Appendix 9.1 Some algebra of international treaties

Let signatories be indexed by s and non-signatories by n.

Non-signatories

Non-signatories choose z_n to solve

$$\frac{dB(Z)}{dZ}\frac{dZ}{dz_n} = \frac{dC(z_n)}{dz_n}$$

Noting that $dZ/dz_n = 1$, and that – given our assumption of symmetry – all countries' efficient abatement will be identical, the solution can be written as

$$\frac{dB(Z)}{dZ} = \frac{dC(z_n)}{dz} \qquad (9.4)$$

where $Z = Z_n + Z_s$, $Z_n = (N - k)z_n$ and $Z_s = kz_s$.

Signatories

Choose abatement levels that maximise aggregate payoffs of all signatories:

$$\text{Max } \prod_s = kB(Z) - \sum_{i=1}^{k} C(z_j)$$

The solution requires

$$k\frac{\mathrm{d}B(Z)}{\mathrm{d}Z}\left[\frac{\partial Z}{\partial Z_s}\cdot\frac{\partial Z_s}{\partial Z_j}+\frac{\partial Z}{\partial Z_n}\cdot\frac{\partial Z_n}{\partial Z_s}\cdot\frac{\partial Z_s}{\partial Z_j}\right]=\frac{\mathrm{d}C(z_j)}{\mathrm{d}z_j}$$

for all $j = 1,\ldots, k$ (9.5)

$$k\frac{\mathrm{d}B(Z)}{\mathrm{d}Z}\left[1\cdot\frac{\partial Z_s}{\partial Z_j}+1\cdot\frac{\partial Z_n}{\partial Z_s}\cdot\frac{\partial Z_s}{\partial Z_j}\right]=\frac{\mathrm{d}C(z_j)}{\mathrm{d}z_j}$$

for all $j = 1,\ldots, k$

What determines $\partial Z_n/\partial Z_s$? It is chosen so that signatories would not wish to revise their choices after the choices of non-signatories. Those non-signatory choices are determined by 9.4 above.

Totally differentiating 9.4 and noting that $\mathrm{d}Z = \mathrm{d}Z_s + \mathrm{d}Z_n$ and $\mathrm{d}z_n = \mathrm{d}Z_n/(N - k)$ we obtain

$$\frac{\partial Z_n}{\partial Z_s}=\frac{\frac{\mathrm{d}^2B(Z)}{\mathrm{d}Z^2}\cdot(N-k)}{\frac{\mathrm{d}^2C(z)}{\mathrm{d}z^2}-\frac{\mathrm{d}^2B(Z)}{\mathrm{d}Z^2}.(N-k)} \qquad (9.6)$$

Then substitute equation (9.6) into (9.5), and add (9.4). This gives two equations which we shall not reproduce here, but will just label as equations (9.7) and (9.8).

A self-enforcing agreement also requires that

- no signatory can gain by unilaterally withdrawing from the agreement;
- no non-signatory can gain by unilaterally acceding to the agreement;

which together imply that

$$\prod_s (k^*) \geq \prod_n (k^* - 1) \text{ and } \prod_s (k^*) \geq \prod_s (k^* + 1) \tag{9.9}$$

Equations 9.7, 9.8 and 9.9 give us three equations in 3 unknowns from which we can solve for z_n*, z_s* and k*.