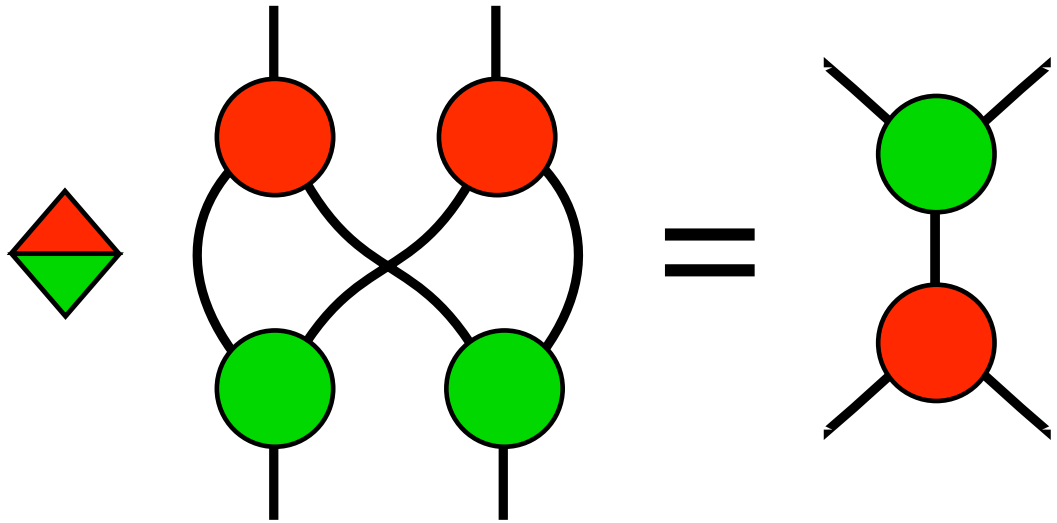
Interacting Quantum Observables



Ross Duncan
Oxford University Computing Laboratory

Interacting Quantum Observables

Joint work with
Bob Coecke

Motivations

$$A \Rightarrow B$$

Motivations

$$A \Rightarrow B \qquad\qquad !A \multimap B$$

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Hilbert space,
unitary transforms,
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?

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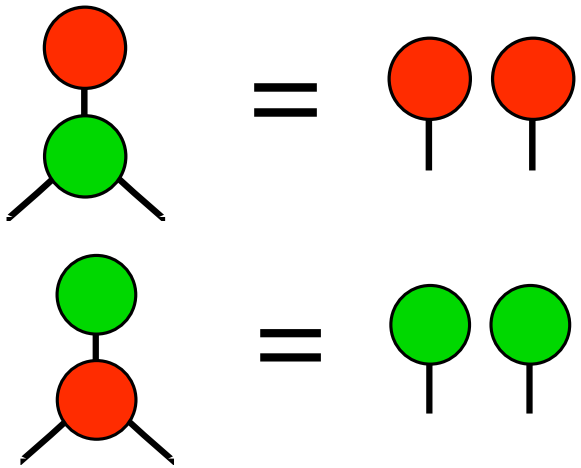


Motivations

1. Better Theory
   - Usual formulation of QM has lots of "junk"
   - Which axioms make quantum mechanics quantum?
   - Are there other models which have some of these properties?
2. Better Tools
   - Reason more easily about quantum processes
     - formal tools!
   - Synthesise quantum programs more easily
   - Type systems for quantum programs?

Quantum Mechanics (Abridged)

1. Quantum states are represented by unit vectors in a complex Hilbert space.

$$|0\rangle\,, |1\rangle\,, \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \in \mathbb{C}^2 \;=:\; \mathcal{Q}$$

2. The state space formed by combining two or more systems is the tensor product of their individual state spaces

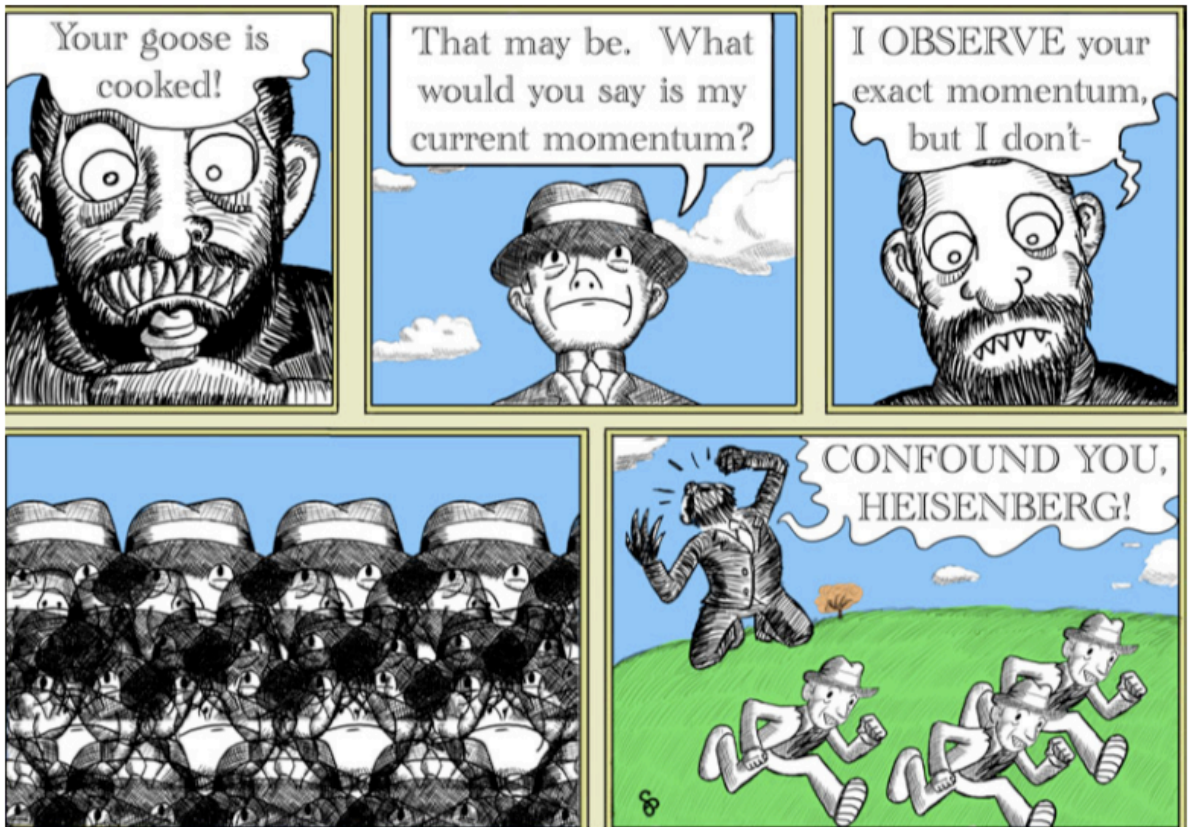
$$|010\rangle := |0\rangle \otimes |1\rangle \otimes |0\rangle \;\in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \;=\; \mathcal{Q}^3$$

3. For each discrete time step, an undisturbed quantum system evolves according to a unitary operator acting on its state space

$$X, Z, H : \mathcal{Q} \to \mathcal{Q} \qquad \wedge X, \wedge Z : Q^2 \to Q^2$$

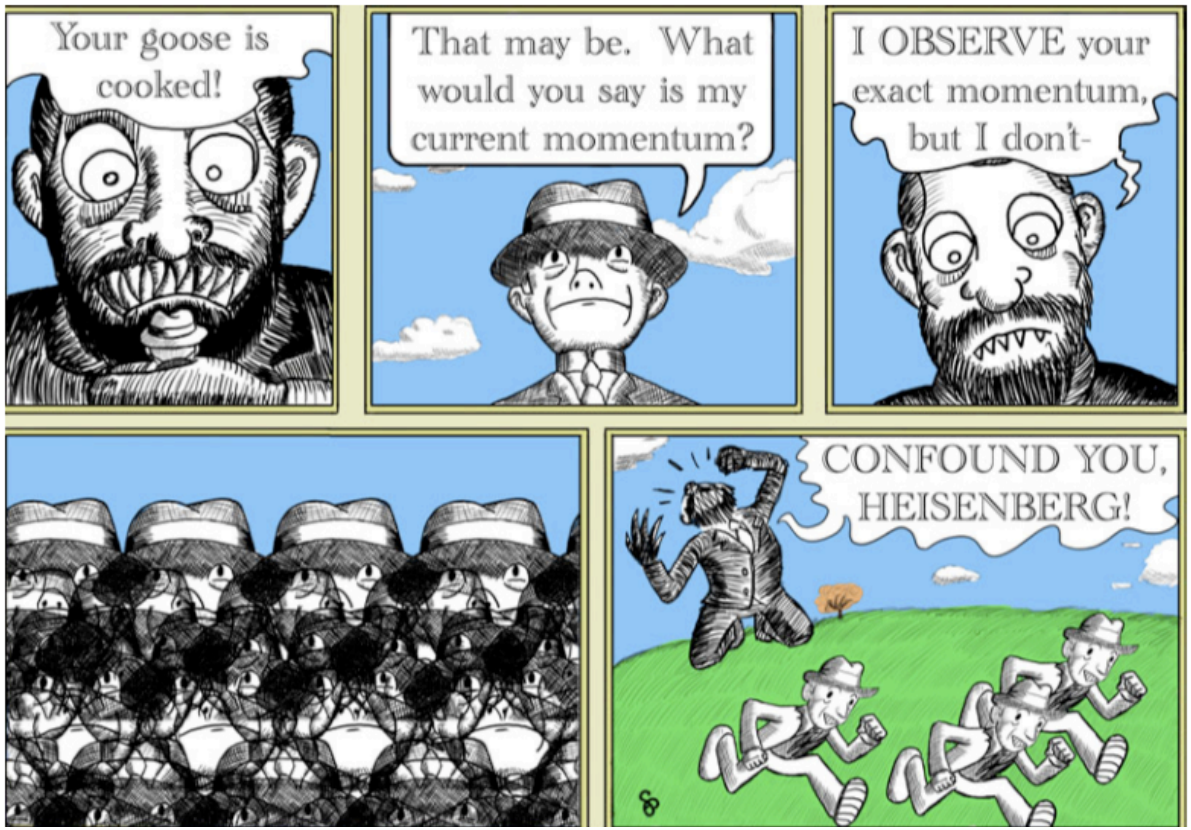
...**but** the quantum state is not directly accessible... more on this later!

Quantum Observables



What is "classical" anyway?

Quantum Observables



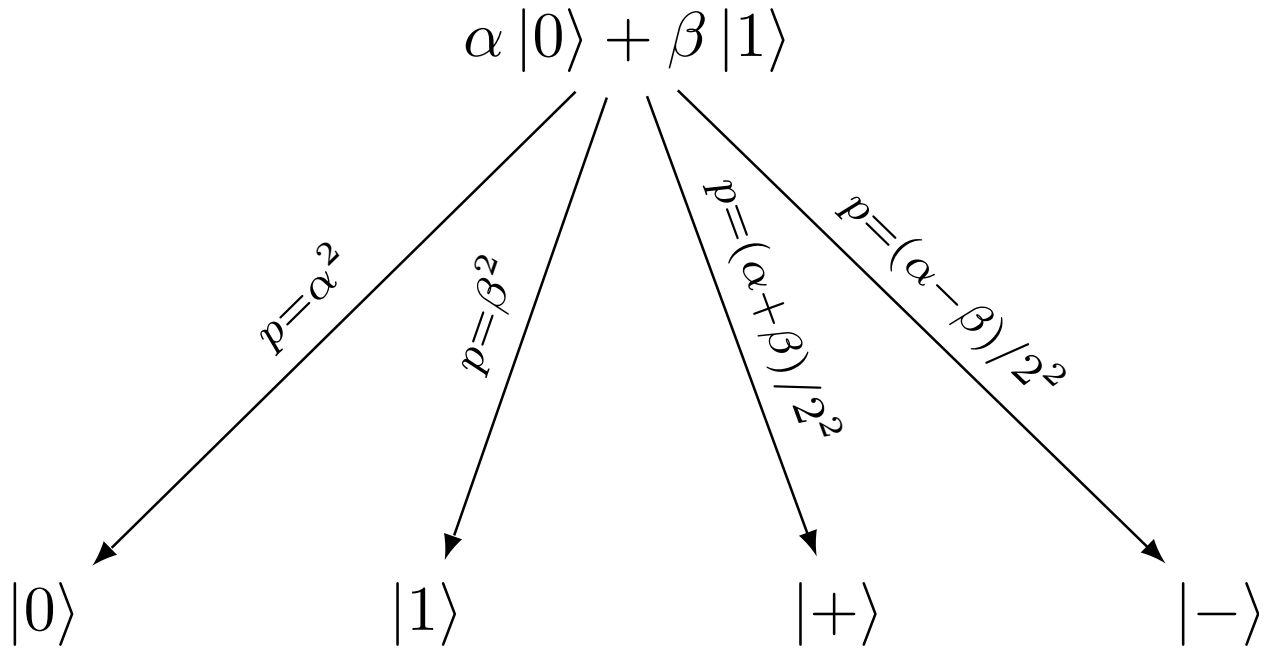
What is "classical" anyway?

http://dresdencodak.com

X and Z Spins

We can measure the spin of qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



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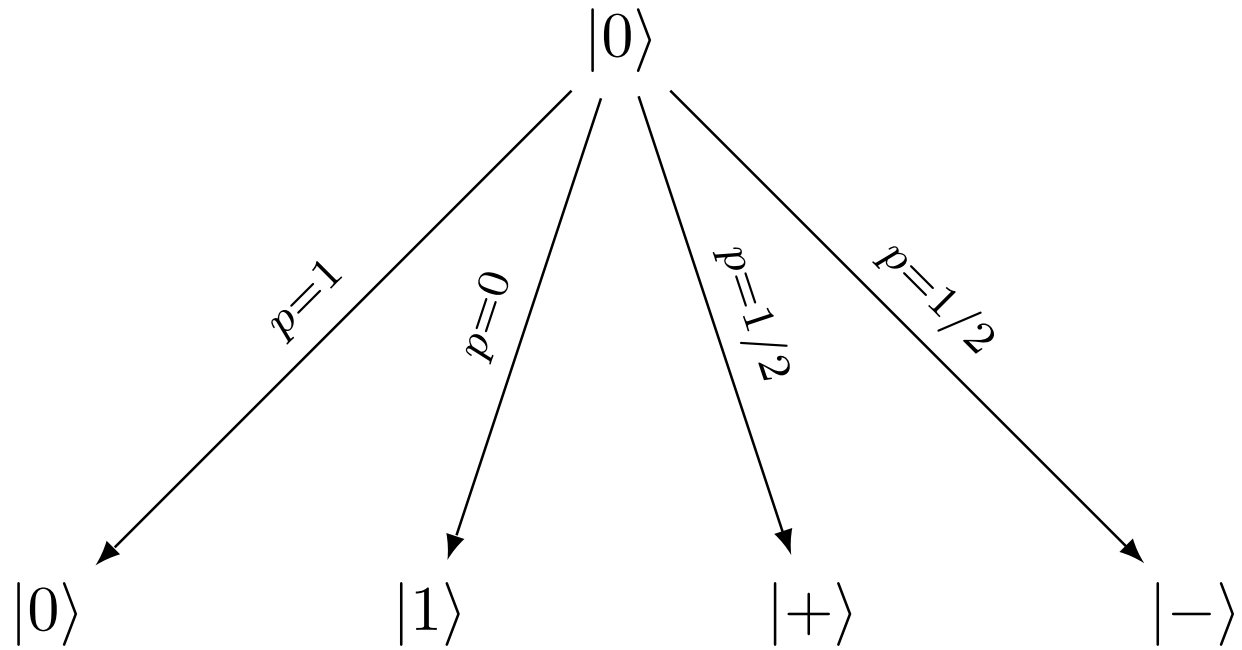

