

## Using Excel to do matrix algebra computations to illustrate the material on ambient pollution standards in Chapter 5, section 5.8

### The notation used: a reminder

J distinct receptors, each being indexed by the subscript j (so j = 1, 2, ..., J)

N distinct pollution sources, each being indexed by the subscript i (so i = 1, 2, ... , N)

Transfer coefficient  $d_{ji}$  describes the impact on pollutant concentration at receptor j attributable to source i.

$M_i$  is the total emissions from source i.

$A_j$  is the total level, or concentration rate, of pollution at location j, given by:

$$A_j = \sum_{i=1}^N d_{ji} M_i \quad (5.6)$$

Sheet 1

This computes the  $A_j$  values for the numerical example in Section 5.8. There we have  $N = 2$  sources and  $J = 4$  receptors. All eight  $d_{ji}$  coefficients form the  $J \times N$  matrix,  $\mathbf{D}$ . Denoting the vector of emissions from the two sources as  $\mathbf{M}$  and the vector of ambient pollution levels in the four receptors as  $\mathbf{A}$  we have

$$\mathbf{A} = \mathbf{D} \mathbf{M} \quad (5.8)$$

or

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \\ d_{41} & d_{42} \end{bmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \quad (5.9)$$

Thus we have

$$A_1 = d_{11}M_1 + d_{12}M_2 \quad (5.7a)$$

$$A_2 = d_{21}M_1 + d_{22}M_2 \quad (5.7b)$$

$$A_3 = d_{31}M_1 + d_{32}M_2 \quad (5.7c)$$

$$A_4 = d_{41}M_1 + d_{42}M_2 \quad (5.7d)$$

Knowledge of the **M** vector and the **D** matrix allow us to calculate ambient pollution levels at each receptor. If, for example, **D** and **M** are

$$\mathbf{D} = \begin{bmatrix} 0.7 & 0.1 \\ 0.9 & 0.2 \\ 0.3 & 0.2 \\ 0.1 & 0.0 \end{bmatrix} \text{ and } \mathbf{M} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

then  $A_1 = 9$ ,  $A_2 = 13$ ,  $A_3 = 7$  and  $A_4 = 1$ , as shown in the marked block of cells H8 to H11, and labelled as **A**. Sheet 1 has been set up as a complete, finished sheet, with the appropriate Excel commands already built into the worksheet. In a moment, we will illustrate how a worksheet like this can be set up from scratch. But first, notice that if you change any of the values of the  $d_{ji}$  coefficients (in cells B8 to C11) or of the  $M_i$  (in cells F8 and F9), the solution values of **A** will be automatically recomputed.

Now we show how to construct an Excel spreadsheet from scratch to do calculations of this kind. The example used in this case is that given in Appendix 5.2. The problem there is one in which there are two emissions sources, and two pollution receptors. The **D** matrix of transition coefficients is given by

$$\mathbf{D} = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$$

First of all, we just note that the socially-efficient solution of this example gives  $M_1 = 13$ ,  $M_2 = 6$ ,  $A_1 = 50$  and  $A_2 = 51$ . Let us demonstrate that, for the given matrix **D**, the values of **A** must be  $A_1 = 50$  and  $A_2 = 51$  if  $M_1 = 13$  and  $M_2 = 6$ .

Here are the necessary steps:

1. Move to a blank worksheet in the workbook (or create a new sheet)
2. Write the arrays of numbers for the D and M matrices in appropriate places. For example, place the four elements of the 2×2 D matrix in cells B8 to C9 and the two elements of the M vector in cells F8 and F9.
3. Decide where the solution values for A will be placed. Here A will be a 2×1 vector, so we could place those values in cells H8 and H9 (when they get calculated).
4. Move the cursor to the top cell of the location you have chosen for the A vector (in this case, to cell H8)
5. In the formula box, insert the = sign (to let Excel know you are about to enter a formula) and then select the MMULT function.
6. In the dialogue box that appears, for Array 1 give the cell references for the D matrix. That is: B8:C9
7. For Array 2, give the cell references for the M vector that is to post-multiply D. That is: F8:F9
8. Click OK to accept. The value 50 should now appear in cell F8
9. Using the mouse, select and highlight all cells into which the values of the A vector should be placed. That is, F8 and F9. Leave these cells highlighted.
10. Move the mouse pointer to the end of the formula that appears in the formula box, and left click the mouse once.
11. Now press F2, followed by keys CNRL + SHIFT + ENTER together. The two solution values for A (50 and 51 should now appear).

(A finished example of this exercise is given in sheet *Example in App 5.2* if you wish to verify that your sheet is correct.)

**PLEASE NOTE:** Matrix algebra command syntax may vary with later versions of Excel. Use the Excel HELP function for the MMULT command to check appropriate syntax.

We now move on to computing the efficient solution to this problem. This is done analytically in Appendix 5.2 itself, using the marginal efficiency condition. An alternative approach is to use Excel's SOLVER routine to find the answer by a linear programming technique.

Here are the appropriate steps

1. Copy the spreadsheet you have just constructed. In this copied sheet, change the values of M1 and M2 to 9 and 17 (These numbers are arbitrary, and just move the system away from its socially-efficient solution.)
2. In cell E12, enter a formula which calculates the net benefit. This given by

$$=344*F8-(7/2)*F8^2+344*F9-(7/2)*F9^2-(H8^2)/2-(H9^2)/2$$

and is obtained by integrating the respective marginal cost and marginal benefit functions to get the total cost and benefit functions, using the following information:

- The marginal damage of pollution function is  $MD(A) = A$  and is identical everywhere.

- The marginal benefit of emissions function,  $MB(M)$ , is identical for each firm, and is given by  $MB(M_i) = a - b \cdot M_i$  where we assume  $a = 344$  and  $b = 7$ .

Thus our formula corresponds to NB where

$$NB = \sum_{i=1}^N B_i(M_i) - \sum_{j=1}^J D_j(A_j) \quad (5.10)$$

3. Finally we use SOLVER to select values for  $M_1$  and  $M_2$  to maximise NB, the value of NB calculated in cell
4. This gives the efficient solution  $M_1 = 13$  and  $M_2 = 6$ .

An interesting property of the solution to Equation set 5.13 is that not only will the efficient emission level differ from firm to firm, but also the efficient ambient pollution level will differ among receptors.

In practice, environmental regulators might deem that is unethical for  $A$  to vary from place to place. So, they might impose an additional constraint on the problem to reflect this ethical position. One form of constraint is that the pollution level in no area should exceed some maximum level  $A^*$  (that is  $A_j^* \leq A^*$  for all  $j$ ). Another, stricter, version would be the requirement that  $A$  should be the same over all areas (that is  $A_j^* = A^*$  for all  $j$ ). In the latter case, the net benefit function to be maximised is

$$L = \sum_{i=1}^N B_i(M_i) - \sum_{j=1}^J D_j(A^*) \quad (5.14)$$

By imposing additional constraints, maximised net benefit is lower in Equation 5.14 than in Equation 5.10. An efficiency loss has been made in return for achieving an equity goal.