Mutually Unbiased Embeddings of Classical Logic

Ross Duncan
Oxford University Computing Lab
Joint work with Bob Coecke
Motivation

- Quantum observables may be incompatible: position/momentum, polarisation, spin ...

- In traditional quantum logic approaches these observables are simply *incomparable* in the lattice.

- However if one wants to *compute* with quantum mechanics we need know how these observables relate to each other.

We want to extract the *constructive content* from the incompatibility of quantum observables.
Classical Structures

*Classical Structures* capture the algebraic aspects of copying and deleting:

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

An example:

\[ \delta : |i\rangle \mapsto |ii\rangle \quad \epsilon : \sum_i |i\rangle \mapsto 1 \]
Classical Structures

\[ V = \quad V = \quad Y = \quad Y = \]

\[ V = \quad \therefore \]

4
Cloning

Consider the map:

\[ \delta_Z : Q \rightarrow Q \otimes Q :: |i\rangle \mapsto |ii\rangle \]

\( \delta_Z \) is the cloning map for the basis \(|0\rangle, |1\rangle\).

Obviously \( \delta_Z \) is cannot clone all states:

\[ \delta_Z |+\rangle = \delta_Z (|0\rangle + |1\rangle) = |00\rangle + |11\rangle \]

However, since quantum states are indistinguishable up to global phase the vectors \( e^{i\alpha} |0\rangle \) and \( e^{i\beta} |1\rangle \), are also cloned, when viewed as quantum states; hence can view \( \delta \) as fixing an observable, i.e. an axis of the Bloch sphere.
Deleting

Q: How to “erase” a quantum state $|\psi\rangle$ known to be in some given basis?

A: Use a measurement which gives no information about the existing state — i.e measurement in a basis $\{b_i\}$ such that

$$|\langle b_i | \psi \rangle| = |\langle b_j | \psi \rangle|$$

$$\Rightarrow |\langle b_i | a_k \rangle| = |\langle b_j | a_k \rangle|$$

$$\Rightarrow |\langle b_i | a_k \rangle| = \frac{1}{\sqrt{d}} \text{ (in finite dim.)}$$

Hence the idea of Mutually Unbiased Bases arise very naturally from the idea of deleting a classical value embedded in a quantum state space.
With respect to the $Z$-basis (i.e. $|0\rangle, |1\rangle$), the unbiassed states are exactly:

$$
\epsilon^\alpha_Z = (|0\rangle + e^{i\alpha} |1\rangle)/2
$$

However if we compose $\epsilon^\alpha_Z$ with $\delta_Z$:

$$(\text{id} \otimes \epsilon^\alpha_Z) \circ \delta_Z = Z_{-\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

Therefore the “copying” and “erasing” operations when combined induce a non-trivial action on states, unless $\alpha = 0$ (Will come back to this a bit later).

Thus, we have a classical structure:

- $\delta_Z$ is the *cloning* map for the basis $|0\rangle, |1\rangle$.
- $\epsilon_Z$ is the *uniform deleting* of this basis.
The monoid of points

Let \((A, \delta, \epsilon)\) be a classical structure; since \((\delta^\dagger, \epsilon^\dagger)\) is a commutative monoid we have a commutative multiplication on points \(\psi, \phi : I \to A\), defined by

\[
\phi \circ \psi = \delta^\dagger \circ (\phi \otimes \psi)
\]
The monoid “currying”

Hence we have a lifting operation:

\[ \Lambda : C(I, A) \to C(A, A) :: \psi \mapsto \delta^\dagger \circ (\psi \otimes \text{id}_A) \]

This is homomorphism of monoids so:

Also: \((\Lambda(\psi))^\dagger = \Lambda(\psi_*)\)
Generalised Spider Theorem

Theorem. Any map constructed from $\delta$, $\epsilon$, some points $\psi_i$ and the adjoints of all of these, is determined only by its type and a product $\bigodot_i \psi_i$: 
Unbiassed Points

**Proposition 1.** Let \( \{|i\rangle\}_i \) be a basis for some finite dimensional Hilbert space, and let 
\[
\delta : |i\rangle \mapsto |ii\rangle
\]
be a cloning map for this basis; let \( |\psi\rangle \) be any point. Then \( \Lambda(\psi) \) is unitary if and only if \( |\psi\rangle \) is unbiassed with respect to \( \{|i\rangle\} \).

This allows us to define unbiassedness in the abstract setting.

**Definition 2.** A point \( \psi : I \rightarrow A \) is unbiassed for relative to \( (A, \delta, \epsilon) \) if \( \Lambda(\psi) \) is unitary.

Note then that the set of unbiassed points form an abelian group with respect to the \( \odot \) operation.
Classical Points

Definition 3. A point \( i : I \to A \) is called classical relative to \((A, \delta, \epsilon)\) if \( \delta \circ i = i \otimes i \).

The classical points are eigenvectors is a suitable sense:
Complementary Classical Structures

Definition 4. Two classical structures \((A, \delta_X, \epsilon_X)\) and \((A, \delta_Z, \epsilon_Z)\) in a \(\dagger\)-SMC are called complementary if they obey the following rules:

- whenever \(z_i : I \to A\) is classical for \((\delta_X, \epsilon_X)\) it is unbiased for \((\delta_Z, \epsilon_Z)\);
- whenever \(x_j : I \to A\) is classical for \((\delta_Z, \epsilon_Z)\) it is unbiased for \((\delta_X, \epsilon_X)\);
- \(\epsilon_X^\dagger\) is classical for \((\delta_Z, \epsilon_Z)\) and \(\epsilon_Z^\dagger\) is classical for \((\delta_X, \epsilon_X)\).
Bialgebraic Laws for Complementary Classical Structures