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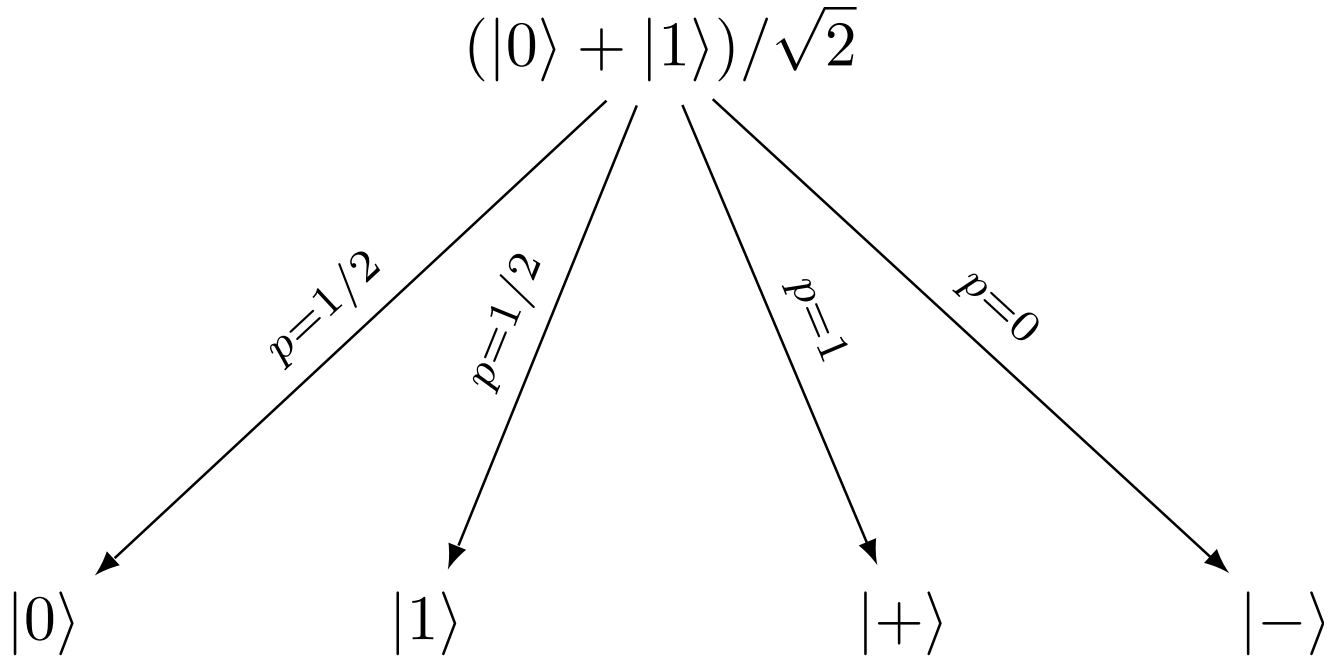
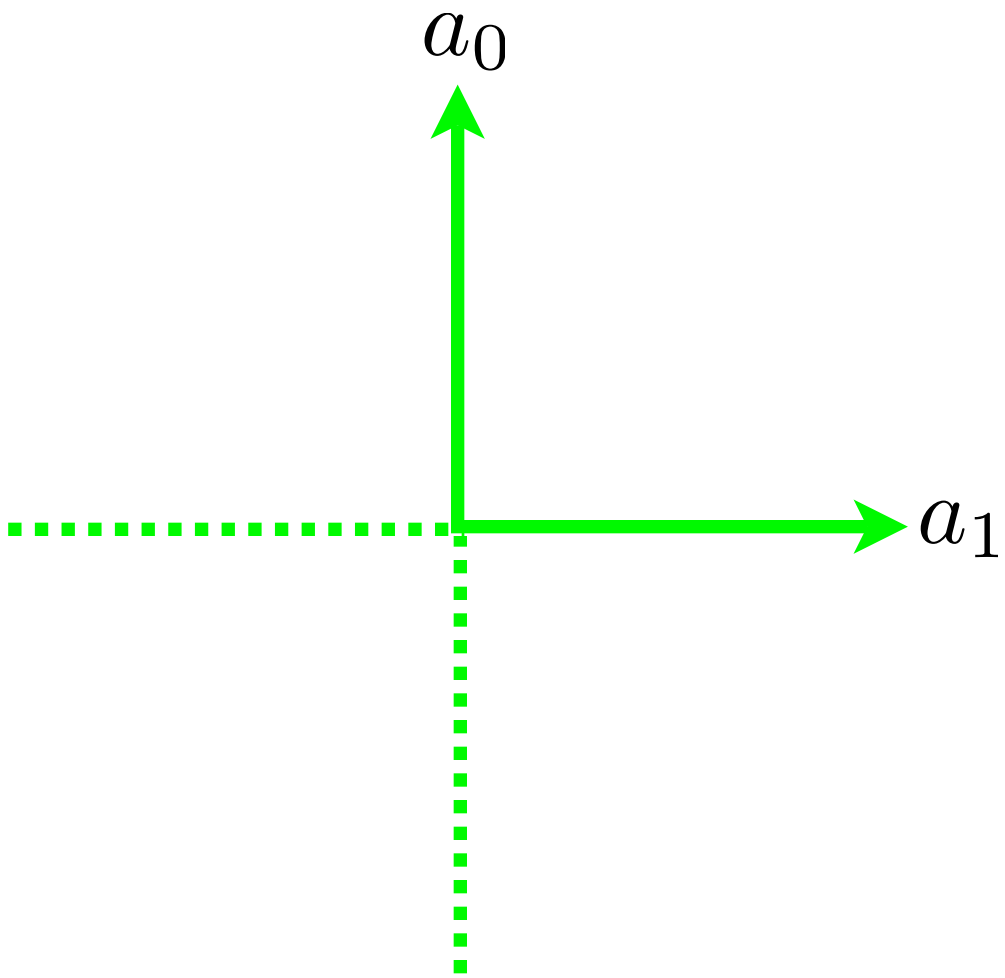


Complementary Observables



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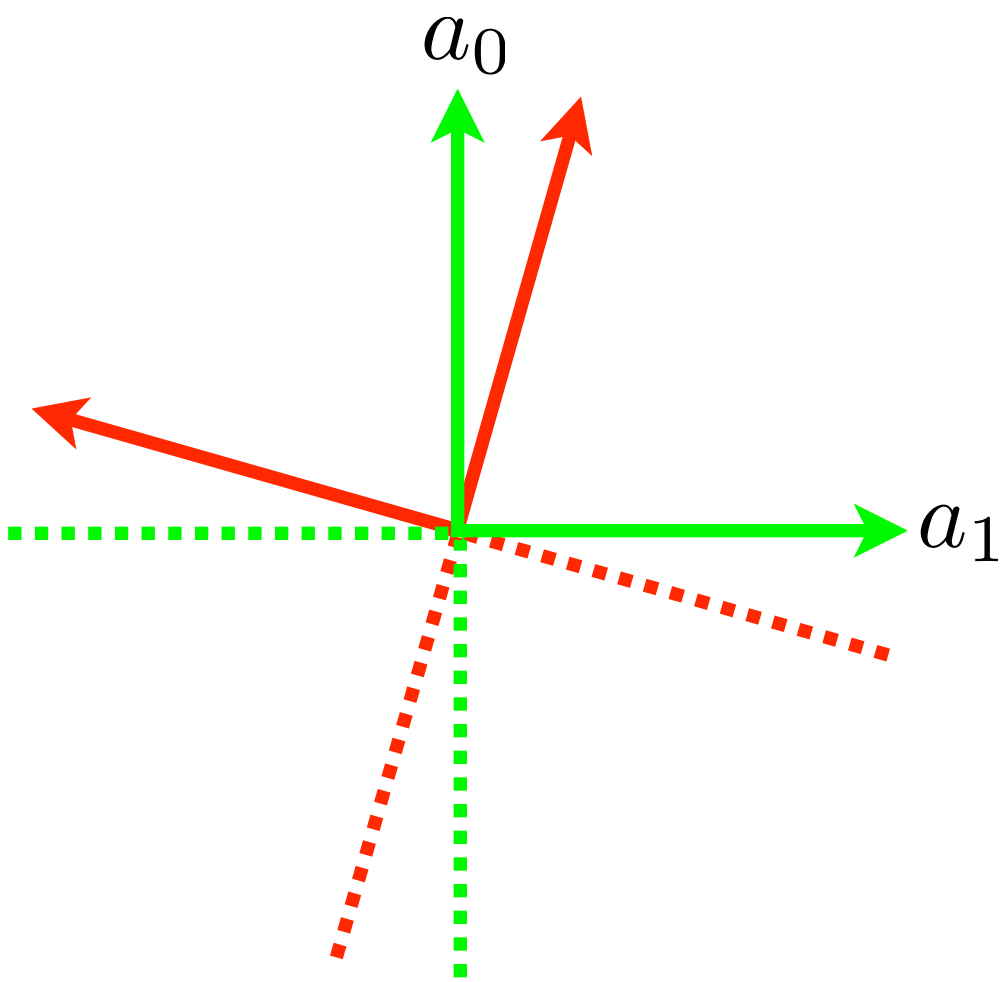

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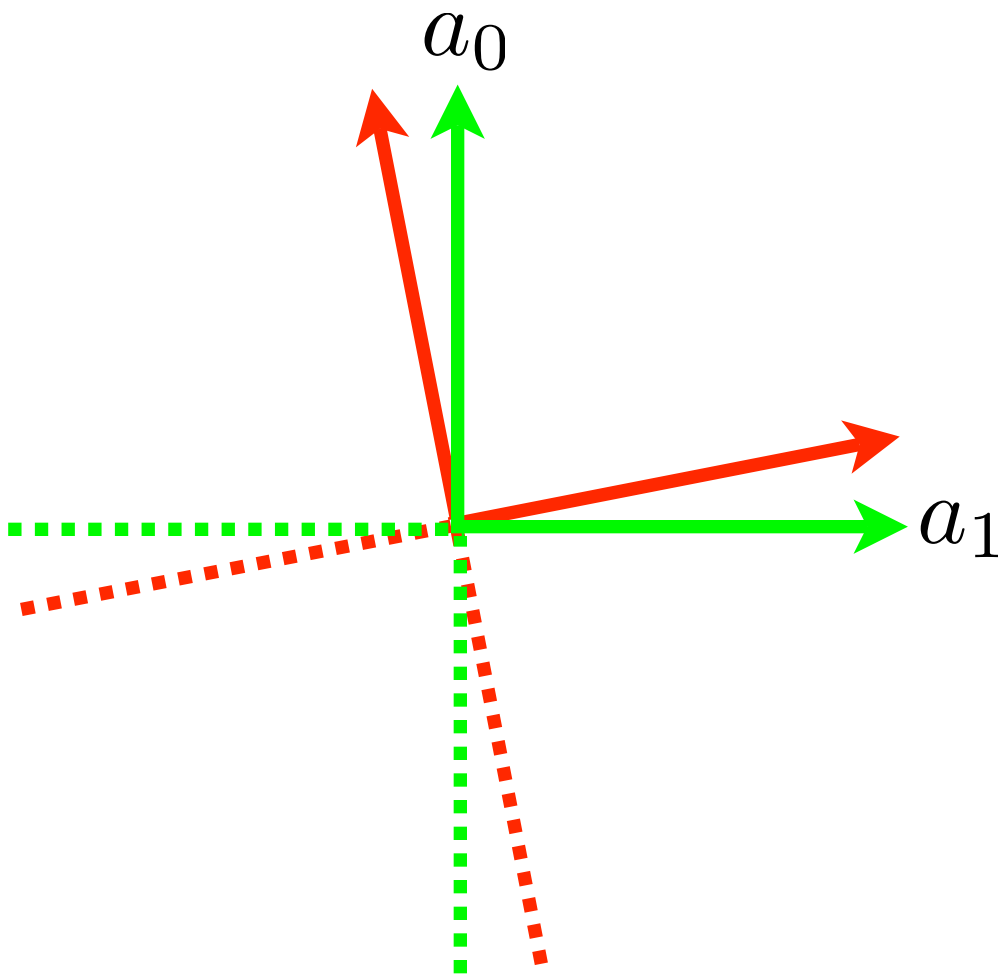

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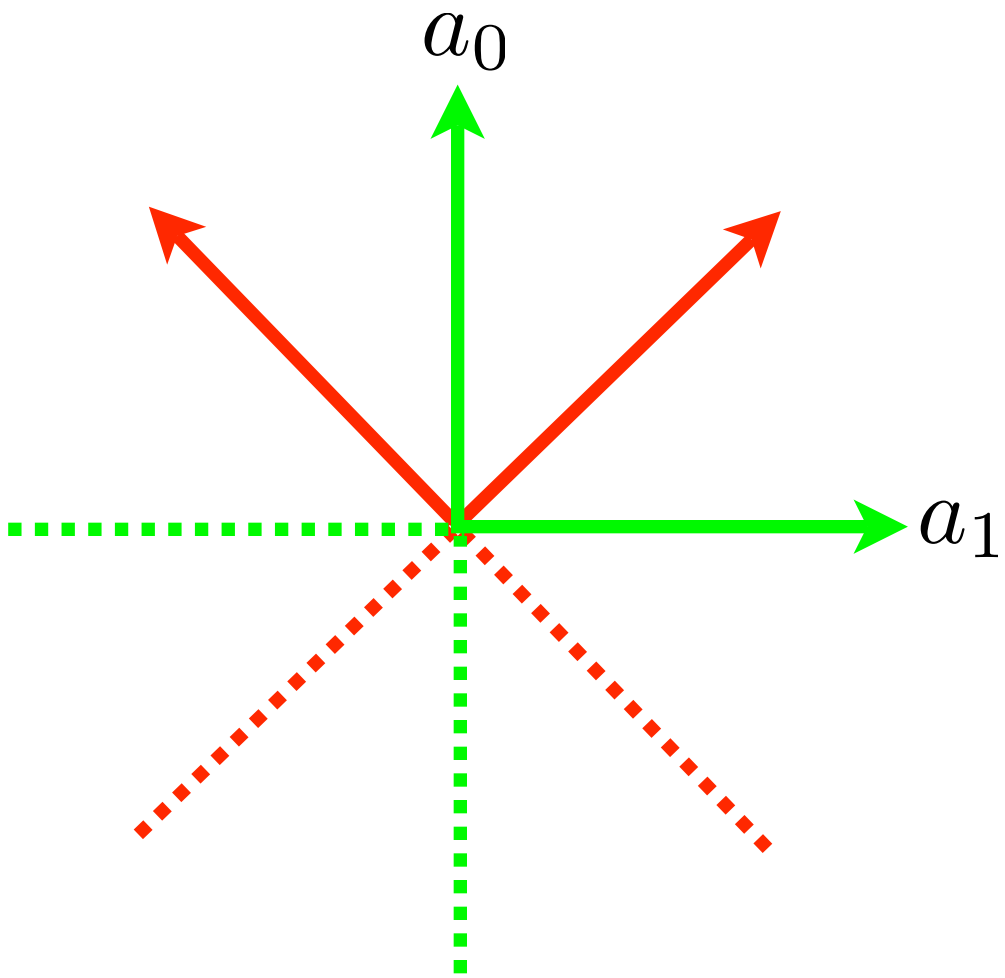

