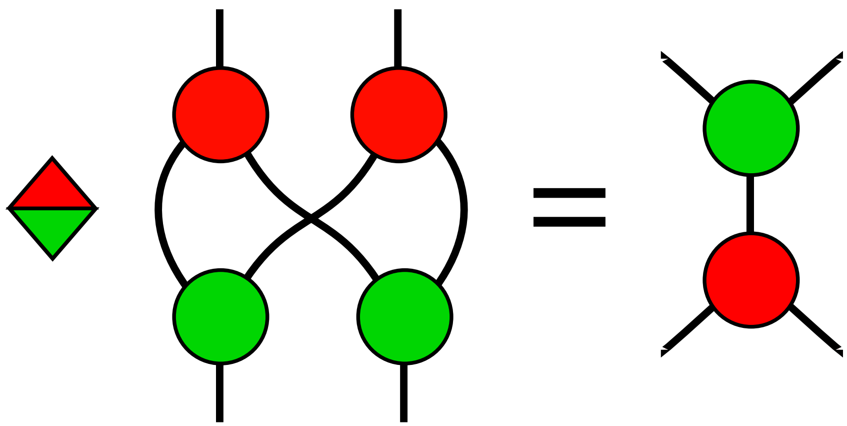
Mutually Unbiased Embeddings of Classical Logic



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Joint work with Bob Coecke

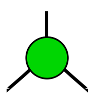
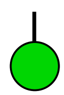
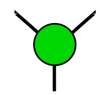
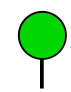
Motivation

- Quantum observables may be incompatible: position/momentum, polarisation, spin ...
- In traditional quantum logic approaches these observables are simply *incomparable* in the lattice.
- However if one wants to *compute* with quantum mechanics we need know how these observables relate to each other.

We want to extract the *constructive content* from the incompatibility of quantum observables.

Classical Structures

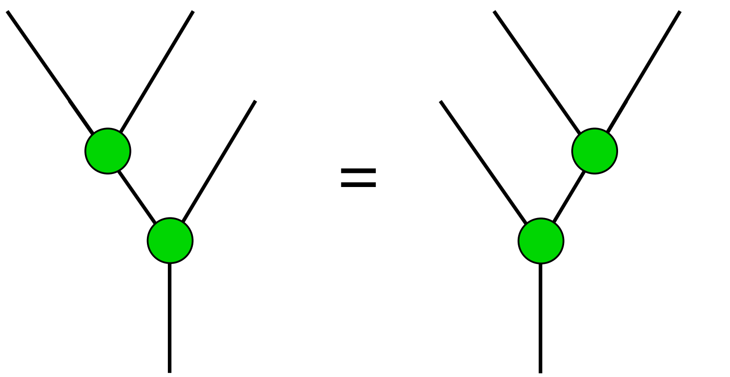
Classical Structures capture the algebraic aspects of copying and deleting;

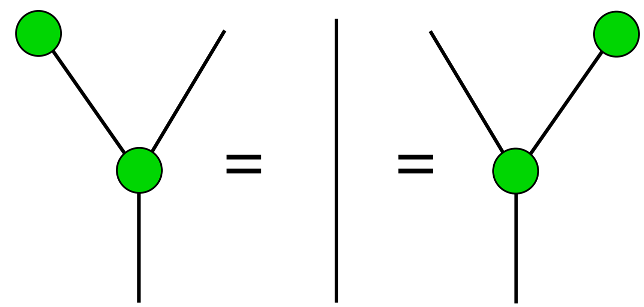
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

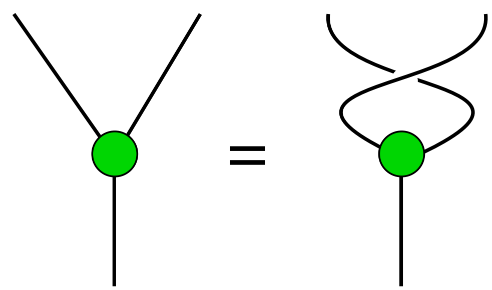
An example:

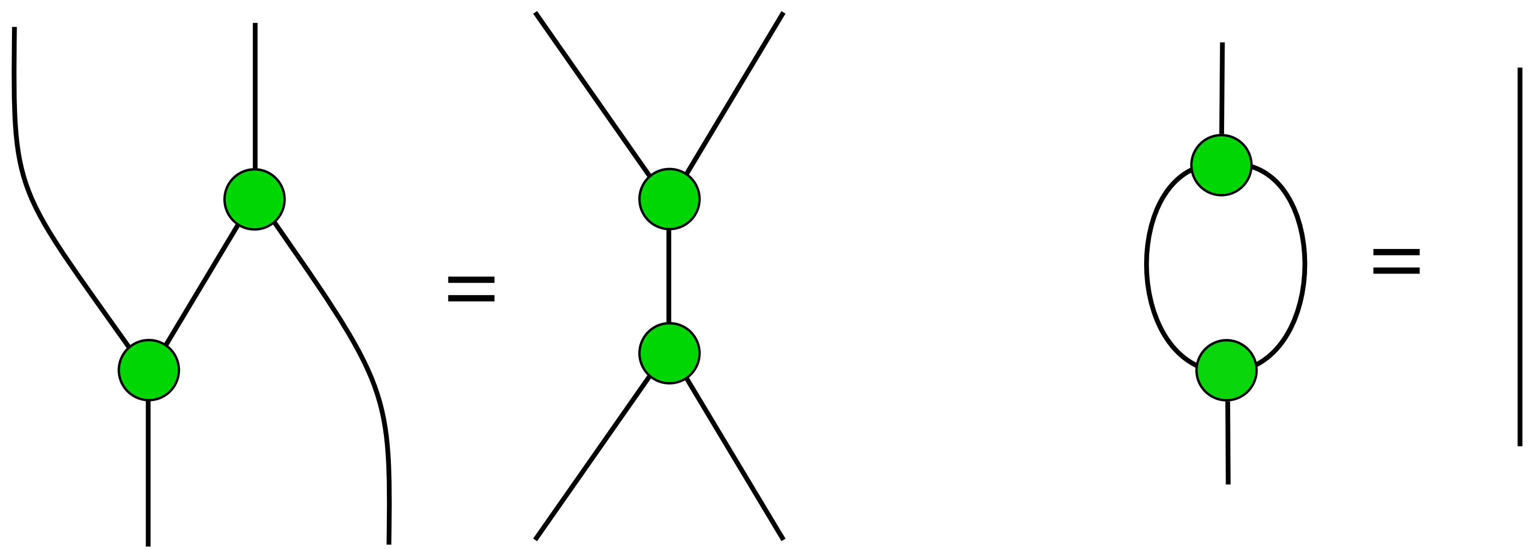
$$\delta : |i\rangle \mapsto |ii\rangle \qquad \epsilon : \sum_i |i\rangle \mapsto 1$$

Classical Structures









Cloning

Consider the map:

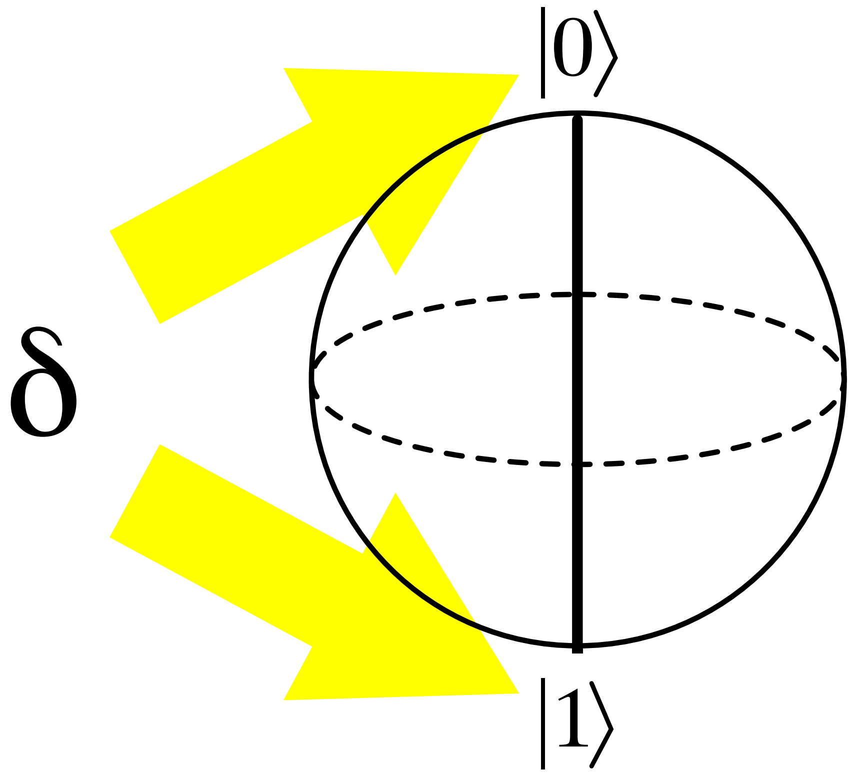
$$\delta_Z : Q \to Q \otimes Q :: |i\rangle \mapsto |ii\rangle$$

δ_Z is the *cloning* map for the basis $|0\rangle, |1\rangle$.

