm's demand for oil as depending on factor prices and the level of output. Using equation 8.50 again gives

$$R = (X / L^a)^{1/b}$$

which, used in equation 8.51 to eliminate R, leads to

$$L = [(a/b)(P/W)]^{b/(a+b)} X^{1/(a+b)} \tag{8.53}$$

for the firm's demand for labour.

Equations 8.52 and 8.53 are known as 'conditional factor demands', since they give demands conditional on the output level. To get, unconditional, factor demand equations we need to determine the profit-maximising output level. With P_X for the price of output, profits are

$$\pi = P_X X - WL - PR$$

and substituting from the conditional factor demand equations, this is

$$\pi = P_X X - W[(a/b)(P/W)]^{b/(a+b)} X^{1/(a+b)}$$
$$+P[(b/a)(W/P)]^{a/(a+b)} X^{1/(a+b)}$$

or

$$\pi = P_X X - ZX^{1/(a+b)} \tag{8.54}$$

where

$$Z = W[(a/b)(P/W)]^{b/(a+b)} + P[(b/a)(W/P)]^{a/(a+b)} \tag{8.55}$$

Taking the derivative of equation 8.54 with respect to X and setting it equal to zero gives

$$P_X - [1/(a+b)]ZX^{\{1/(a+b)\}-1} \tag{8.56}$$

as necessary for profit maximisation. Solving equation 8.56 for X gives the profit-maximising output level as:

$$X = [\{(a+b)P_X\}/Z]^{\{(a+b)/(1-a-b)\}} \tag{8.57}$$

Equation 8.57 is a supply function giving profit-maximising output as depending on the output price, P_X, and factor prices, P and W.

As in the chapter here, CGE models frequently employ the assumption of constant returns to scale in production. In this case, there is no supply function. For the Cobb–Douglas production function, constant returns to scale means $a + b = 1$, which means that the exponent of X in equation 8.56 is zero. This means that the equation does not involve X and cannot be solved for it to give a supply function. For $a + b = 1$, equation 8.56 becomes

$$P_X = Z$$

which, using equation 8.55 with $a + b = 1$, is:

$$P_X = W[(a/b)(P/W)]^{(1-a)} + P[(b/a)(W/P)]^{a} \tag{8.58}$$

Now, using $a + b = 1$ in equations 8.52 and 8.53 gives

$$R = [(\{1-a\}/a)(W/P)]^{a} X$$

and

$$L = [(a/\{1-a\})(P/W)]^{(1-a)} X$$

so that, dividing by X,

$$U_R = R/X = [(\{1-a\}/a)(W/P)]^{a} \tag{8.59}$$

and

$$U_L = L/X = [(a/\{1-a\})(P/W)]^{(1-a)} \tag{8.60}$$

These two equations give the use of each factor per unit output as functions of the relative prices of the factors. If we knew the level of output X, we could use them to derive factor demands, by multiplying U_R and U_L by the output level.

On rearrangement, equation 8.58 with $a + b = 1$ is

$$\begin{aligned} P_X = {} & W^{a} P^{(1-a)} (a/\{1-a\})^{(1\text{-}a)} \\ & + P^{(1-a)} W^{a} (\{1-a\}/a)^{a} \end{aligned}$$

so that it and equations 8.59 and 8.60 mean that

$$P_X = WU_L + PU_R \tag{8.61}$$

which is the unit cost equation. It has the implication, well known from basic microeconomics, that for constant returns to scale in production, profits are zero at all levels of output.

To see this, note that

$$WU_L + PU_R = W(L/X) + P(R/X) = (WL + PR)/X$$

which is average cost, so that equation 8.61 says that price equals average cost. Although demonstrated here for a Cobb–Douglas production function, this result holds for any constant-returns-to-scale production function. With constant returns to scale, the firm produces to satisfy demand, using the factor input mix that follows from cost minimisation, and makes zero profit.