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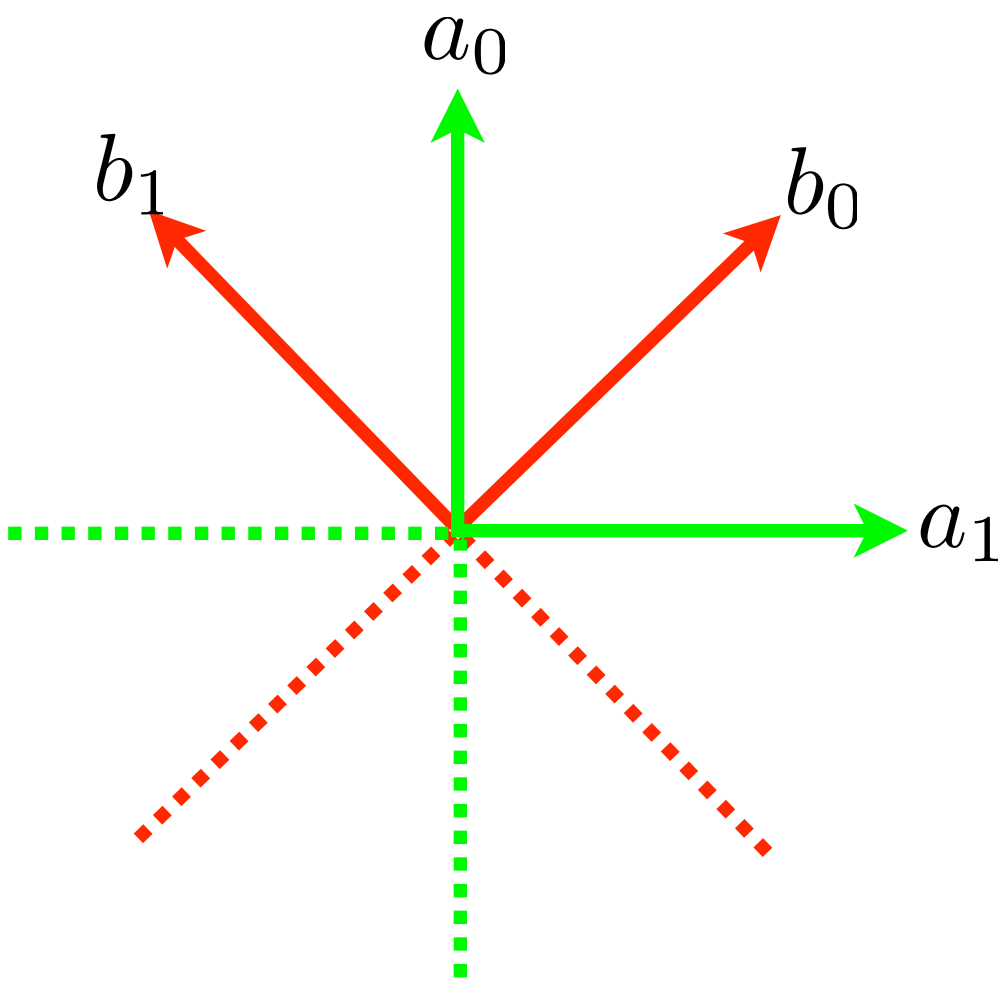

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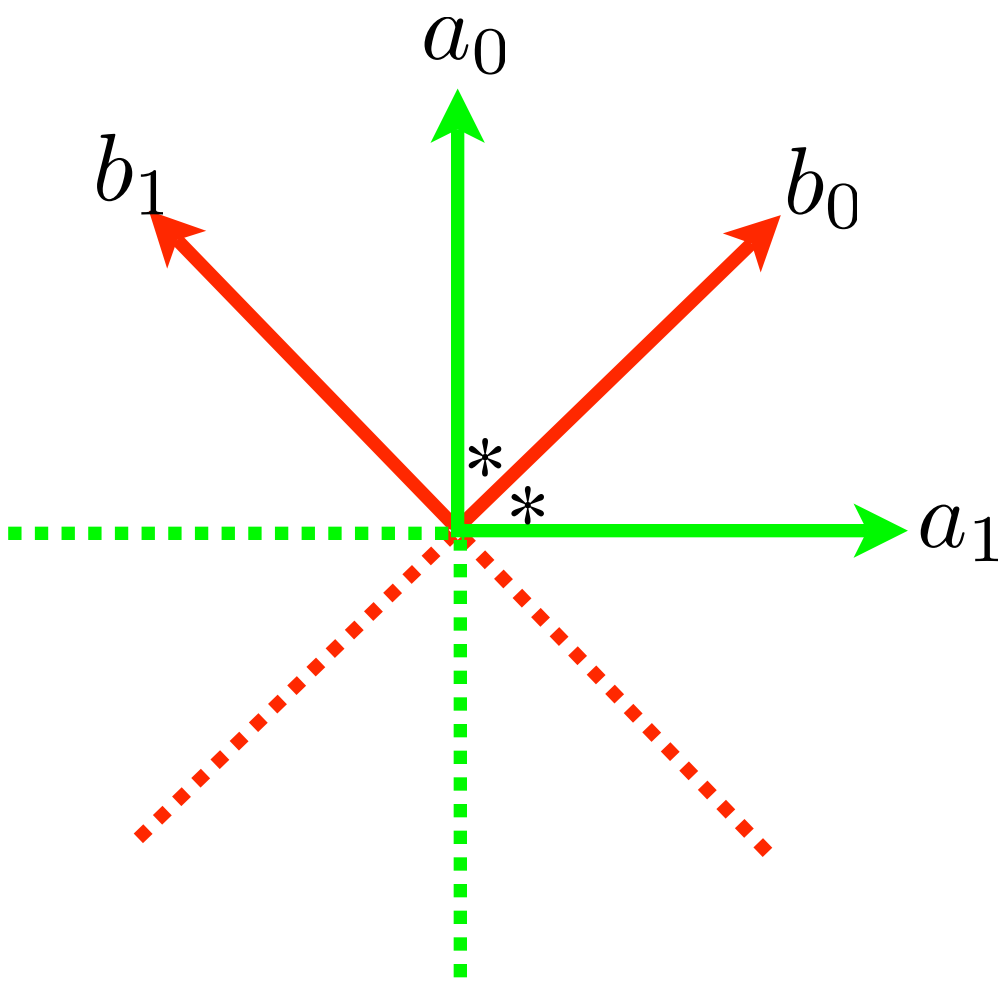

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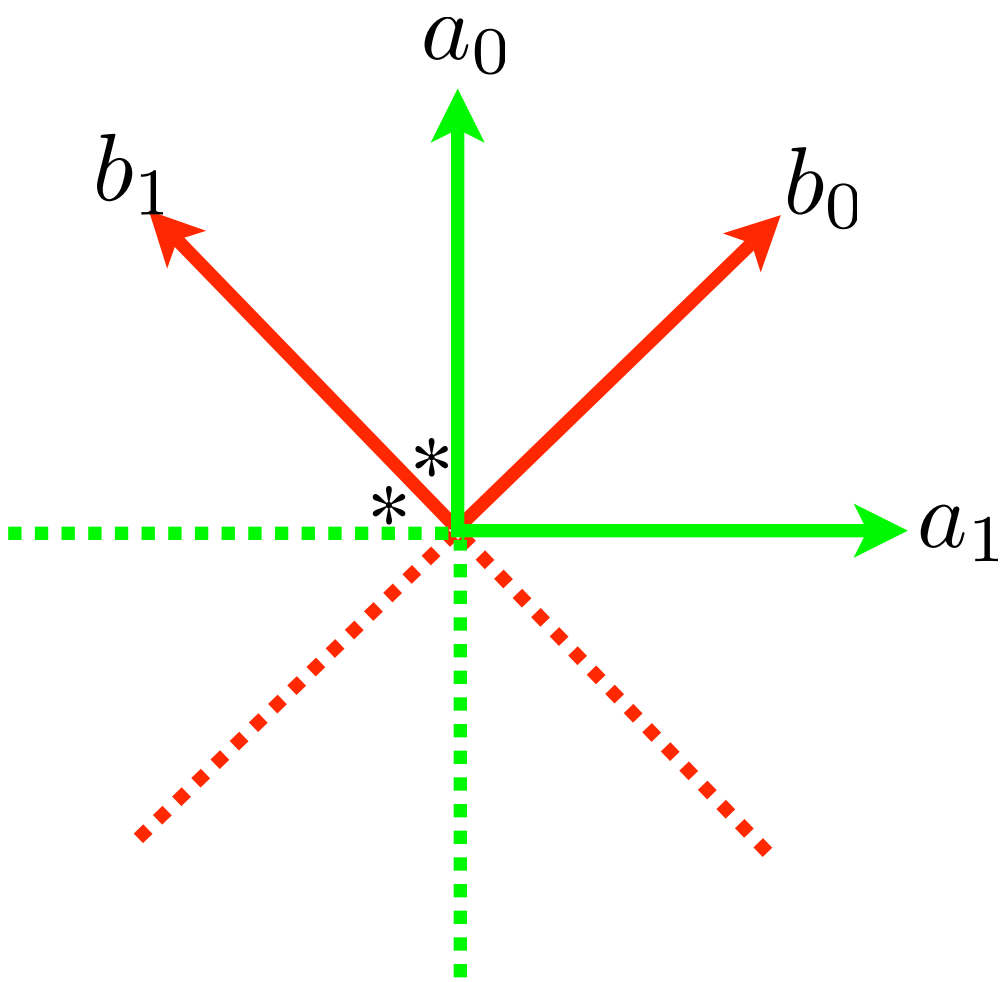

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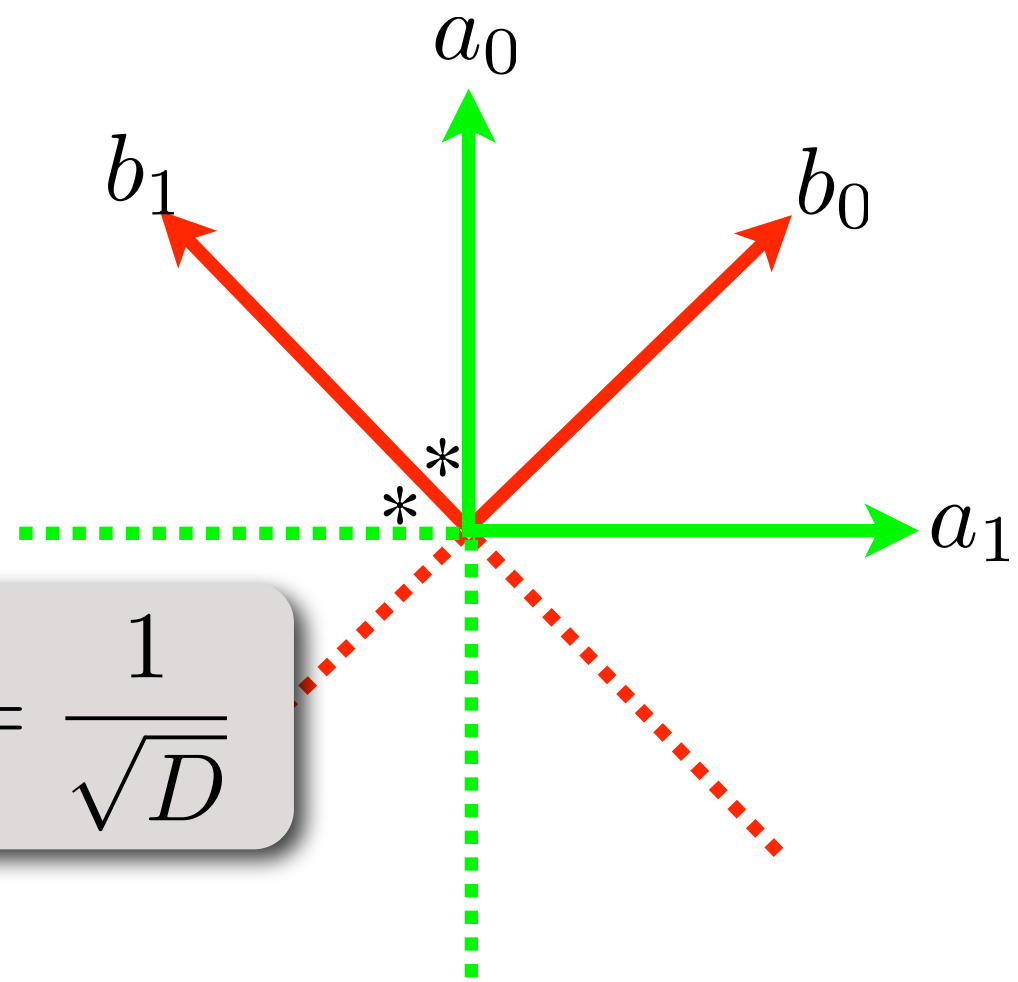


$$\langle a_i \mid b_j \rangle = \frac{1}{\sqrt{D}}$$

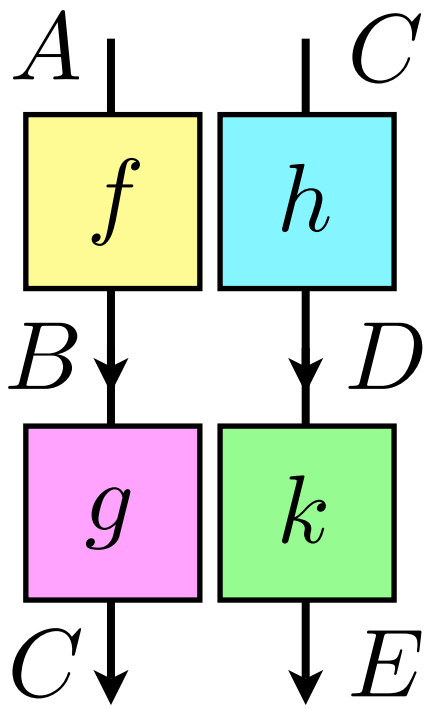
Our Approach

We aim to extract the *informatic content* from the incompatibility of quantum observables:

- basis of monoidal categories : no-cloning, no-deleting;
- classical structures : what does a copying operation look like?
- incompatible observables : how do classical operations which act on complementary states interact?
- What do these general structures look like in the particular case of qubits?

At each stage we will be able to do bit more quantum computing!

Monoidal Categories

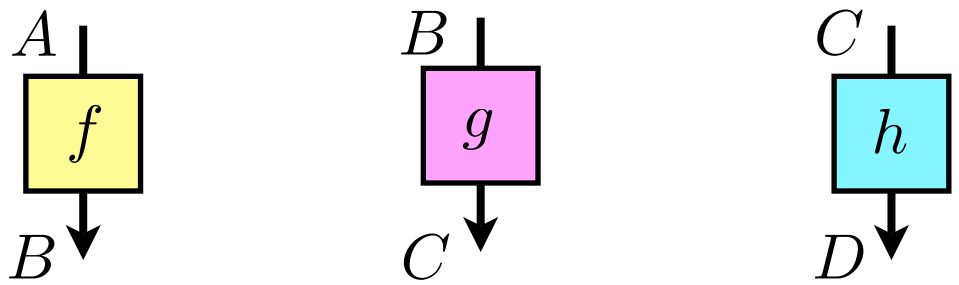


A theory of interacting processes

Categories

A *category* consists of *objects* $A, B, C,$ etc, and *arrows* between them:

$$f : A \to B \qquad g : B \to C \qquad h : C \to D$$



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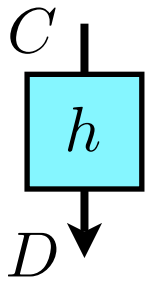
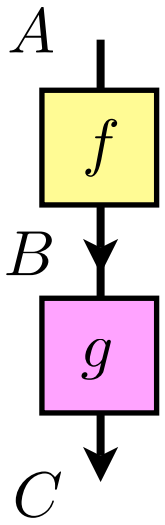


$$g \circ f : A \to C$$



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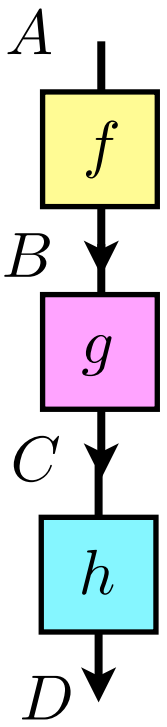

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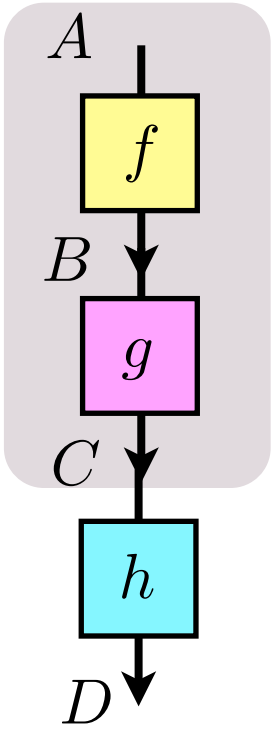