Obviously δ_Z is cannot clone all states:

$$\delta_Z |+\rangle = \delta_Z(|0\rangle + |1\rangle) = |00\rangle + |11\rangle$$

However, since quantum states are indistinguishable up to global phase the *vectors* $e^{i\alpha} |0\rangle$ and $e^{i\beta} |1\rangle$, are also cloned, when viewed as quantum states; hence can view δ as fixing an *observable* i.e. an axis of the Bloch sphere.


$|0\rangle$
δ
$|1\rangle$

Deleting

Q: How to "erase" a quantum state $|\psi\rangle$ known to be in some given basis?

A: Use a measurement which gives *no information* about the existing state — i.e measurement in a basis $\{b_i\}$ such that

$$\begin{aligned} |\langle b_i \mid \psi\rangle| &= |\langle b_j \mid \psi\rangle| \\ \Rightarrow \quad |\langle b_i \mid a_k\rangle| &= |\langle b_j \mid a_k\rangle| \\ \Rightarrow \quad |\langle b_i \mid a_k\rangle| &= \frac{1}{\sqrt{d}} \text{ (in finite dim.)} \end{aligned}$$

Hence the idea of *Mutually Unbiased Bases* arise very naturally from the idea of *deleting* a classical value embedded in a quantum state space.

With respect to the Z-basis (i.e. $|0\rangle, |1\rangle$), the unbiassed states are exactly:

$$\epsilon_Z^\alpha = (|0\rangle + e^{i\alpha}|1\rangle)/2$$

However if we compose ϵ_Z^α with δ_Z:

$$(\mathrm{id} \otimes \epsilon_Z^\alpha) \circ \delta_Z = Z_{-\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

Therefore the “copying” and “erasing” operations when combined induce a non-trivial action on states, unless $\alpha = 0$ (Will come back to this a bit later).

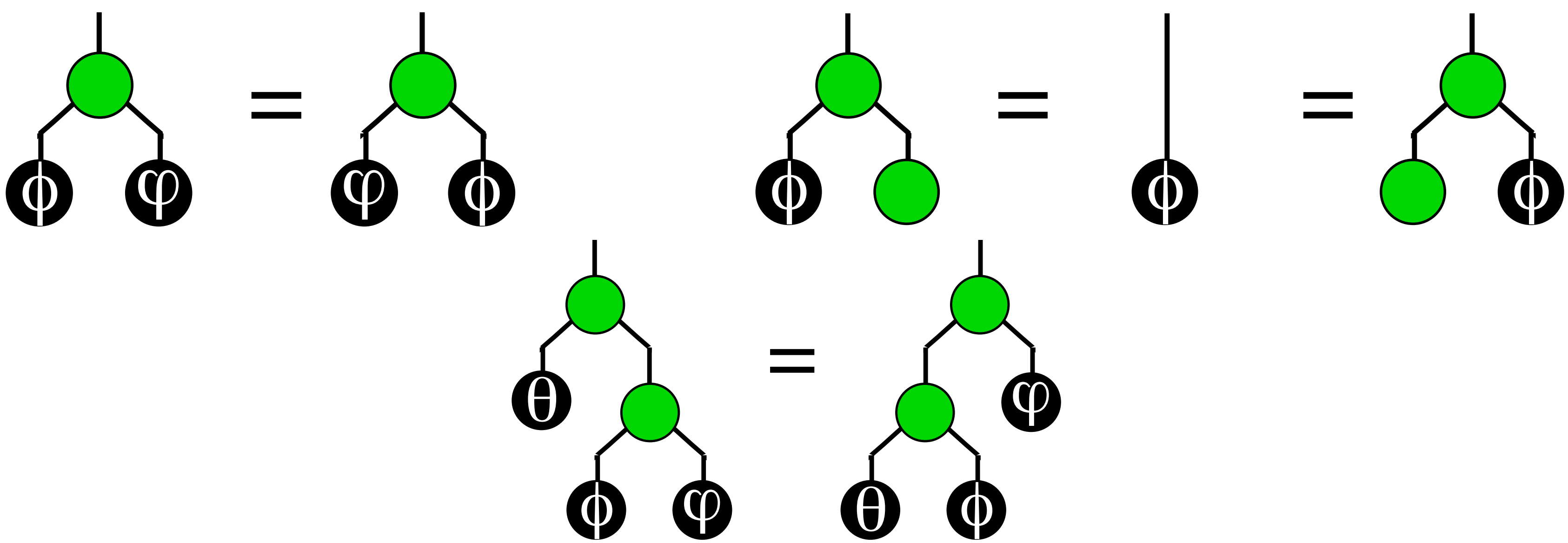
Thus, we have a classical structure:

- δ_Z is the *cloning* map for the basis $|0\rangle, |1\rangle$.
- ϵ_Z is the *uniform deleting* of this basis.

The monoid of points

Let (A, δ, ϵ) be a classical structure; since $(\delta^\dagger, \epsilon^\dagger)$ is a commutative monoid we have a commutative multiplication on points $\psi, \phi : I \to A$, defined by

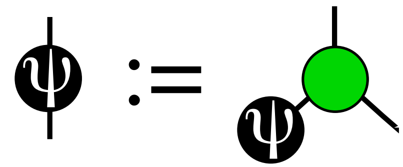
$$\phi \odot \psi = \delta^\dagger \circ (\phi \otimes \psi)$$



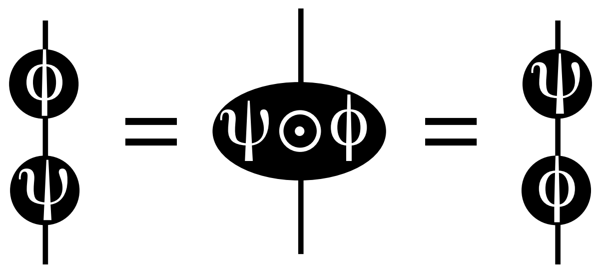
The monoid "currying"

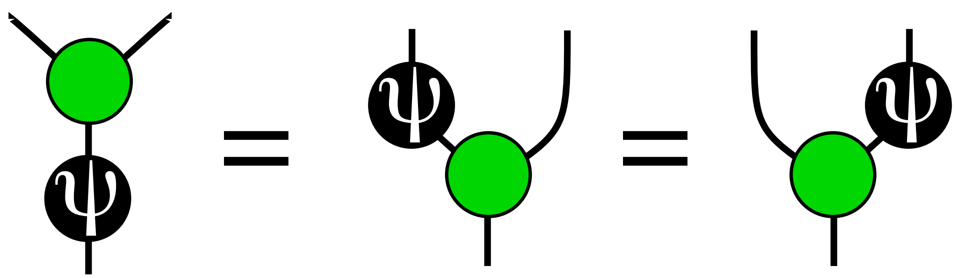
Hence we have a lifting operation:

$$\Lambda : \mathcal{C}(I, A) \to \mathcal{C}(A, A) :: \psi \mapsto \delta^\dagger \circ (\psi \otimes \mathrm{id}_A)$$



This is homomorphism of monoids so:

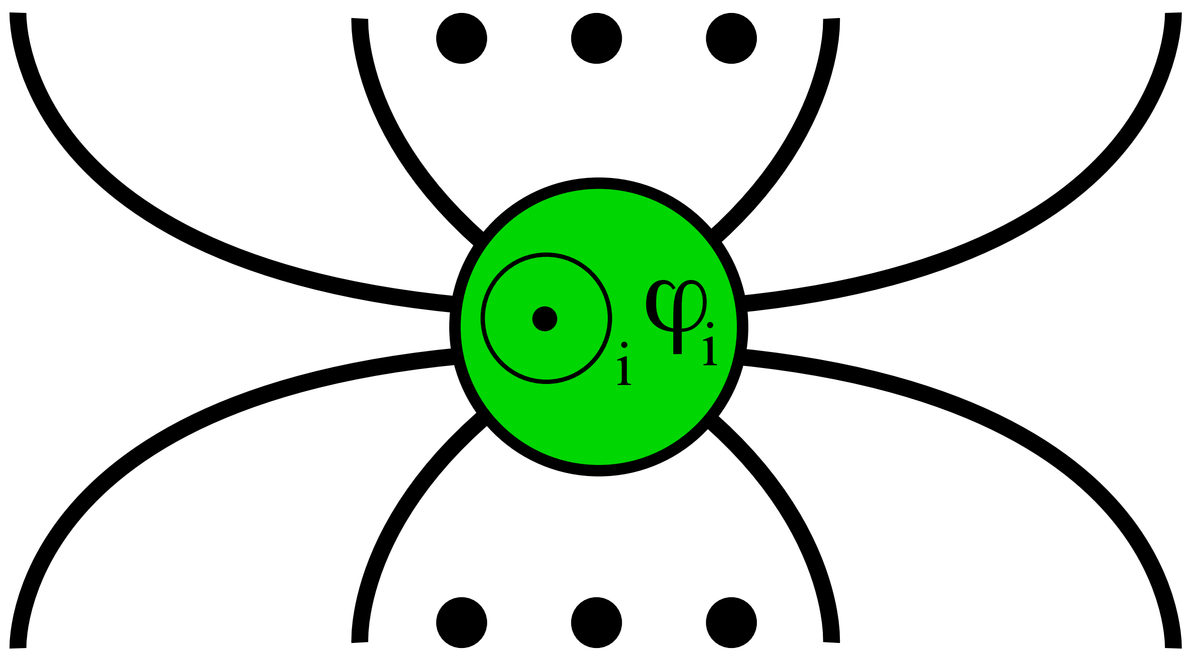




Also: $(\Lambda(\psi))^\dagger = \Lambda(\psi_*)$

Generalised Spider Theorem

Theorem. *Any map constructed from δ, ϵ, some points ψ_i and the adjoints of all of these, is determined only by its type and a product $\bigodot_i \psi_i$:*



Unbiassed Points

Proposition 1. *Let* $\{|i\rangle\}_i$ *be a basis for some finite dimensional Hilbert space, and let*

$$\delta : |i\rangle \mapsto |ii\rangle$$

be a cloning map for this basis; let $|\psi\rangle$ *be any point. Then* $\Lambda(\psi)$ *is unitary if and only if* $|\psi\rangle$ *is unbiassed with respect to* $\{|i\rangle\}$*.*

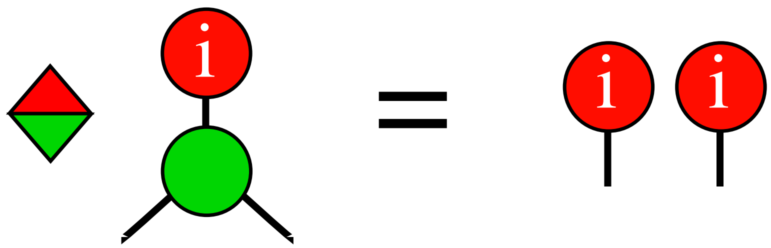
This allows us to *define* unbiassedness in the abstract setting.

Definition 2. A point $\psi : I \to A$ is unbiassed for relative to (A, δ, ϵ) if $\Lambda(\psi)$ is unitary.

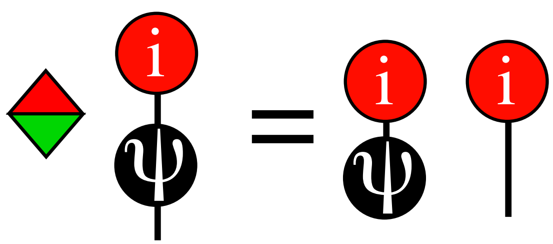
Note then that the set of unbiassed points form an abelian group with respect to the $\odot$ operation.

Classical Points

Definition 3. A point $i : I \to A$ is called *classical* relative to (A, δ, ϵ) if $\delta \circ i = i \otimes i$.



The classical points are eigenvectors is a suitable sense:



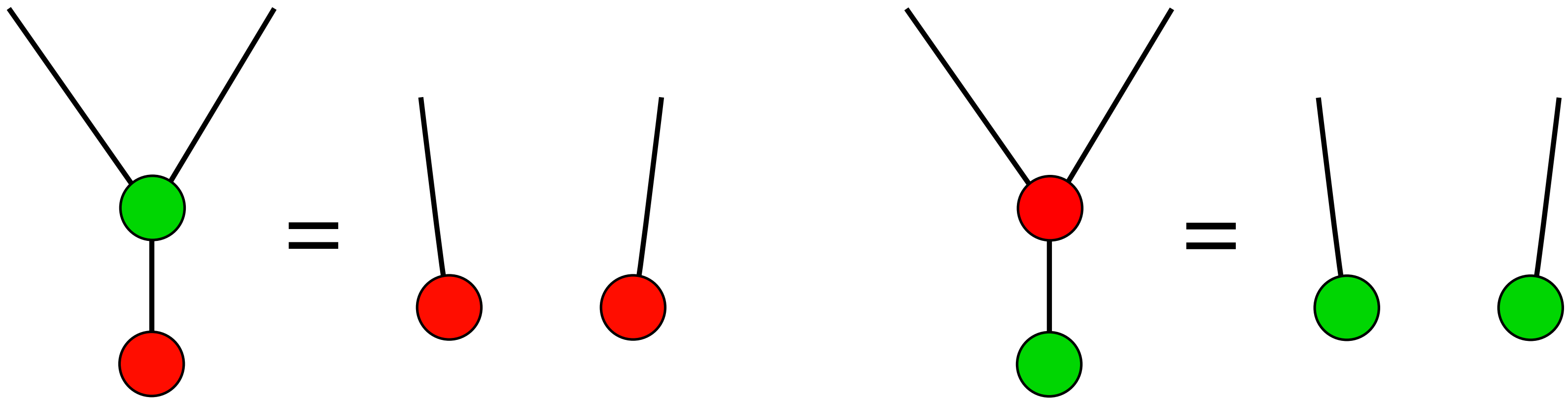
Complementary Classical Structures

Definition 4. *Two classical structures* $(A, \delta_X, \epsilon_X)$ *and* $(A, \delta_Z, \epsilon_Z)$ *in a* $\dagger$*-SMC are called* complementary *if they obey the following rules:*

- *whenever* $z_i : I \to A$ *is classical for* (δ_X, ϵ_X) *it is unbiased for* (δ_Z, ϵ_Z);
- *whenever* $x_j : I \to A$ *is classical for* (δ_Z, ϵ_Z) *it is unbiased for* (δ_X, ϵ_X);
- $\epsilon_X^\dagger$ *is classical for* (δ_Z, ϵ_Z) *and* $\epsilon_Z^\dagger$ *is classical for* (δ_X, ϵ_X).