Cloning Laws:
Bialgebraic Laws for Complementary Classical Structures

Bialgebra Law:
Bialgebraic Laws for Complementary Classical Structures

Dimension Law:

The pair of non-commuting observables fails to be a true bialgebra: every equation has a (hidden) scalar factor. Call this structure a *scaled bialgebra*.
Bialgebraic Laws for Complementary Classical Structures

Dimension Law:

\[ \begin{array}{c}
\text{\includegraphics[width=1cm]{diagram.png}} = \text{\includegraphics[width=1.5cm]{circle.png}}
\end{array} \]

The pair of non-commuting observables fails to be a true bialgebra: every equation has a (hidden) scalar factor. Call this structure a *scaled bialgebra*. 
Scaled Bialgebra Laws

\[\begin{align*}
\downarrow & \downarrow = \quad \downarrow \quad \downarrow \\
\downarrow & \quad \downarrow \quad = \quad \downarrow \\
\end{align*}\]
“Negation”

\[ X_\pi = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad :: \quad \begin{cases} |0\rangle & \mapsto |1\rangle \\ |1\rangle & \mapsto |0\rangle \end{cases} \]

\[
\begin{array}{c}
\begin{tikzpicture}[baseline=(current  bounding  box.center)]
  \node (Q) at (0,0) {$Q$};
  \node (QxQ) at (2,0) {$Q \otimes Q$};
  \node (QxQxQxQ) at (2,2) {$Q \otimes Q \otimes Q \otimes Q$};
  \node (QxQxQxQ) at (2,-2) {$Q \otimes Q \otimes Q \otimes Q$};
  \draw[->] (Q) -- node[above] {$\delta$} (QxQ);
  \draw[->] (QxQ) -- node[above] {$\delta$} (QxQxQxQ);
  \draw[->] (QxQxQxQ) -- node[above] {$\delta$} (QxQxQxQ);
  \draw[->] (QxQxQxQ) -- node[above] {$\delta$} (QxQxQxQ);
\end{tikzpicture}
\end{array}
\]
“Negation”
“Negation”

\[ X :: |0\rangle + e^{i\alpha} |1\rangle \rightarrow e^{i\alpha} |1\rangle + |0\rangle = |0\rangle + e^{-i\alpha} |1\rangle \]
Hadamard as a Mediating Map

We can define the red classical structure in terms of $H$ and the green structure:

We can immediately derive a law for changing the colour of dots by introducing $H$ boxes – in fact this gives a general “colour duality”.

\[ \text{Diagram} \]
Example: A MBQC 1-qubit unitary

Proposition 5. If $U$ is a unitary on $\mathbb{C}^2$ there exist $\alpha, \beta, \gamma$ such that $U = Z_\alpha X_\beta Z_\gamma$.

Here is (part of) a measurement based program to compute this:
Example: A MBQC 1-qubit unitary
Example: A MBQC 1-qubit unitary
Example: A MBQC 1-qubit unitary
Example: A MBQC 1-qubit unitary
Example: A MBQC 1-qubit unitary

\[ = Z_\alpha X_\beta Z_\gamma \]
Example: Quantum Fourier Transform

Among the most important quantum algorithms, the quantum Fourier transform is a key stage of factoring.

$$|j_0j_1 \cdots j_n\rangle \mapsto (|0\rangle + e^{2\pi i \alpha_0} |1\rangle)(|0\rangle + e^{2\pi i \alpha_1} |1\rangle) \cdots (|0\rangle + e^{2\pi i \alpha_n} |1\rangle)$$

where $\alpha_k = 0.j_k \cdots j_n = \sum_{l=k}^{n} j_l/2^k$

For 2 qubits:

$$|00\rangle \mapsto (|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \quad |10\rangle \mapsto (|0\rangle + e^{i\pi} |1\rangle)(|0\rangle + |1\rangle)$$

$$|01\rangle \mapsto (|0\rangle + e^{i\pi/2} |1\rangle)(|0\rangle + e^{i\pi} |1\rangle) \quad |11\rangle \mapsto (|0\rangle + e^{i3\pi/2} |1\rangle)(|0\rangle + e^{i\pi} |1\rangle)$$
Example: Quantum Fourier Transform

\[ j_1 = |1> \]

\[ j_0 = |0> \]
Example: Quantum Fourier Transform
Example: Quantum Fourier Transform
Example: Quantum Fourier Transform

\[ \begin{array}{c}
\pi \\
\hline
H
\end{array} \]

\[ \begin{array}{c}
H \quad \pi \\
\hline
-\pi/4 \\
\pi \\
\hline
\pi/4
\end{array} \]
Example: Quantum Fourier Transform
Example: Quantum Fourier Transform

\[ \pi \quad \text{H} \]

\[ -\pi/4 \quad \pi \quad \pi/4 \]
Example: Quantum Fourier Transform

\[
\begin{align*}
\pi & \quad \rightarrow \quad H \\
-\pi/4 & \quad \rightarrow \quad \pi/4
\end{align*}
\]
Example: Quantum Fourier Transform

\[ \pi \quad \text{H} \quad \pi/4 \quad \pi/4 \]
Example: Quantum Fourier Transform

\[ \pi \rightarrow H \rightarrow \pi/2 \]
Example: Quantum Fourier Transform

\[ \pi \rightarrow \pi/2 \]

which is the correct result! YAY!
To hear more:

Clifford Lectures at Tulane University this week, starting wednesday.

Information flow in physics, geometry, logic and computation

On saturday:

- 9.00: Samson Abramsky, *part 4*
- 10.30: Ross Duncan, *Logic of Complementary Quantum Observables*
- 14.00: Bob Coecke, *Kindergarten Quantum Mechanics*
- 15.30: Samson Abramksy, *part 5*

http://www.math.tulane.edu/~mwm/clifford