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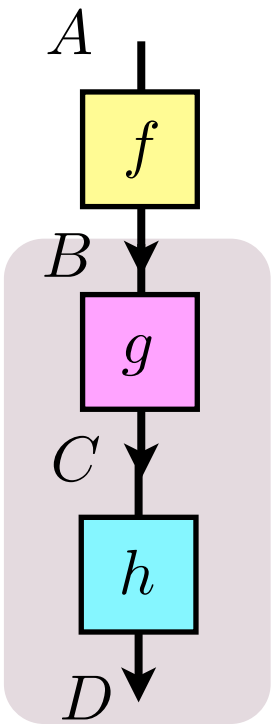

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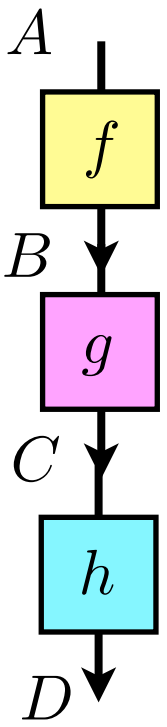
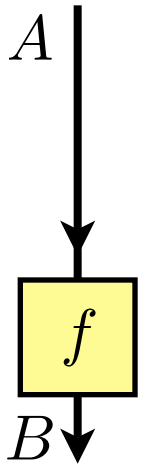


Categories

$$\mathrm{id}_A : A \to A$$

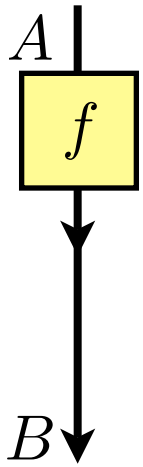
Categories

$$f \circ \mathrm{id}_A : A \to B$$



Categories

$$\mathrm{id}_B \circ f : A \to B$$



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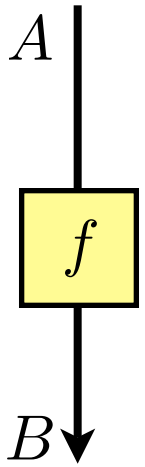
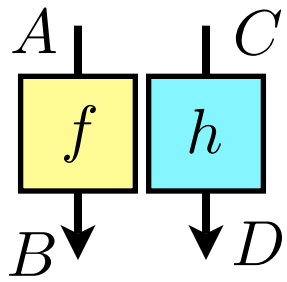


Monoidal Categories

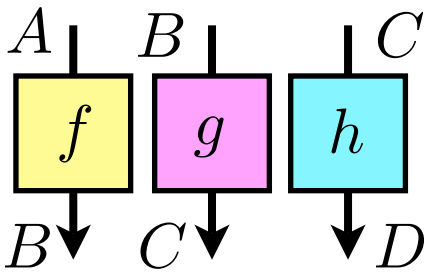
A *strict monoidal category* is a category equipped with a tensor product on both objects and arrows:

$$f \otimes h : A \otimes C \to B \otimes D$$



The tensor is associative:

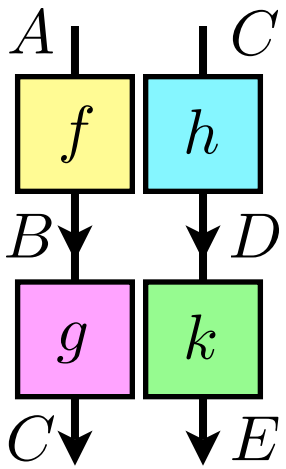
$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$



Monoidal Categories

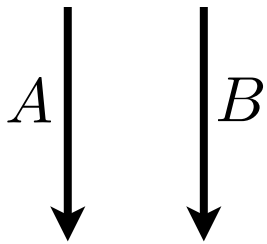
The tensor product is *bifunctorial*, meaning that it preserves composition:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$



and identities:

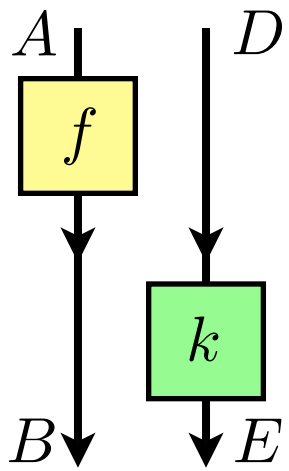
$$\mathrm{id}_{A \otimes B} = \mathrm{id}_A \otimes \mathrm{id}_B$$



Monoidal Categories

In particular we have the following:

$$(\mathrm{id}_B \otimes k) \circ (f \otimes \mathrm{id}_D) = (f \otimes \mathrm{id}_E) \circ (\mathrm{id}_A \otimes k) = f \otimes k$$



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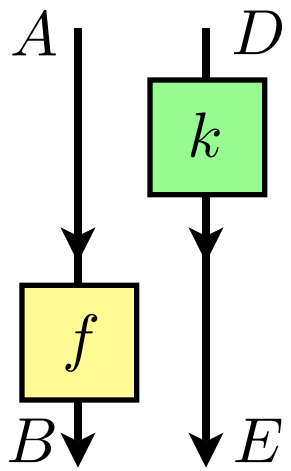

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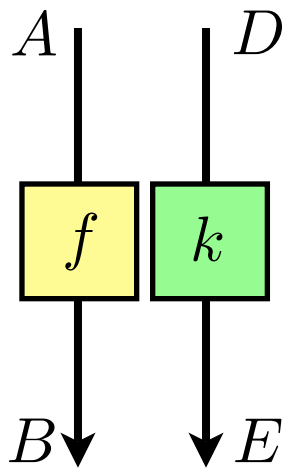


Monoidal Categories

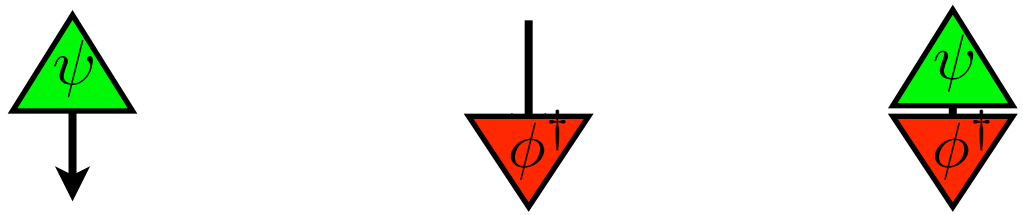
Monoidal categories have a special *unit* object called I which is a left and right identity for the tensor:

$$I \otimes A = A = A \otimes I$$

$$\mathrm{id}_I \otimes f = f = f \otimes \mathrm{id}_I$$

No lines are drawn for I in the graphical notation:

$$\psi : I \to A \qquad \phi^\dagger : A \to I \qquad \phi^\dagger \circ \psi : I \to I$$



Monoidal Categories

The arrows $c : I \to I$ are called *scalars* and they enjoy some special properties:

$$c_1 \otimes c_2 = c_1 \circ c_2$$
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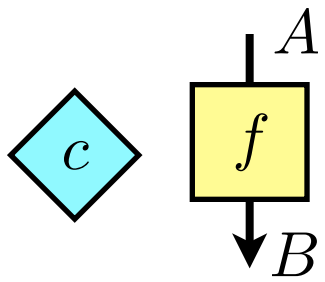
Any arrow $f : A \to B$ can be multiplied by c using the tensor:



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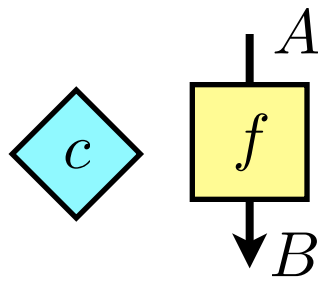

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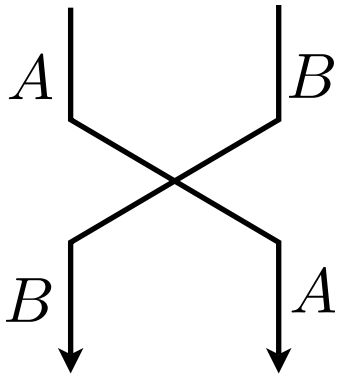
Symmetric Monoidal Categories

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$



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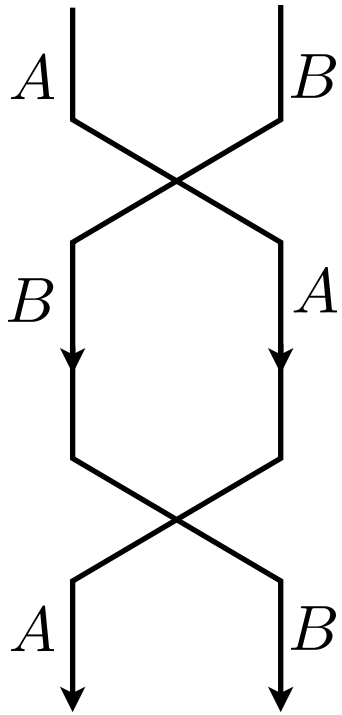

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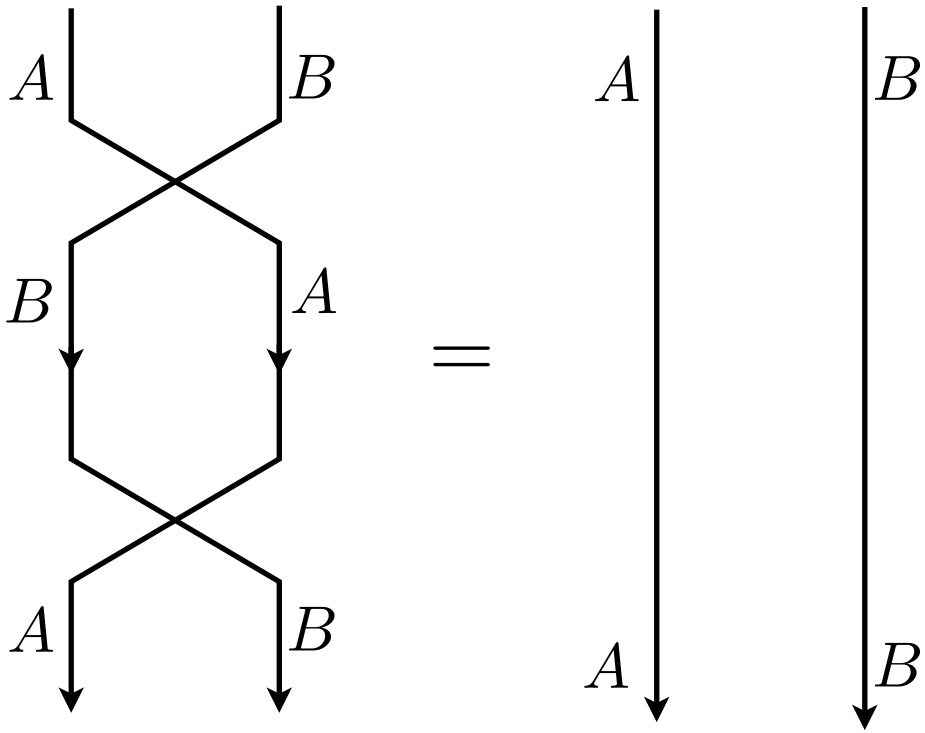

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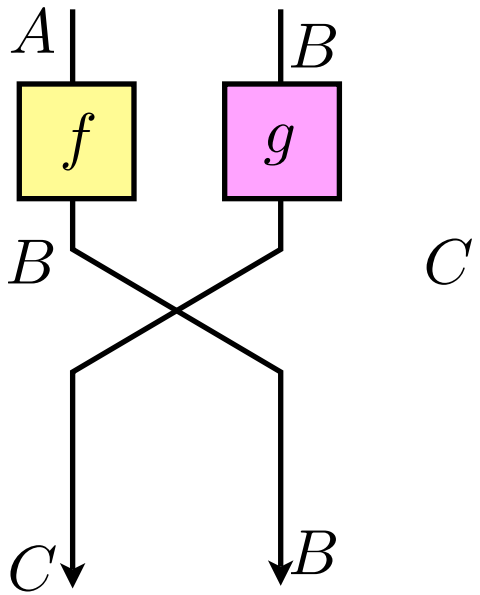

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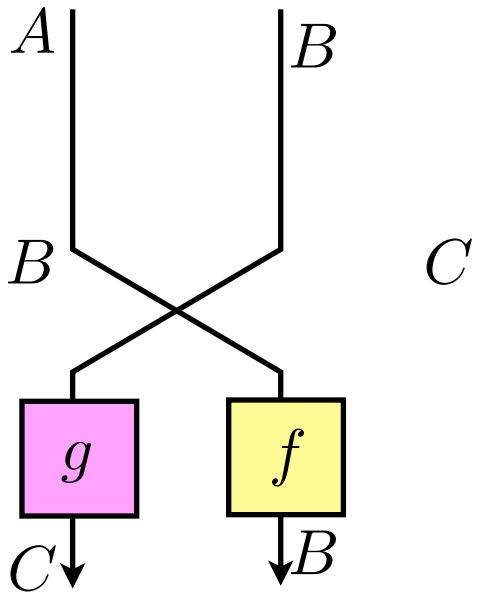
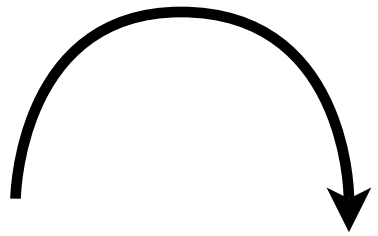
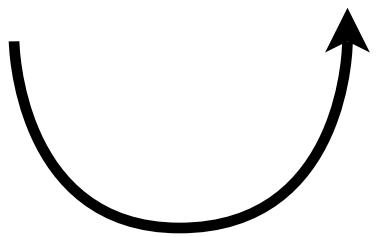


Compact Closure

$d : I \to A^* \otimes A$

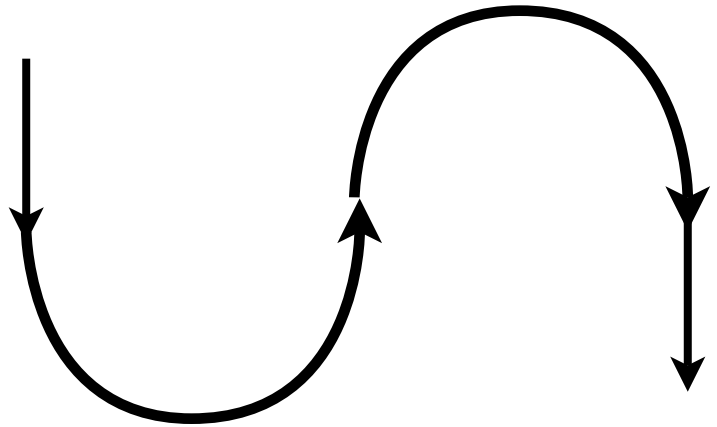


$e : A \otimes A^* \to I$



Compact Closure

$$d : I \to A^* \otimes A \qquad\qquad e : A \otimes A^* \to I$$



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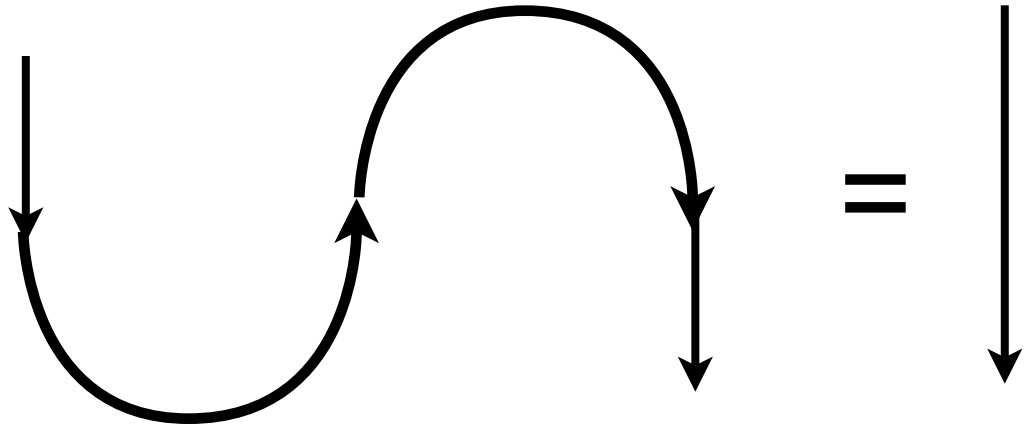