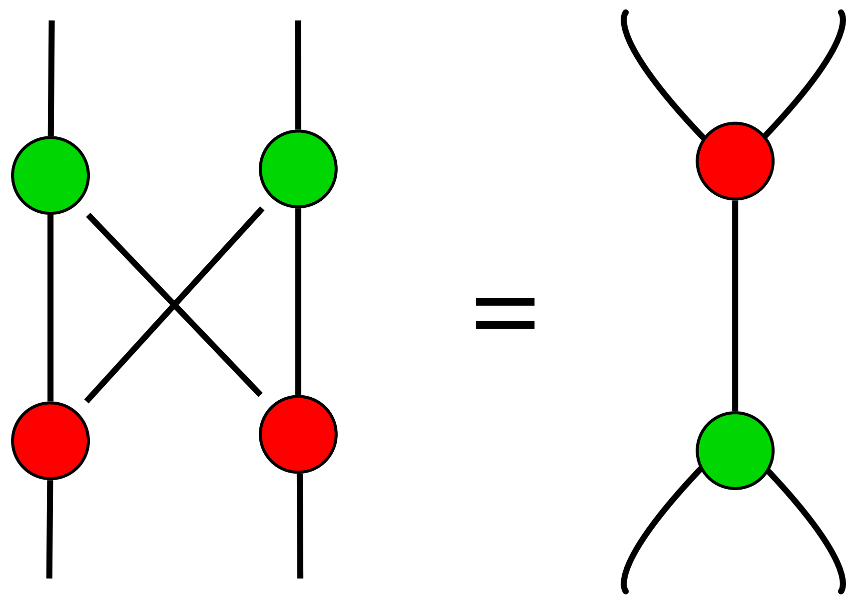
Bialgebraic Laws for Complementary Classical Structures

Cloning Laws:



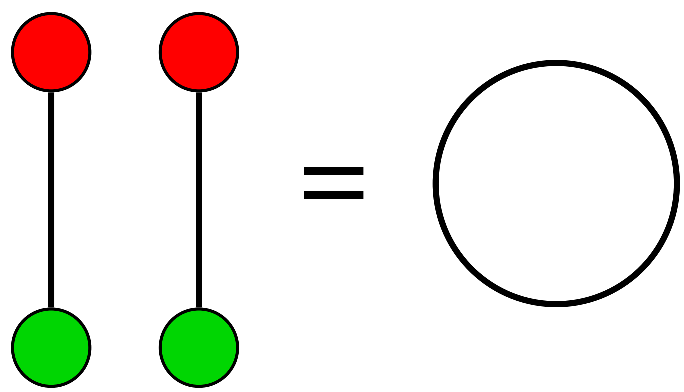
Bialgebraic Laws for Complementary Classical Structures

Bialgebra Law:



Bialgebraic Laws for Complementary Classical Structures

Dimension Law:



The pair of non-commuting observables fails to be a true bialgebra: every equation has a (hidden) scalar factor. Call this structure a *scaled bialgebra*.

Bialgebraic Laws for Complementary Classical Structures

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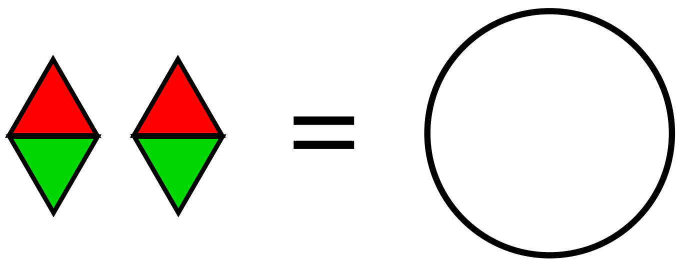

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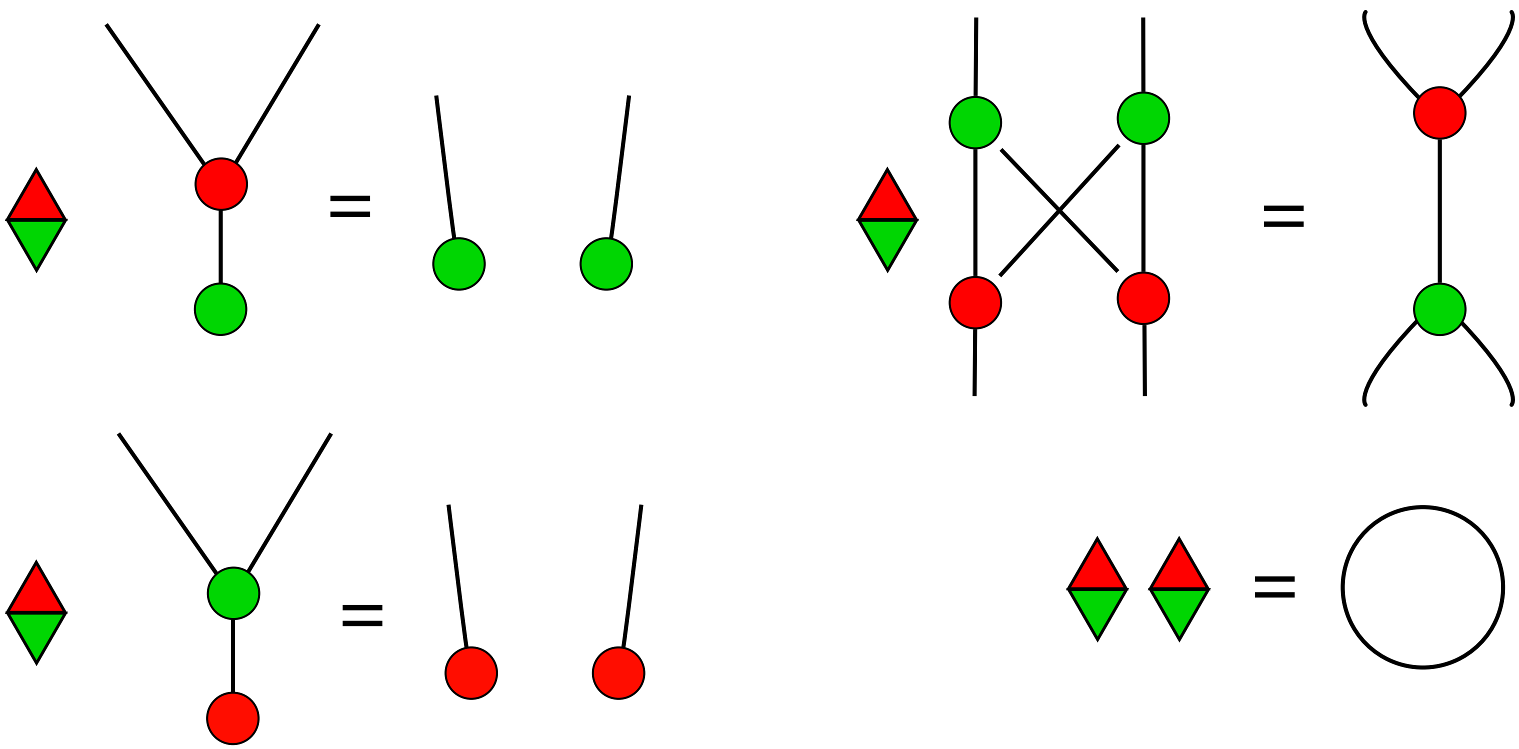
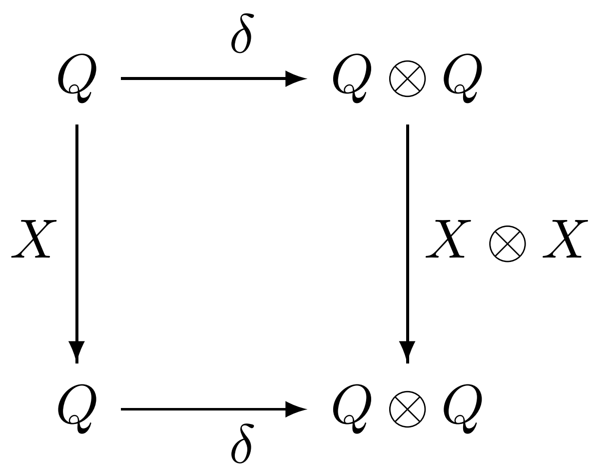
Scaled Bialgebra Laws



“Negation”

$$X_\pi = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad :: \begin{cases} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle \end{cases}$$



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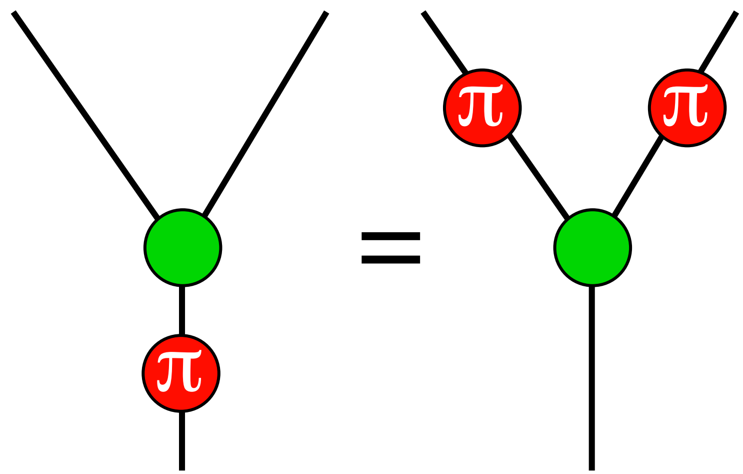
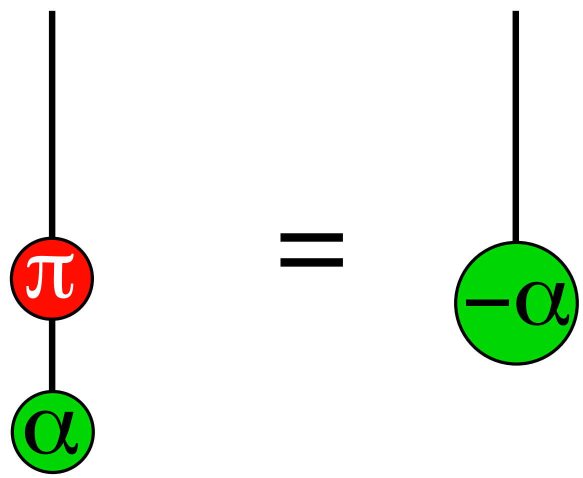