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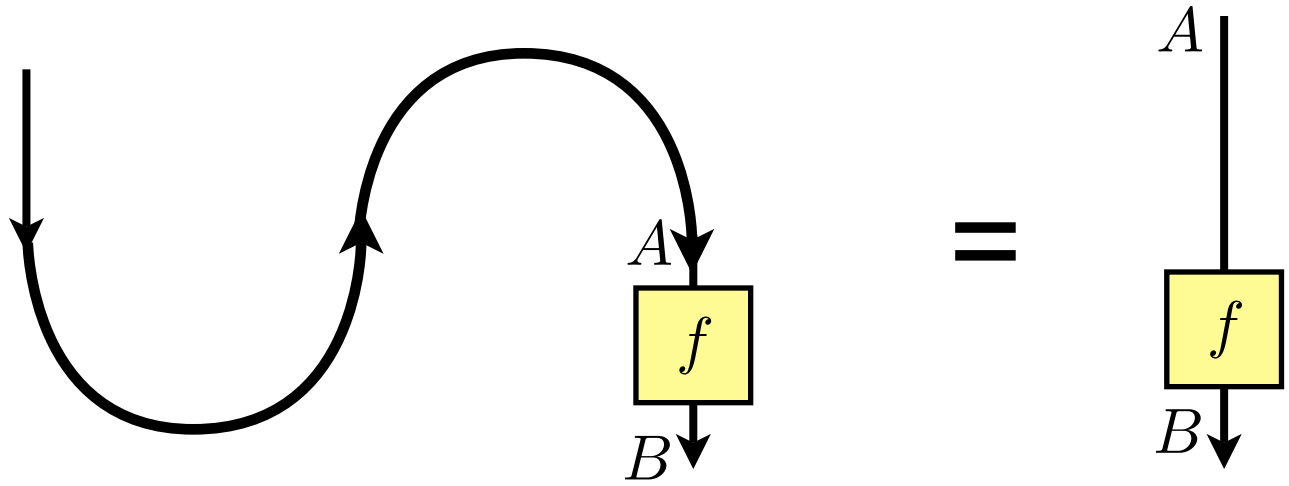

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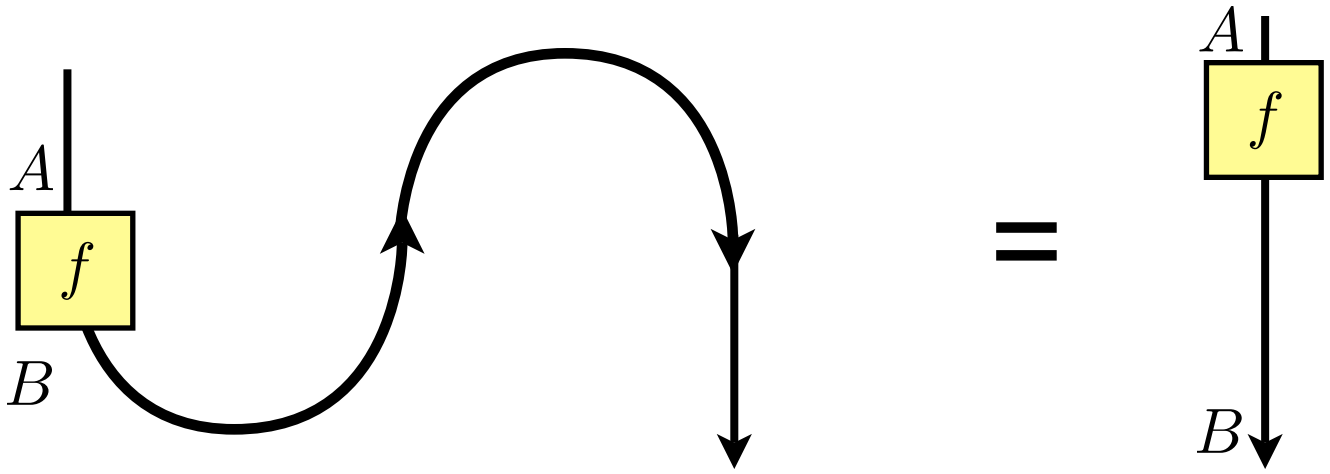


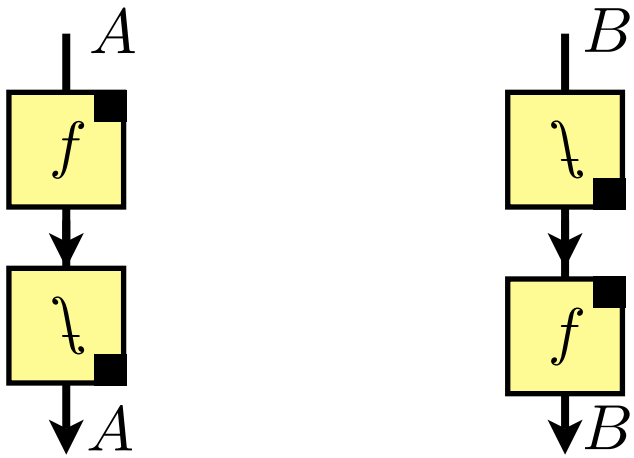
†-Monoidal Categories

A monoidal category is called *†-monoidal* if it is equipped with an involutive functor, $(\cdot)^\dagger$ which reverses the arrows while leaving the objects unchanged, which preserves the tensor structure.

$$f : A \to B \qquad f^\dagger : B \to A$$

†-Monoidal Categories

An arrow $f : A \to B$ is called *unitary* when:



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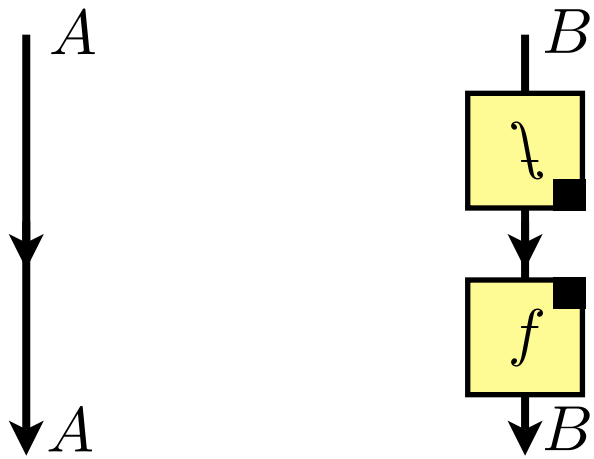

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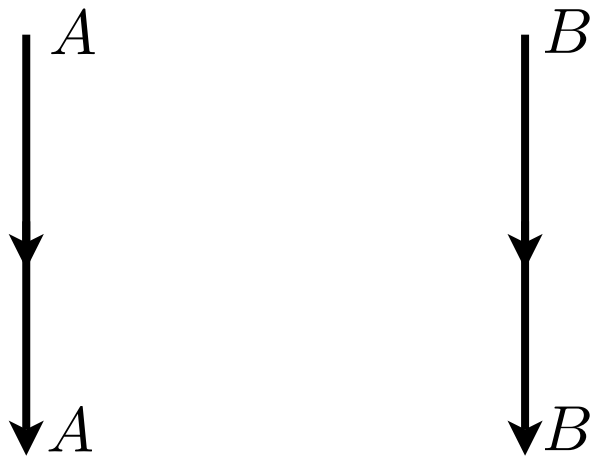
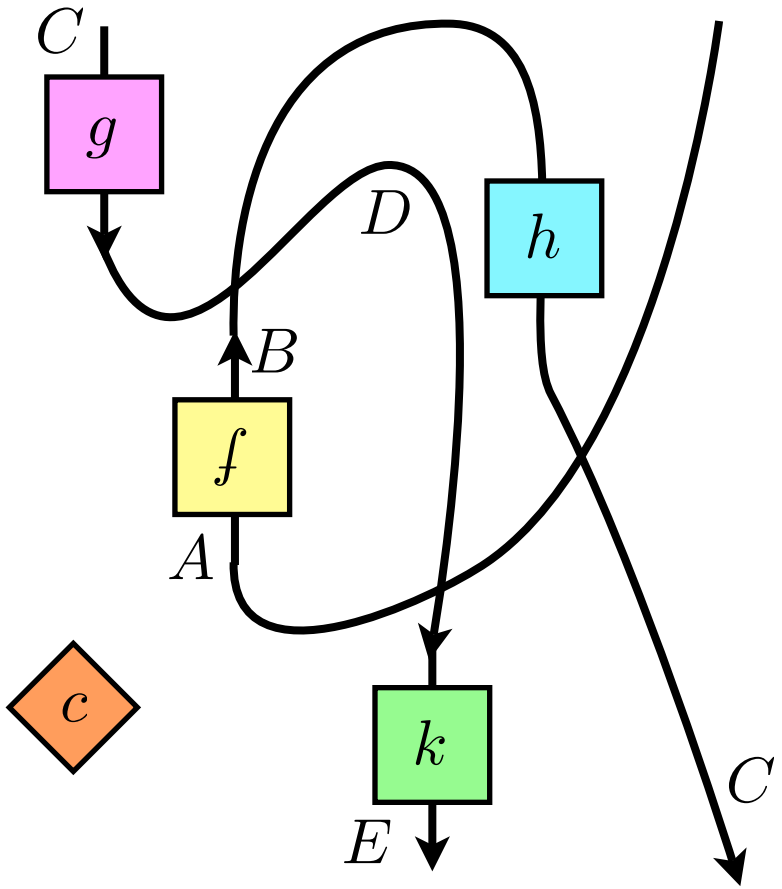


Graphical Calculus Theorem

Thm: one diagram can be deformed to another if and only if their denotations are equal by the structural equations of the category.



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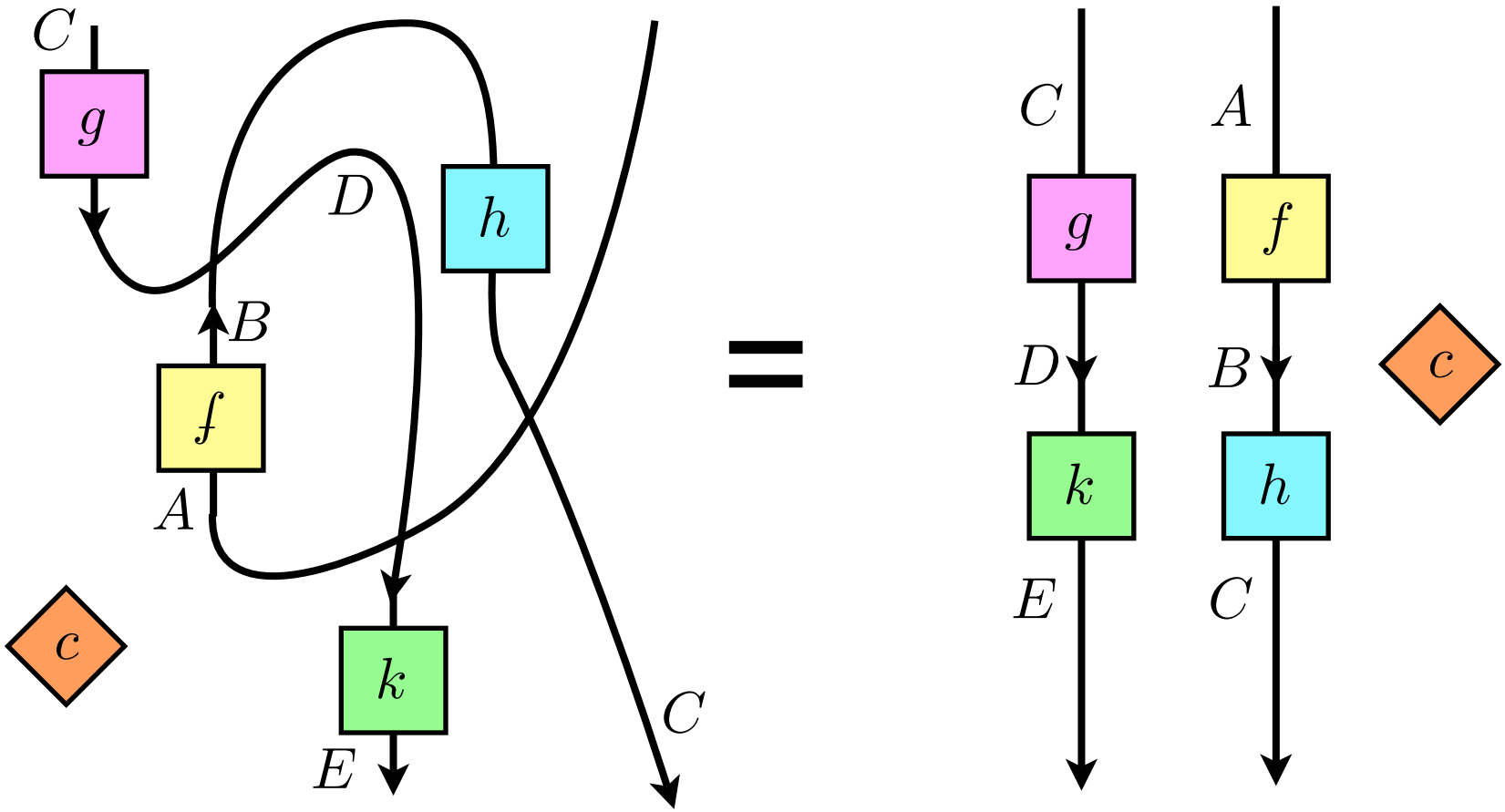


The Category **FDHilb**

FDHilb is the category of finite dimensional complex Hilbert spaces. It is †-monoidal with the following structure.

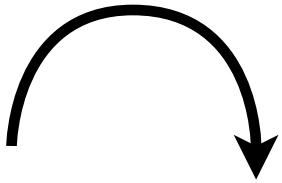
- Objects: finite dimensional Hilbert spaces, A, B, C,etc
- Arrows: all linear maps
- Tensor: usual (Kronecker) tensor product; $I = \mathbb{C}$
- $f^\dagger$ is the usual adjoint (conjugate transpose)

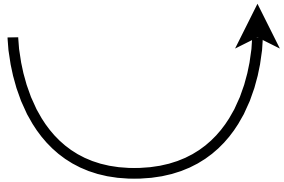
A linear map $\psi : I \to A$ picks out exactly one vector. It is a ket and $\psi^\dagger : A \to I$ is the corresponding bra.

Hence $\psi^\dagger \circ \phi : I \to I$ is the inner product $\langle \psi \mid \phi \rangle$.

Compact Structure of **FDHilb**

In FDHilb the compact structure is given by the maps:





whenever $\{a_i\}_i$ is a basis for A and $\{\overline{a_i}\}_i$ is the corresponding basis for the dual space A^*

Compact Structure of **FDHilb**

In FDHilb the compact structure is given by the maps:

$$d : 1 \mapsto \sum_i a_i \otimes \overline{a_i} \qquad\qquad e : \overline{a_i} \otimes a_i \mapsto 1$$

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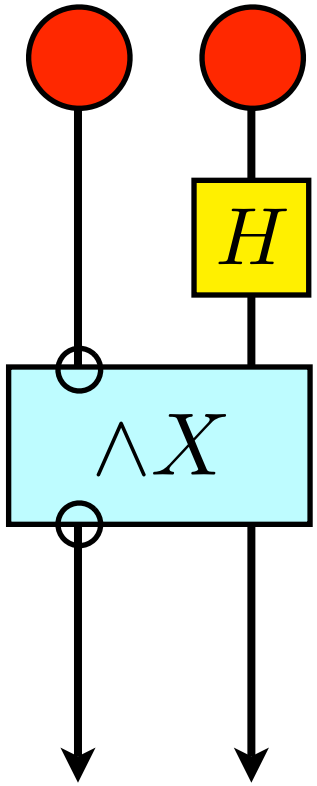
In the case of $\mathbb{C}^2$ the map d picks out the Bell state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

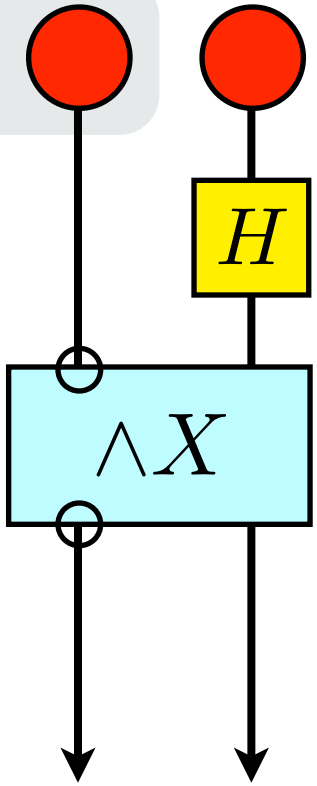
which is the simplest example of *quantum entanglement*.

Example: Preparing a Bell state



Example: Preparing a Bell state

$|0\rangle : I \to Q$



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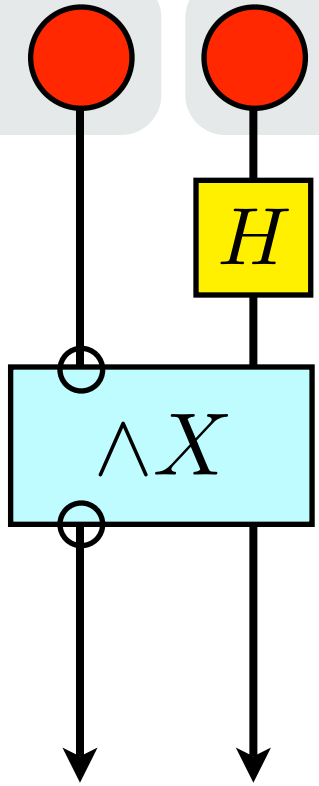

Example: Preparing a Bell state

$|0\rangle : I \to Q$

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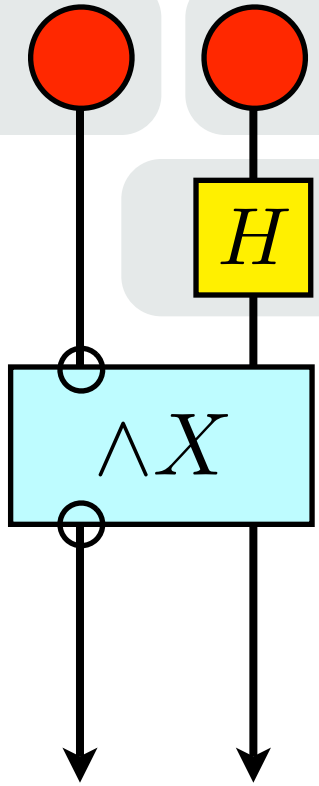
$H : Q \to Q$



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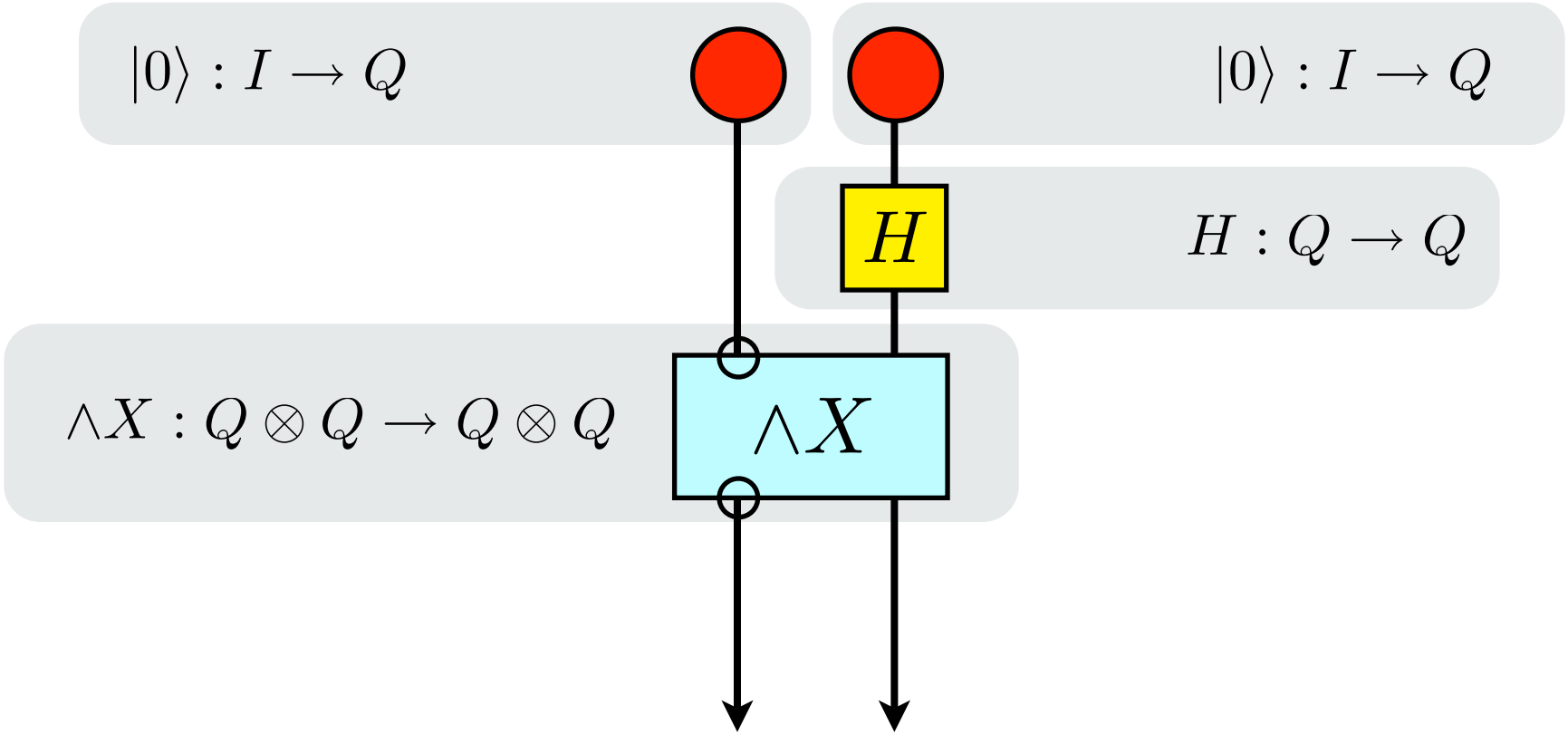

Summary so far

- †-monoidal categories have enough structure to represent all the essential vocabulary of quantum mechanics: state spaces, unitaries, bras and kets, inner product.
- †-monoidal categories have a very nice diagrammatic representation that hides a lot of the underlying equations: try to do quantum mechanics using only diagrams!
- Compact structure gives some kinds of entangled states
- But don't yet have any non-structural equations so it's all a bit too abstract so far.

Summary so far

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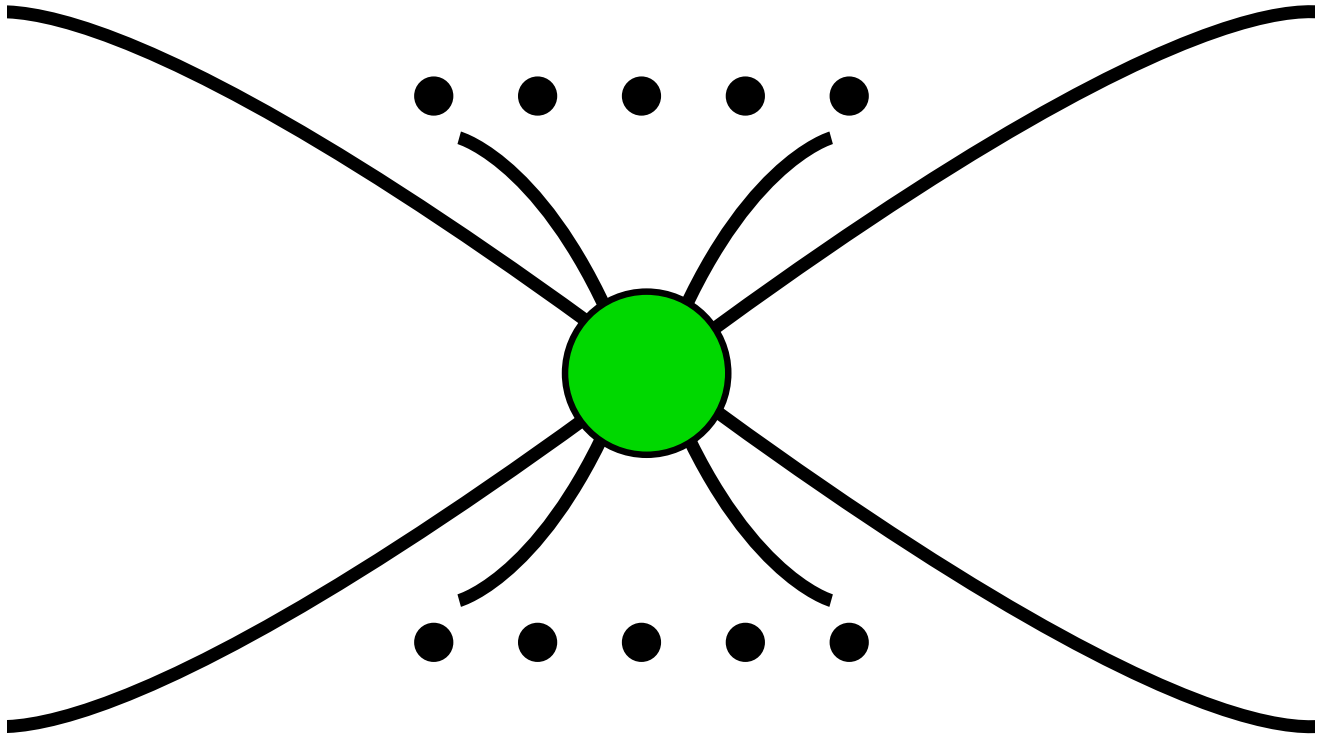
The next step is to axiomatise *bases* and *observables*

Classical Structure



Copying, deleting, and all that

No-Cloning and No-Deleting

Theorem: There are no quantum operations D such that

$$D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$$
$$D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no quantum operations E such that

$$E : |\psi\rangle \mapsto |0\rangle$$
$$E : |\phi\rangle \mapsto |0\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]

"Classical" Quantum States

When can a quantum state be treated as if classical?

- no-go theorems allow copying and deleting of *orthogonal* states;

In other words:

- A quantum state may be copied and deleted if it is an eigenstate of some *known observable*.

No-Cloning and No-Deleting, abstractly

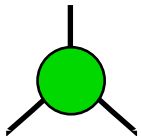
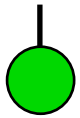
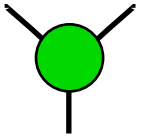
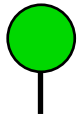
Theorem: if a †-compact category has natural transformations

$$\delta : - \Rightarrow - \otimes -$$

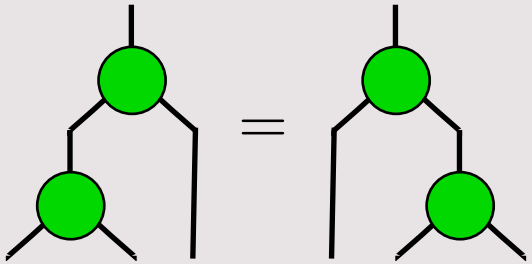
$$\epsilon : - \Rightarrow I$$

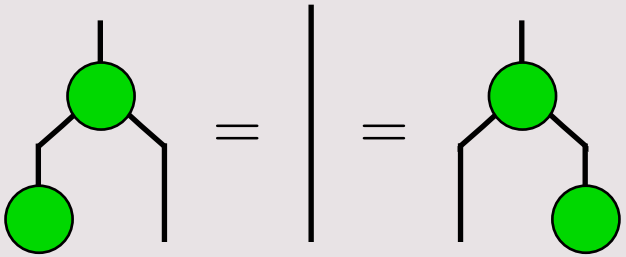
then the category collapses [Abramsky 2007]

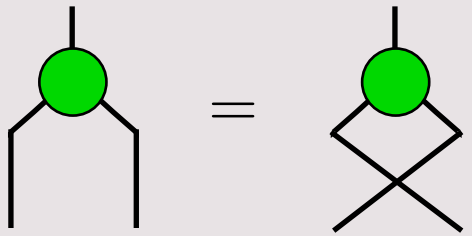
Classical Structures

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Comonoid Laws







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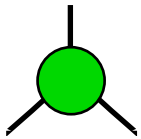
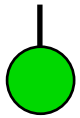
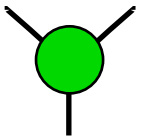
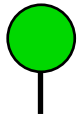
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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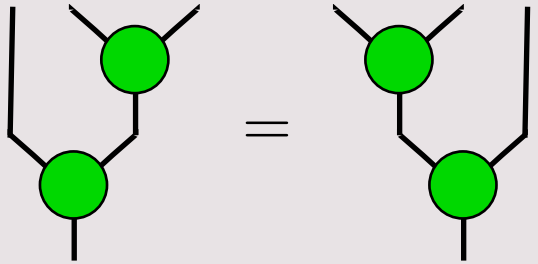
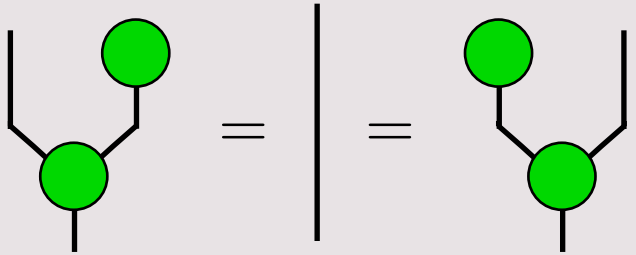
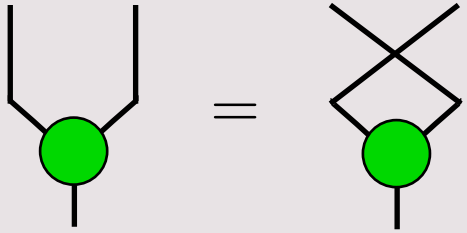
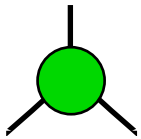
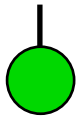
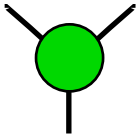
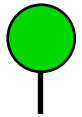






Classical Structures

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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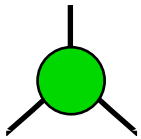
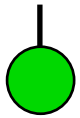
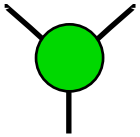
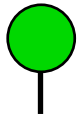
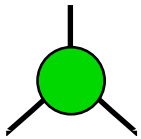
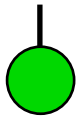
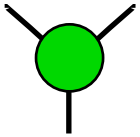
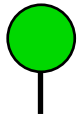
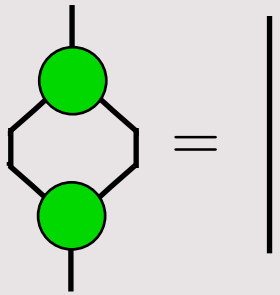
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

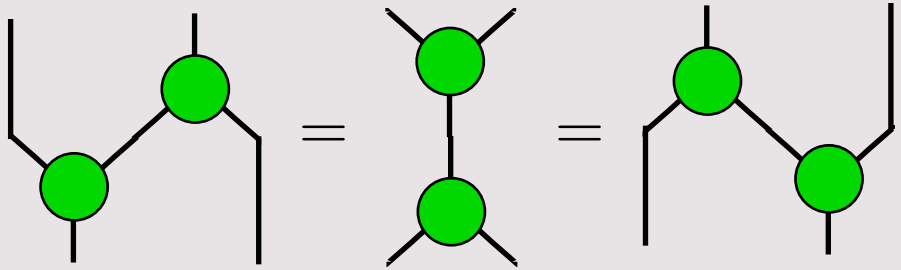
Classical Structures

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Isometry Law



Frobenius Law



Classical Structures

Given any finite dimensional Hilbert space we can define a classical structure by

$$\delta : A \to A \otimes A :: a_i \mapsto a_i \otimes a_i$$
$$\epsilon : A \to I :: \sum_i a_i \mapsto 1$$

Example: $\delta : \begin{array}{l} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{array}$ $\qquad \epsilon : |0\rangle + |1\rangle \mapsto 1$

define a classical structure over qubits; the standard basis is copied and erased. Note however that:

$$\delta(|+\rangle) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

showing that not every state can be cloned.