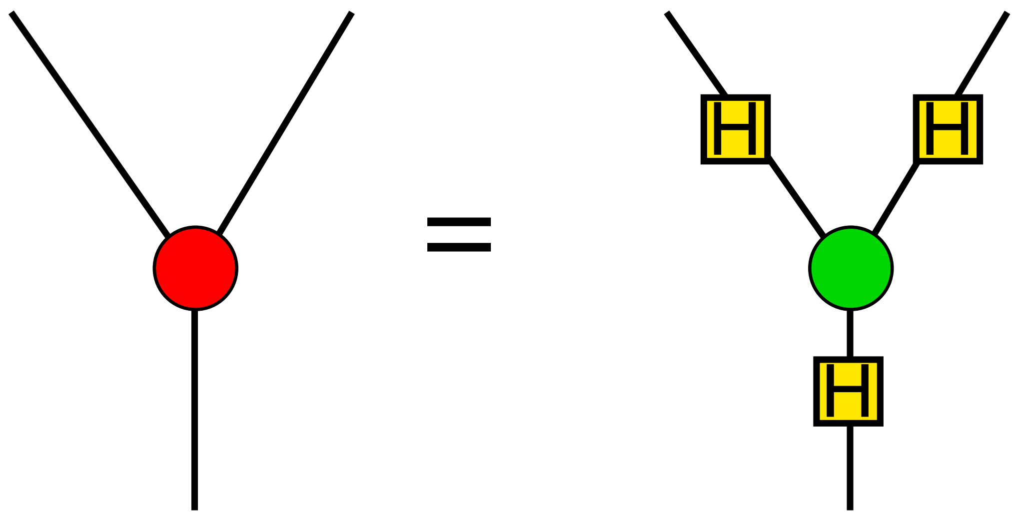
“Negation”

$$X :: |0\rangle + e^{i\alpha}|1\rangle \mapsto e^{i\alpha}|1\rangle + |0\rangle = |0\rangle + e^{-i\alpha}|1\rangle$$



Hadamard as a Mediating Map

We can define the red classical structure in terms of H and the green structure:

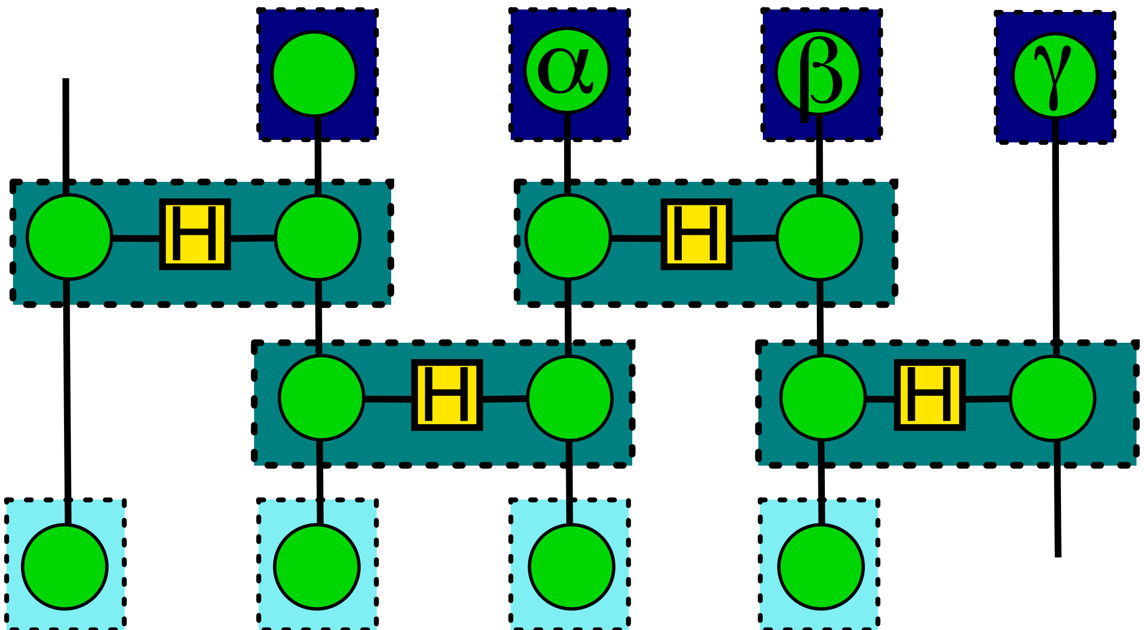


We can immediately derive a law for changing the colour of dots by introducing H boxes – in fact this gives a general “colour duality”.

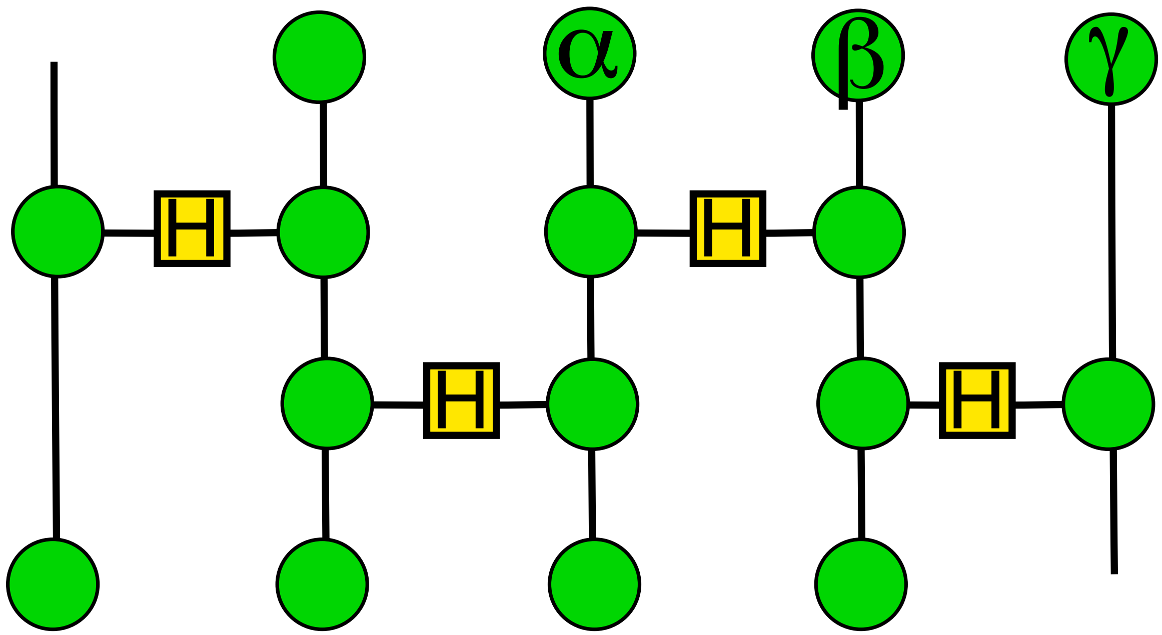
Example: A MBQC 1-qubit unitary

Proposition 5. *If U is a unitary on $\mathbb{C}^2$ there exist α, β, γ such that $U = Z_\alpha X_\beta Z_\gamma$.*

Here is (part of) a measurement based program to compute this:



Example: A MBQC 1-qubit unitary



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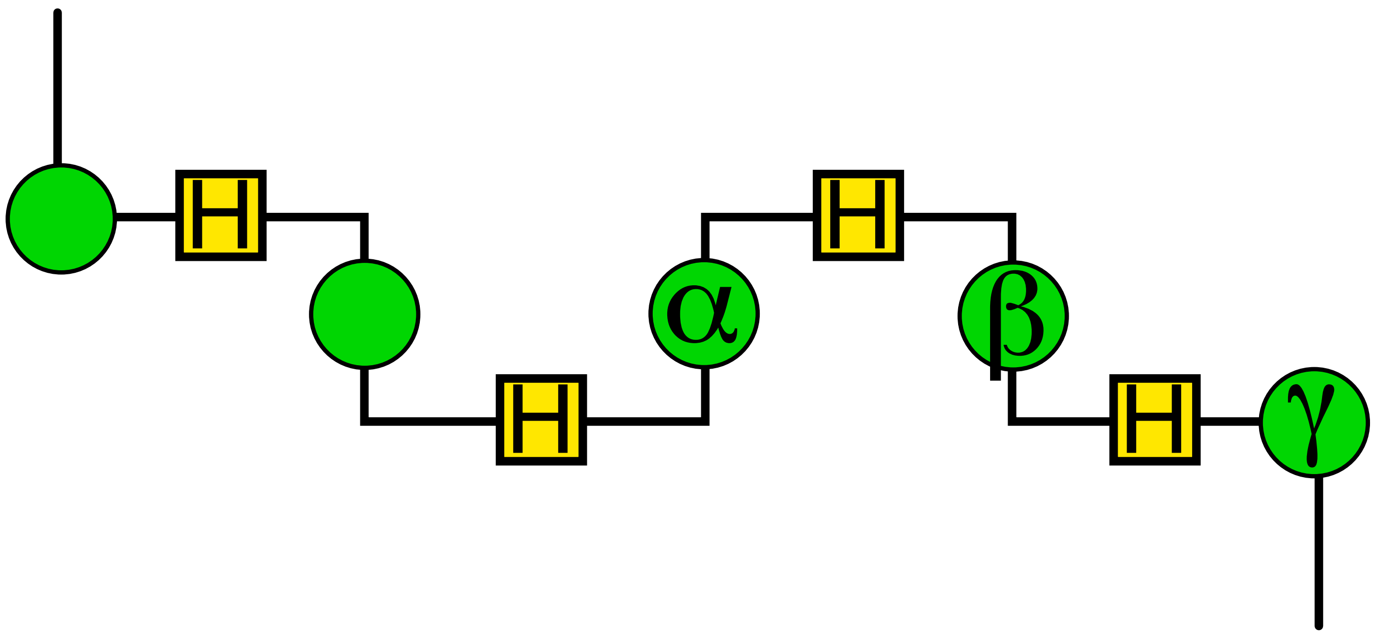

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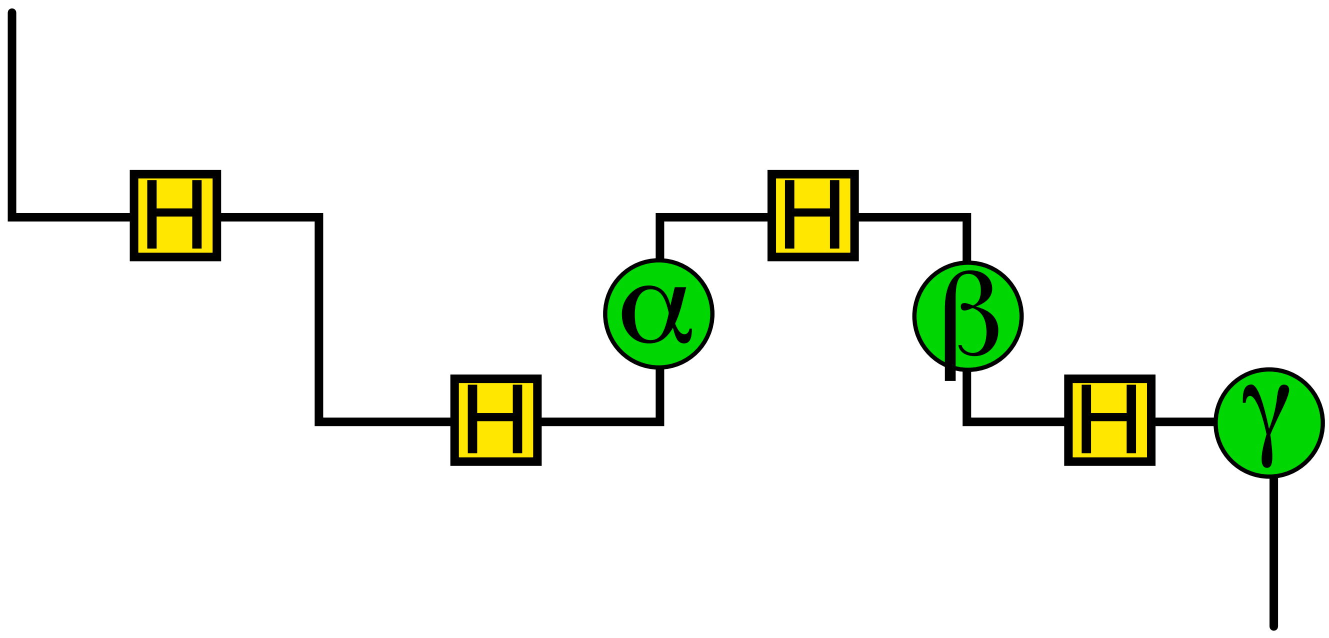

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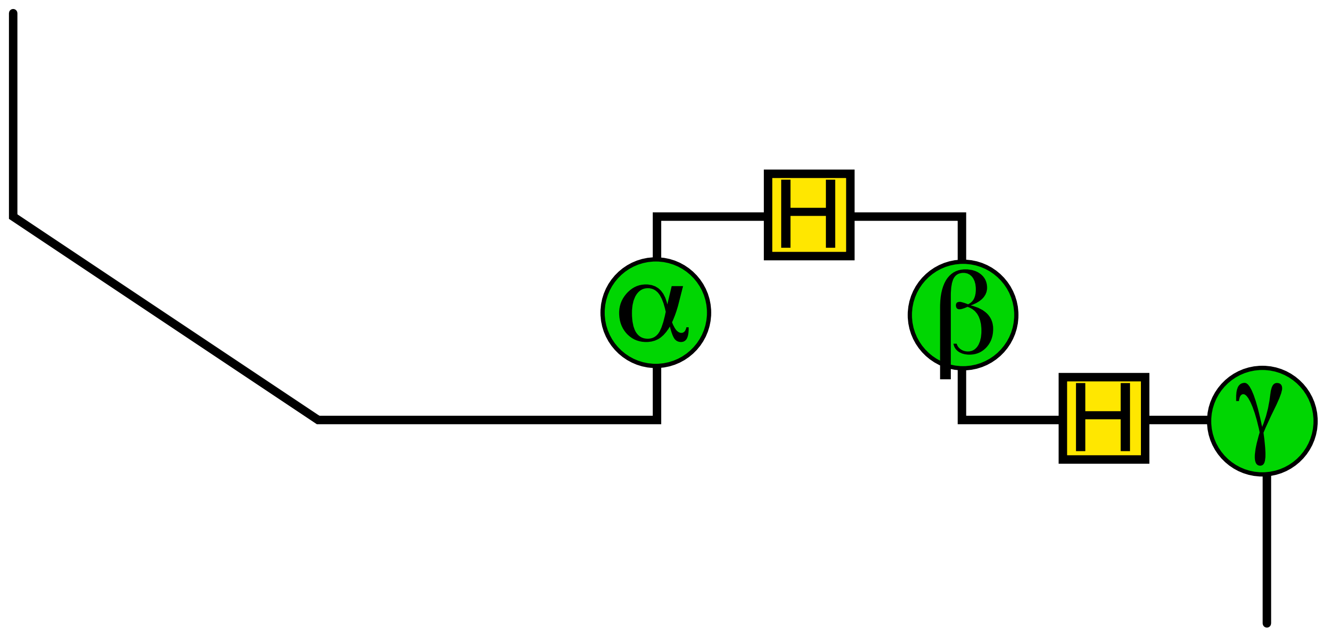