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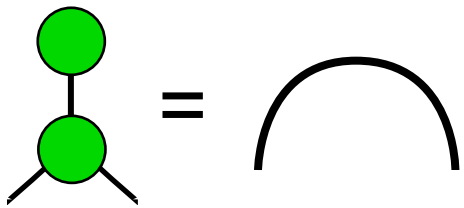

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Each (well behaved) observable defines a basis, hence each observable defines a classical structure!

Spider Theorem

Theorem: any maps constructed from δ and ε , and their adjoints, whose graph is connected, is determined uniquely by the number of inputs and outputs.

Coecke & Paquette 2006

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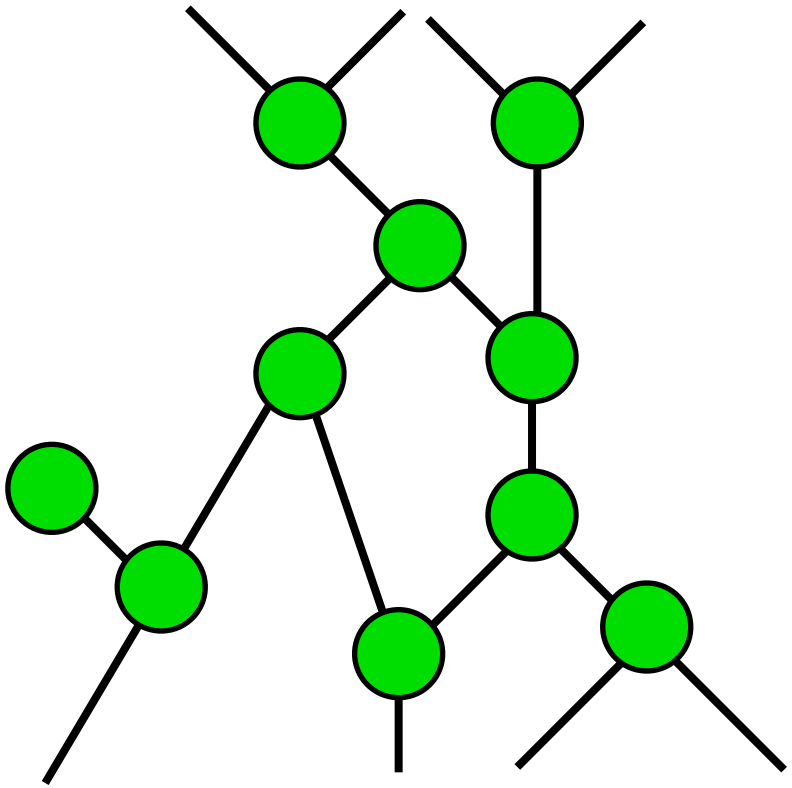

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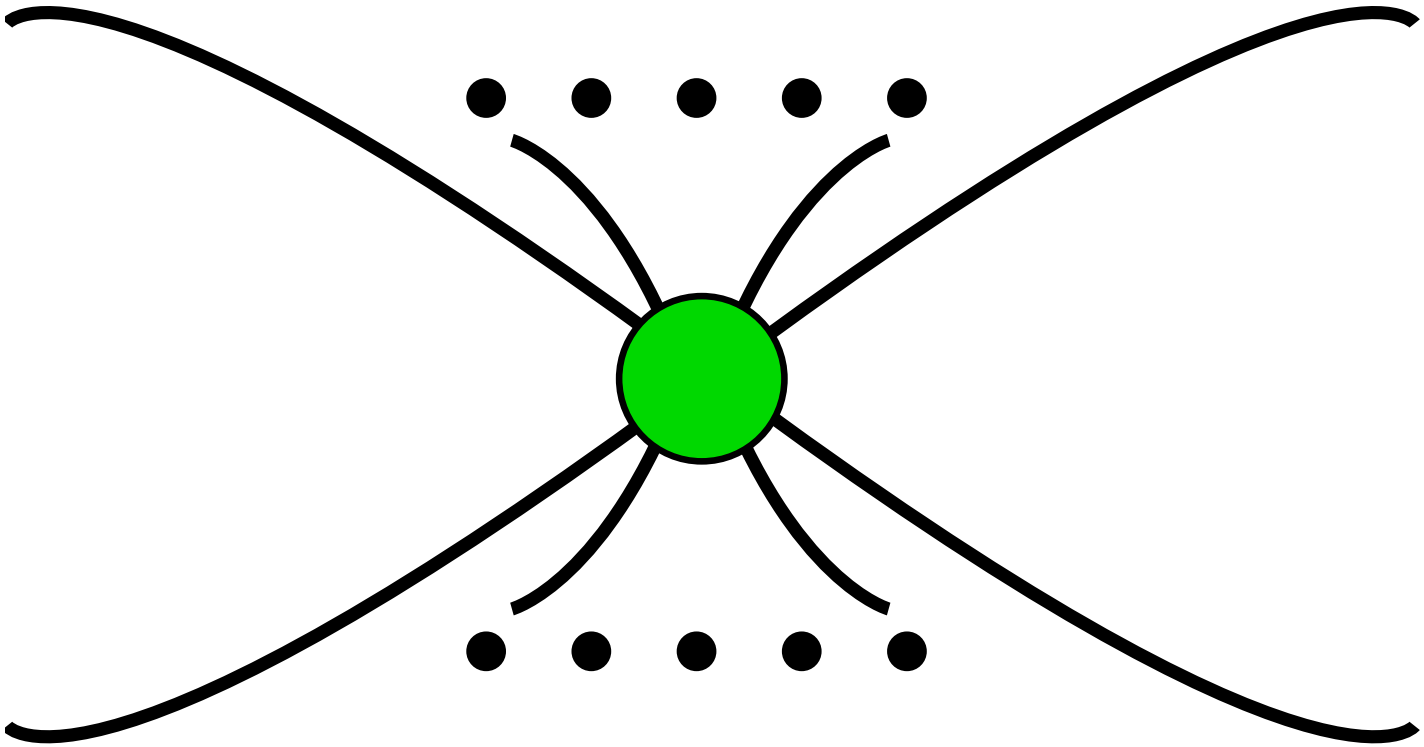
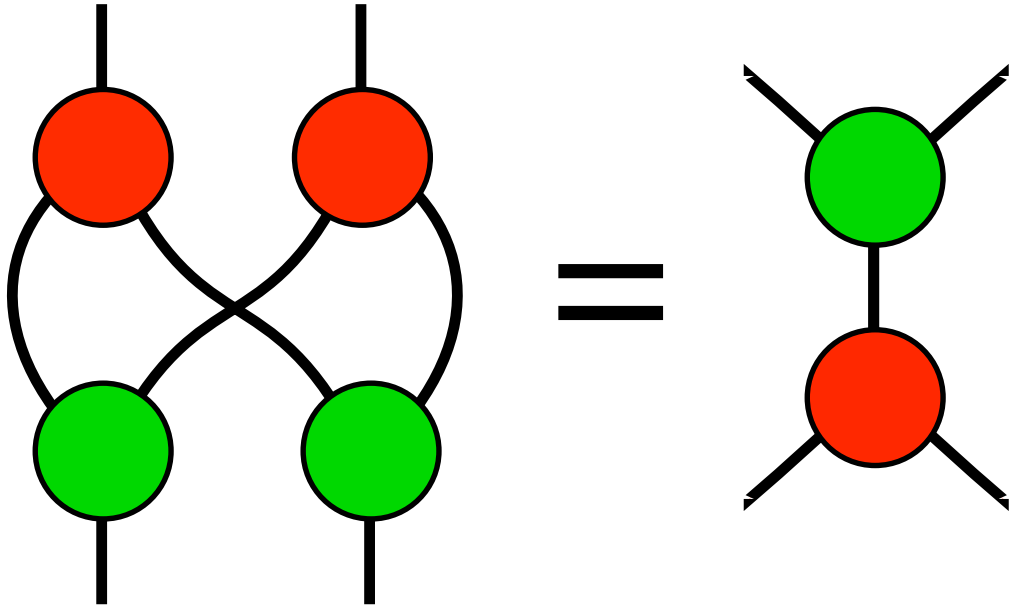


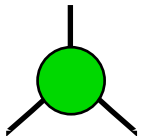
Coecke & Paquette 2006

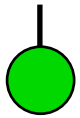
Complementary Observables

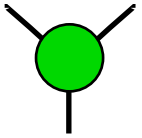


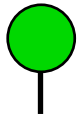
A very general theory of interference

Two kinds of points

$\delta =$ 

$\epsilon =$ 

$\delta^\dagger =$ 

$\epsilon^\dagger =$ 

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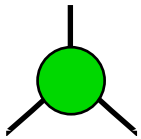
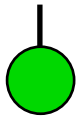
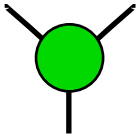
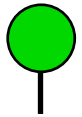
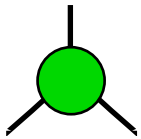
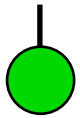
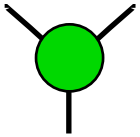
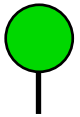
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Classical Points

Those points which can be copied by δ

Two kinds of points

$\delta =$ 
$\epsilon =$ 
$\delta^\dagger =$ 
$\epsilon^\dagger =$ 

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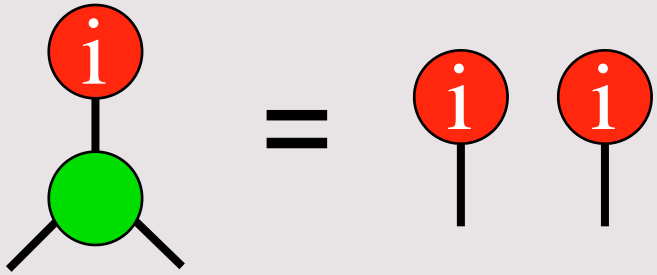

Those points which can be copied by δ

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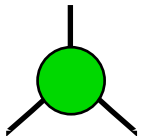
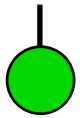
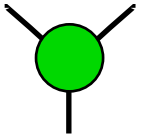
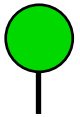
$\delta =$ 
$\epsilon =$ 
$\delta^\dagger =$ 
$\epsilon^\dagger =$ 

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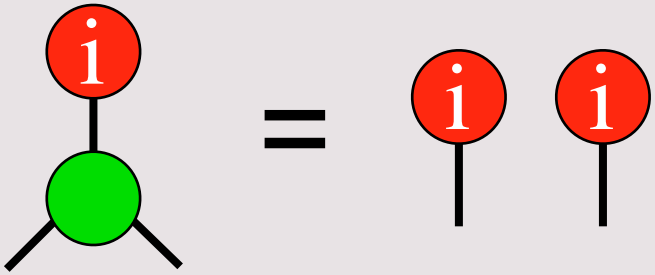

Those points which can be copied by δ

* to keep the pictures tidy, a scalar factor has been omitted (and everywhere from here on)

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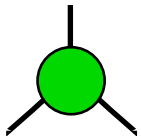
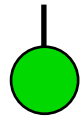
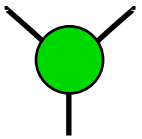
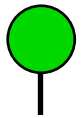
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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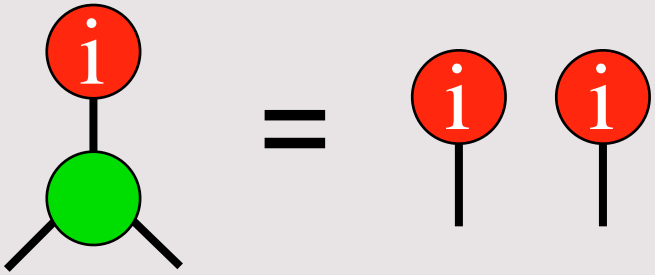

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Unbiased Points

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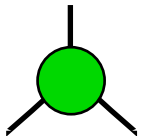
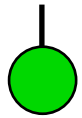
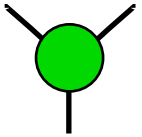
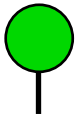
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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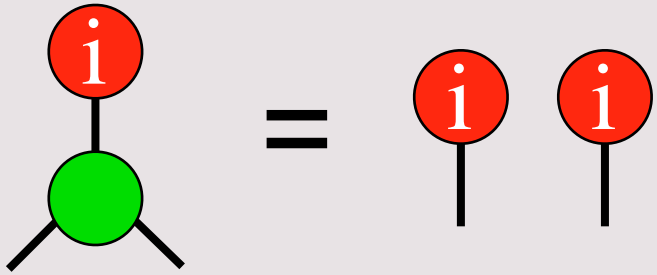

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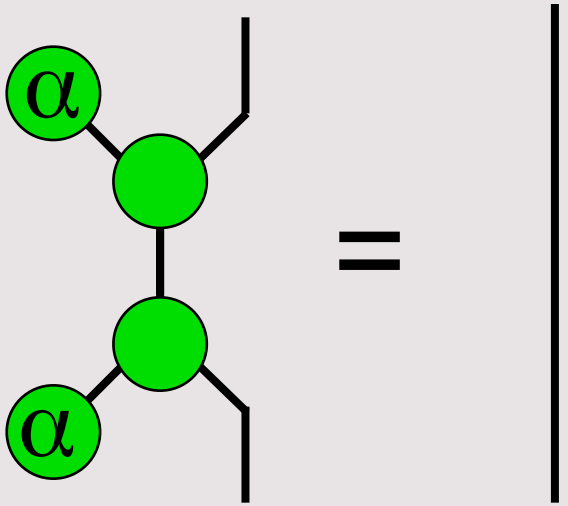

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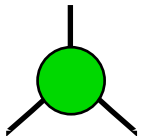
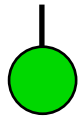
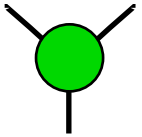
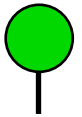
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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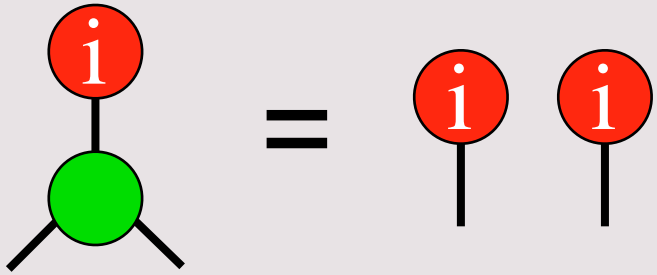

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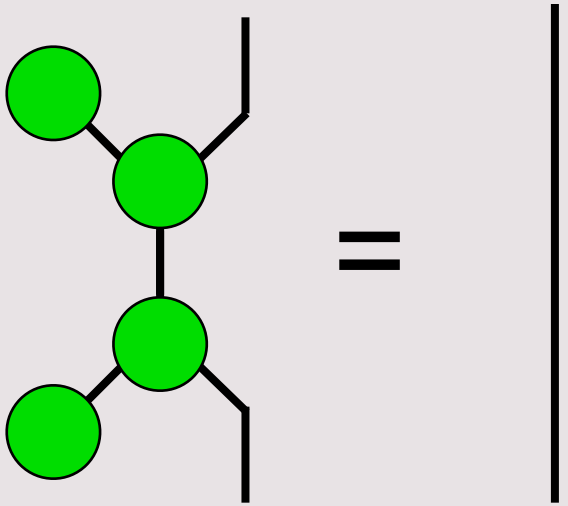
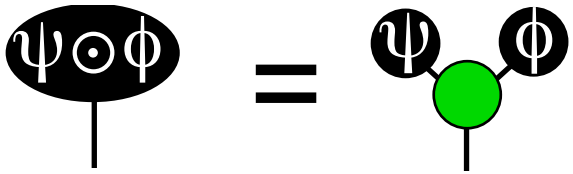