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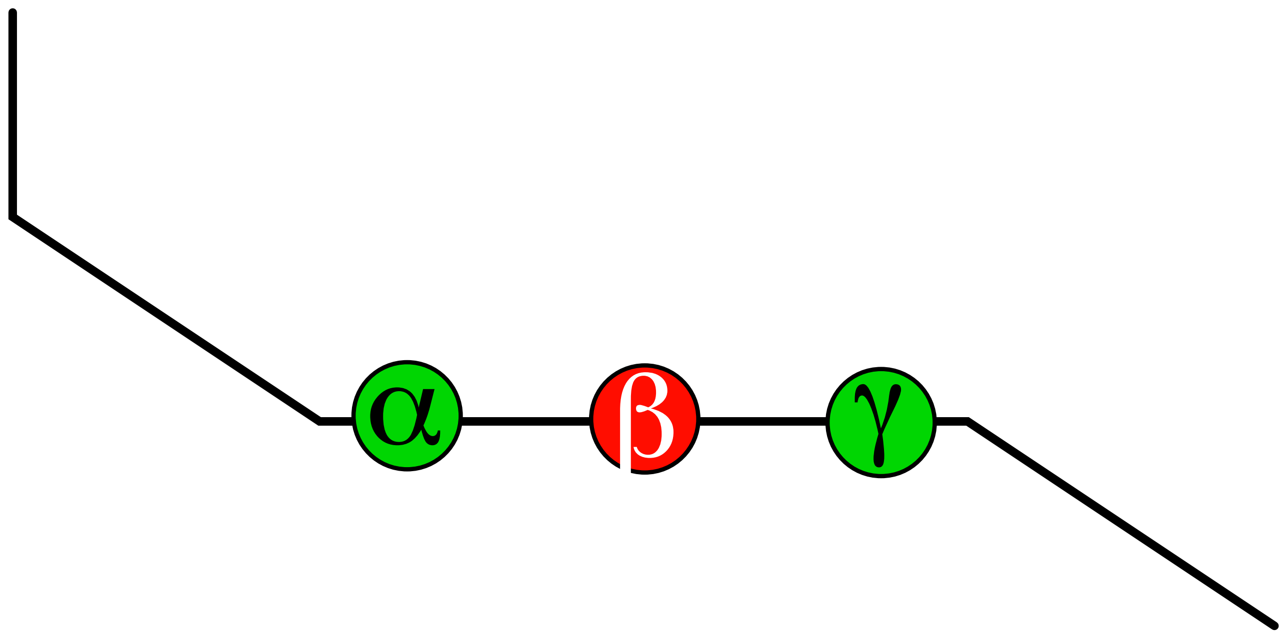


$$= Z_\alpha X_\beta Z_\gamma$$

Example: Quantum Fourier Transform

Among the most important quantum algorithms, the quantum Fourier transform is a key stage of factoring.

$$|j_0 j_1 \cdots j_n\rangle \mapsto (|0\rangle + e^{2\pi i \alpha_0} |1\rangle)(|0\rangle + e^{2\pi i \alpha_1} |1\rangle) \cdots (|0\rangle + e^{2\pi i \alpha_n} |1\rangle)$$

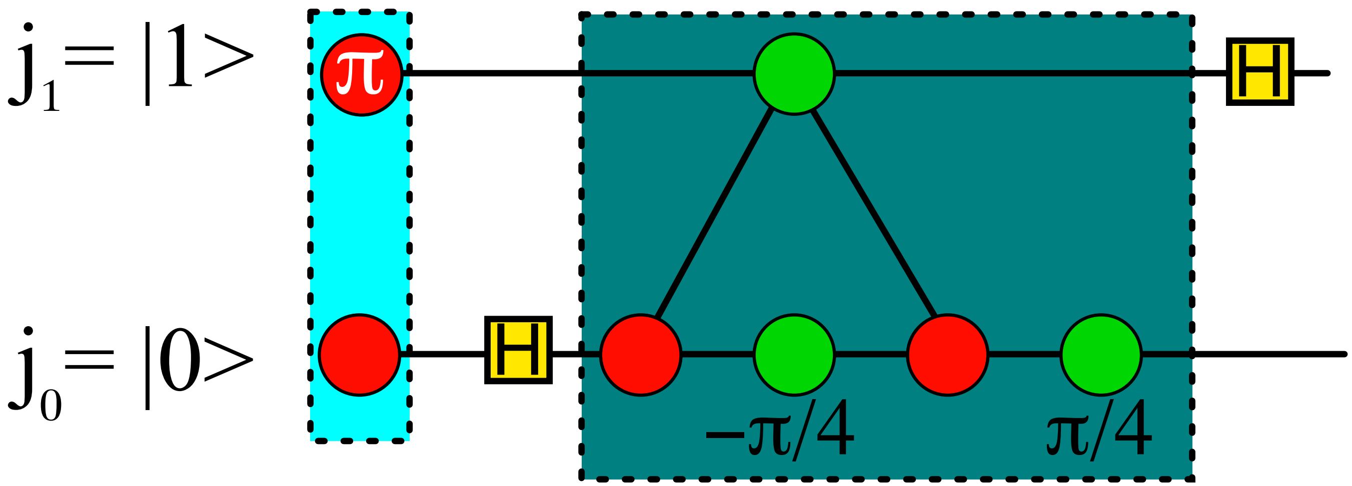
where $\alpha_k = 0.j_k \cdots j_n = \sum_{l=k}^{n} j_l / 2^k$

For 2 qubits:

$$|00\rangle \mapsto (|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \qquad |10\rangle \mapsto (|0\rangle + e^{i\pi} |1\rangle)(|0\rangle + |1\rangle)$$

$$|01\rangle \mapsto (|0\rangle + e^{i\pi/2} |1\rangle)(|0\rangle + e^{i\pi} |1\rangle) \qquad |11\rangle \mapsto (|0\rangle + e^{i3\pi/2} |1\rangle)(|0\rangle + e^{i\pi} |1\rangle)$$

Example: Quantum Fourier Transform



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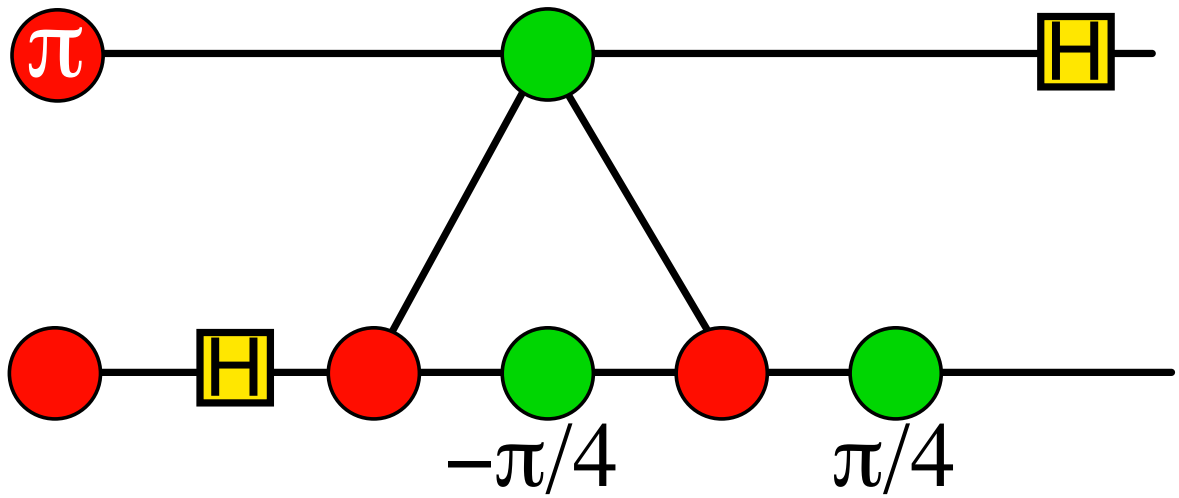

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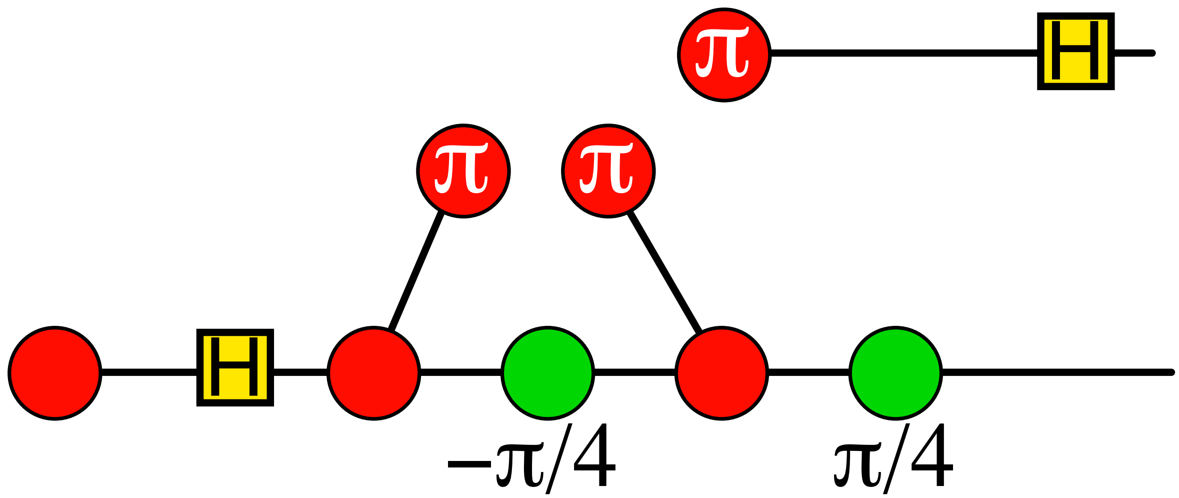

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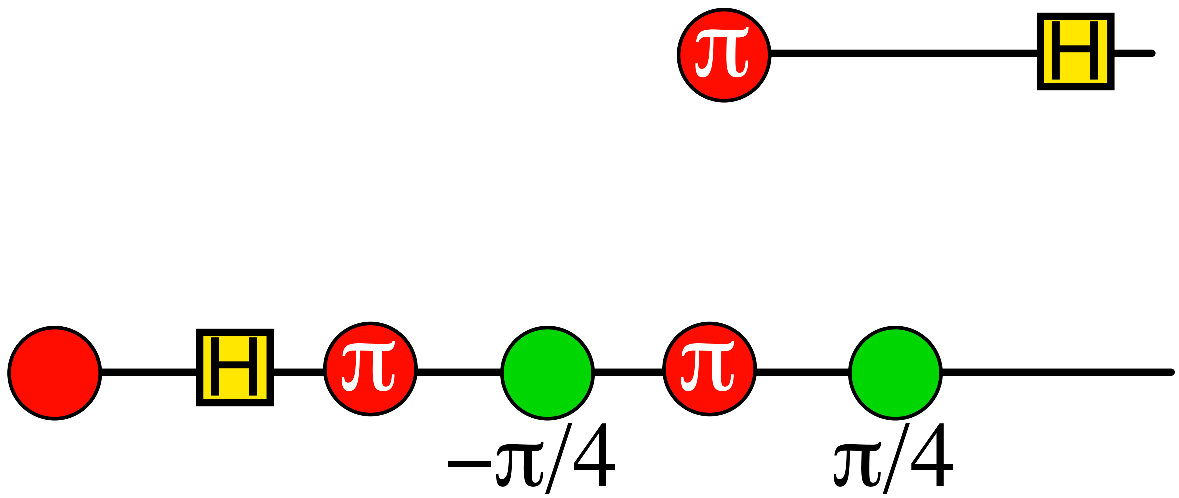

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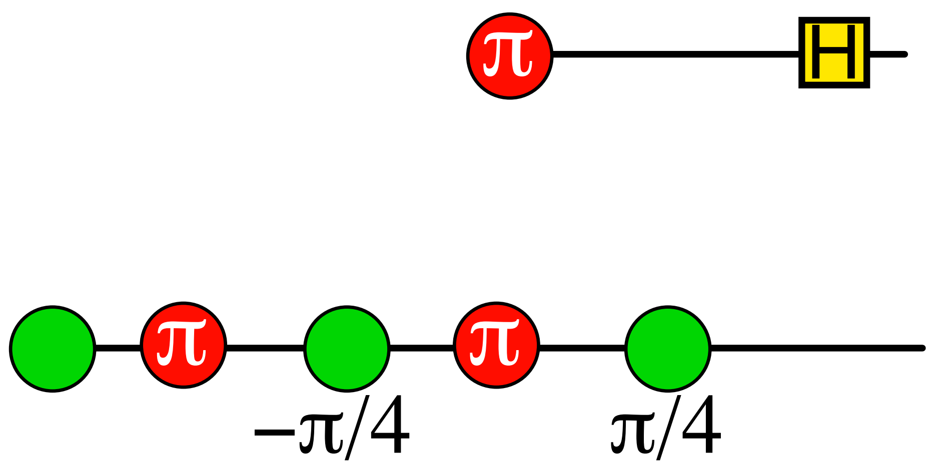

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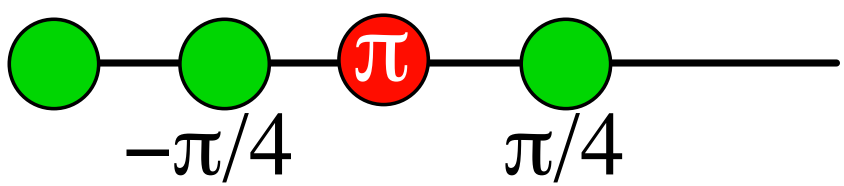

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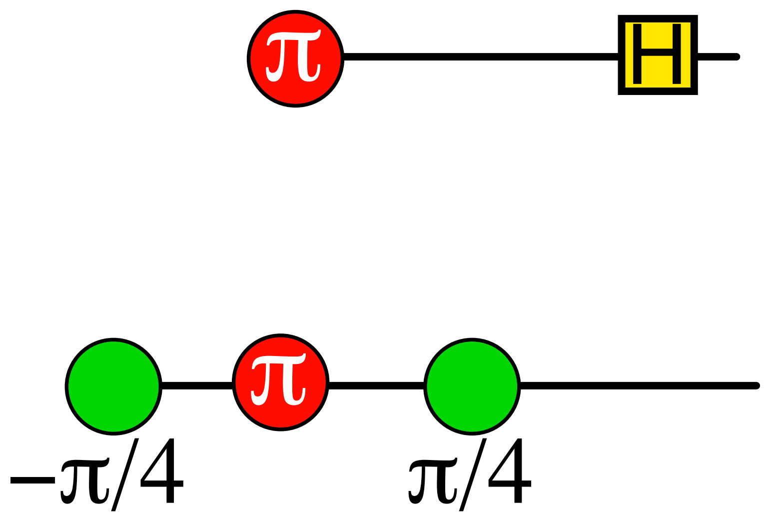

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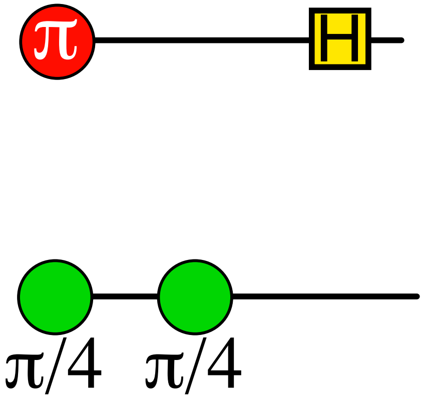

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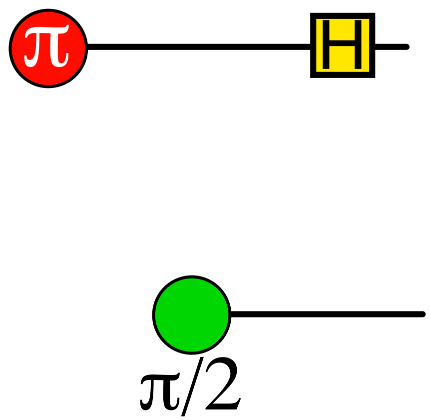

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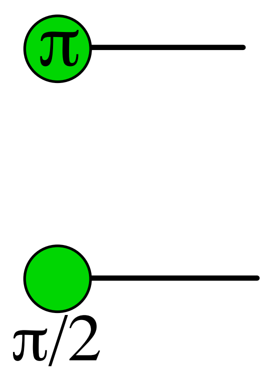

which is the correct result! YAY!

To hear more:

Clifford Lectures at Tulane University this week, starting wednesday.

Information flow in physics, geometry, logic and computation

On saturday:

- 9.00: Samson Abramsky, *part 4*
- 10.30: Ross Duncan, *Logic of Complementary Quantum Observables*
- 14.00: Bob Coecke, *Kindergarten Quantum Mechanics*
- 15.30: Samson Abramksy, *part 5*

`http://www.math.tulane.edu/ mwm/clifford`