Using the monoid operation

Let $\psi, \phi : I \to A$ be points of A; we can combine them using the monoid operation $\delta^\dagger : A \otimes A \to A$

$$\psi \odot \phi := \delta^\dagger \circ (\psi \otimes \phi)$$



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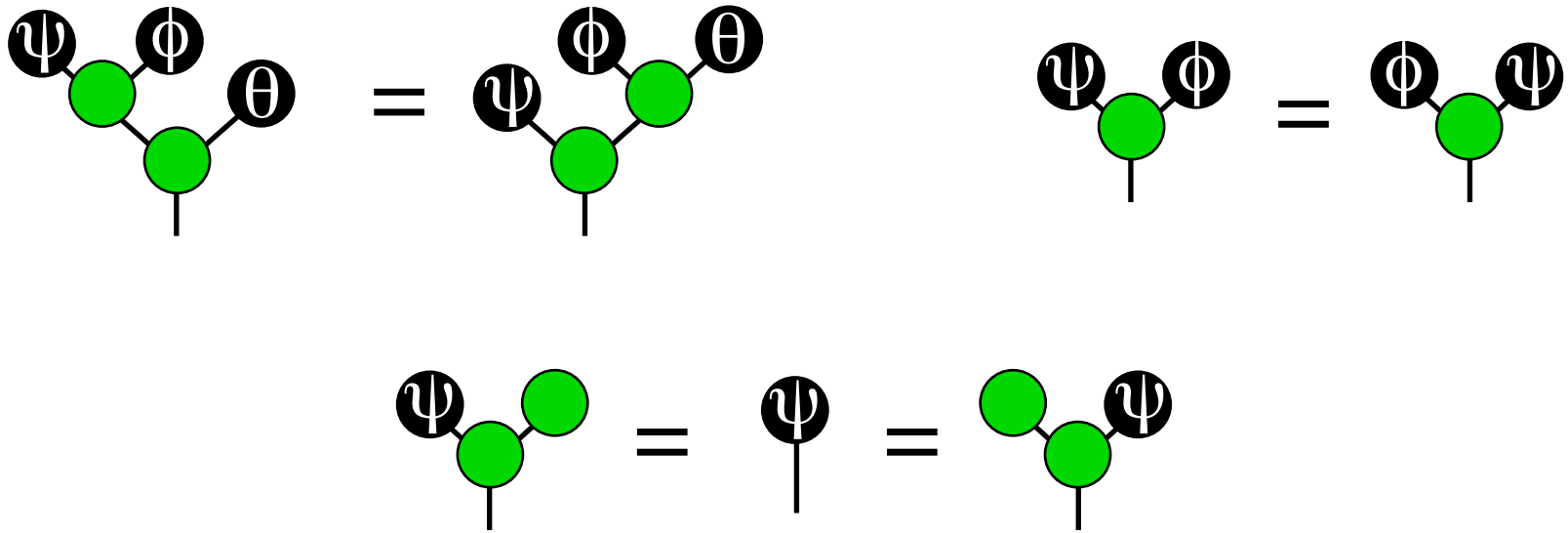
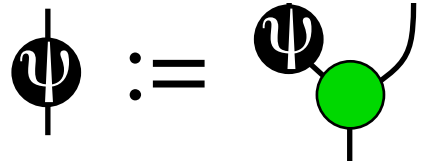


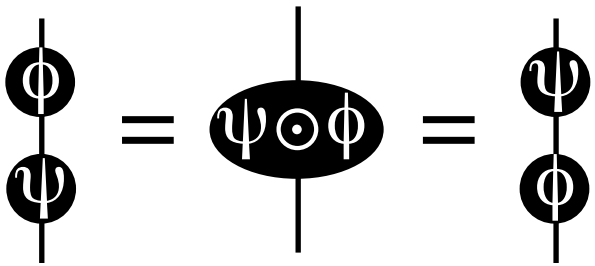
Using the monoid operation

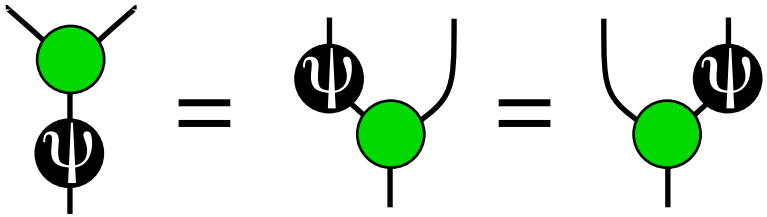
Moreover, each point $\psi : I \to A$ can be lifted to an endomorphism $\Lambda(\psi) : A \to A$

$$\Lambda(\psi) := \delta^\dagger \circ (\psi \otimes \mathrm{id}_A)$$



This yields a homomorphism of monoids so we have:





Generalised Spider Theorem

Theorem: any maps constructed from δ, ε, some points $\psi_i : I \to A$ and their adjoints, whose graph is connected, is determined by the number of inputs and outputs and the product $\bigodot_i \psi_i$.

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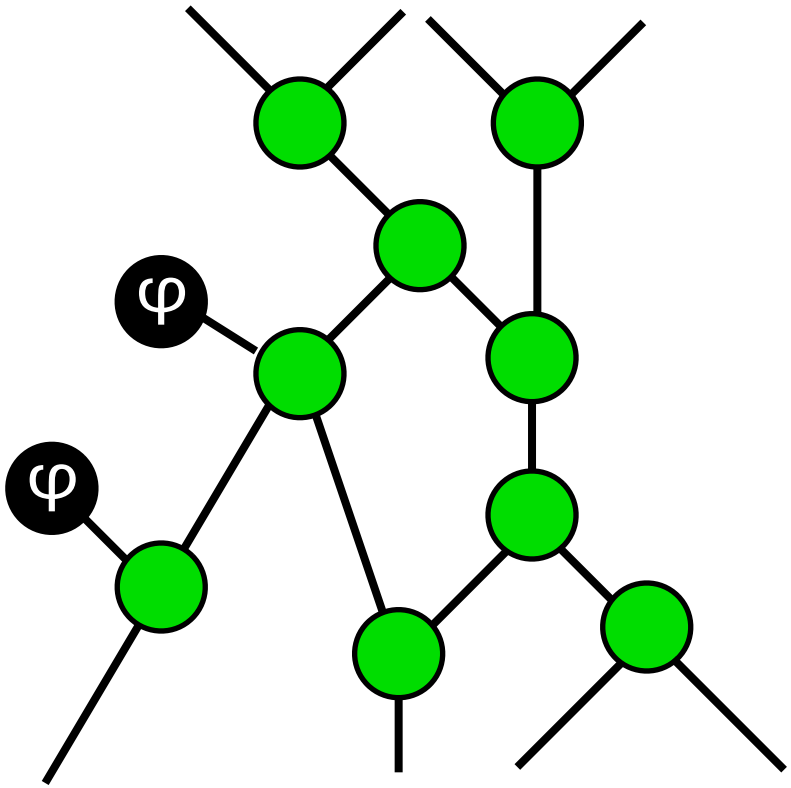

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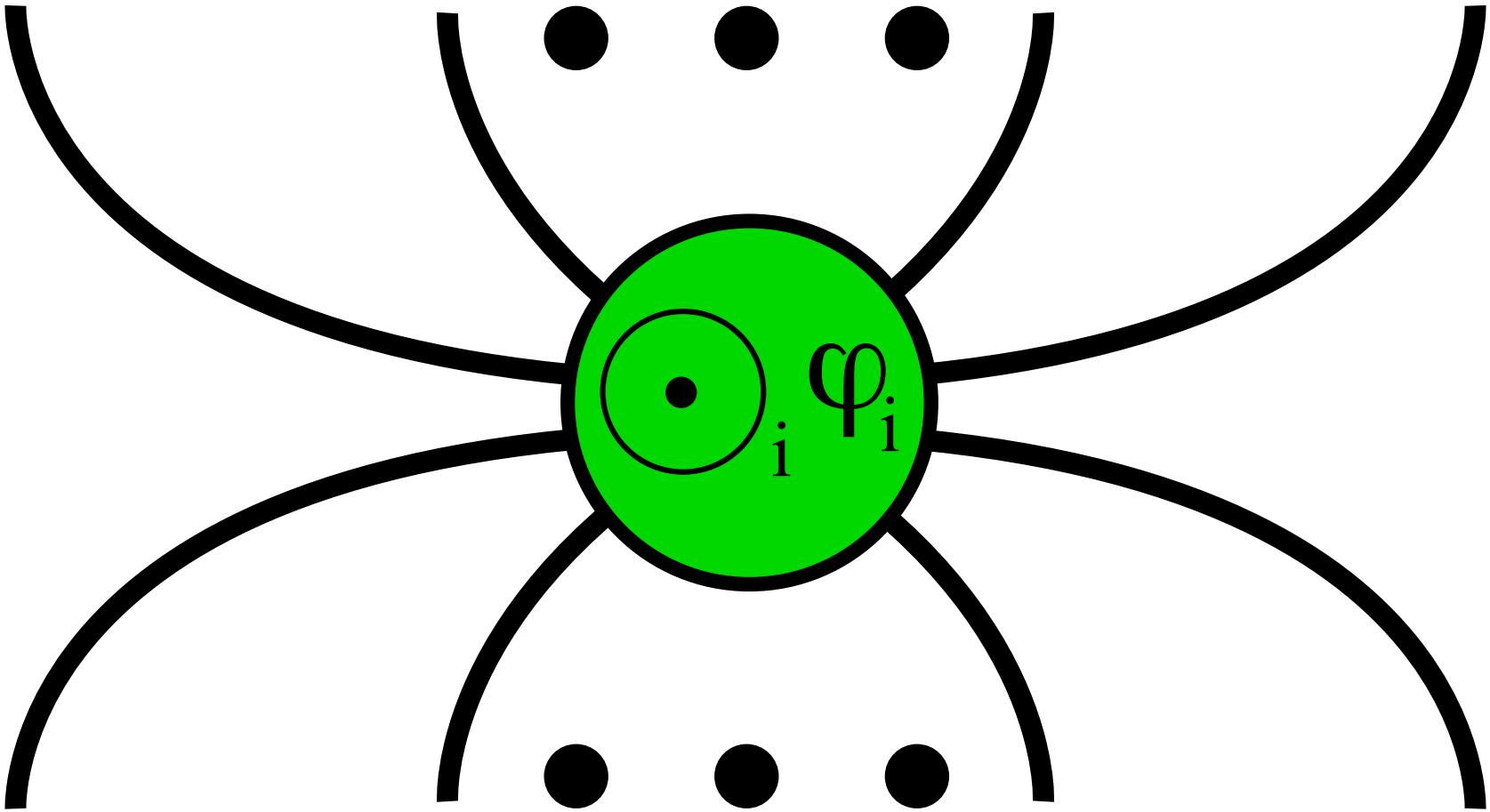
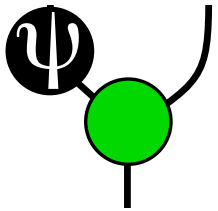


Unbiased Points

Q: When is 

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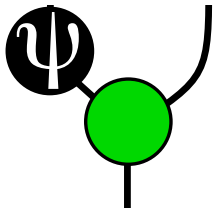
Q: When is 

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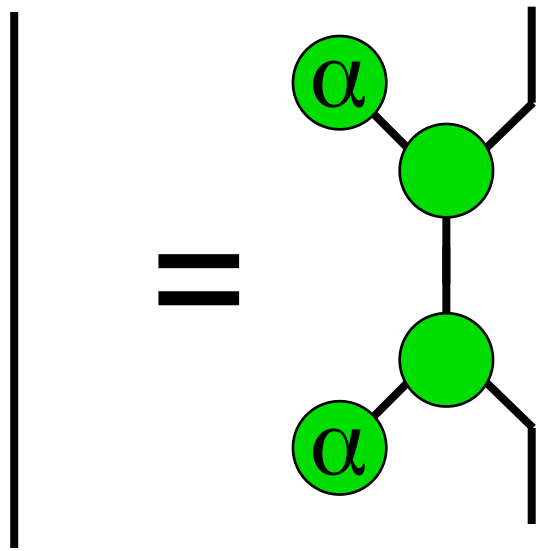
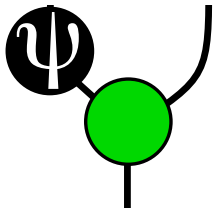


Unbiased Points

Q: When is 

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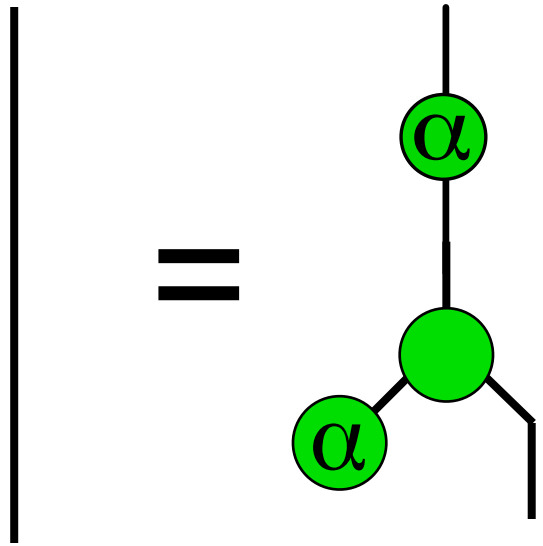
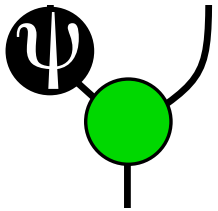


Unbiased Points

Q: When is 

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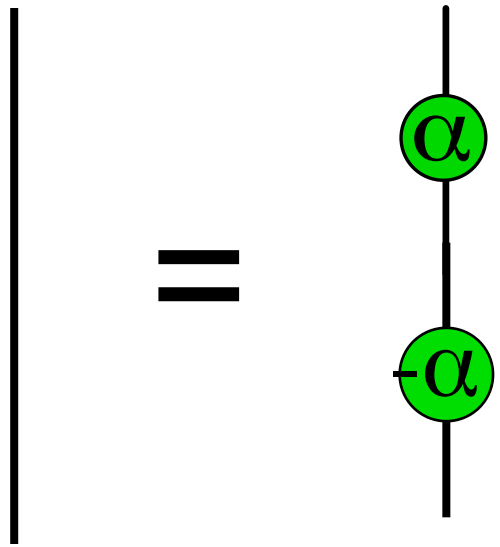
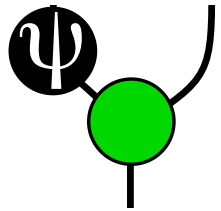


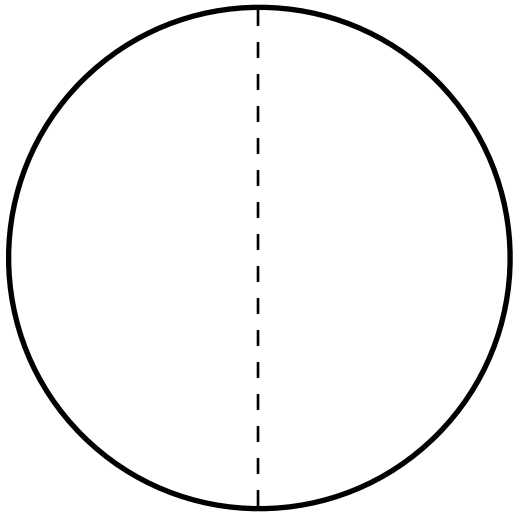
Unbiased Points

Q: When is  unitary?

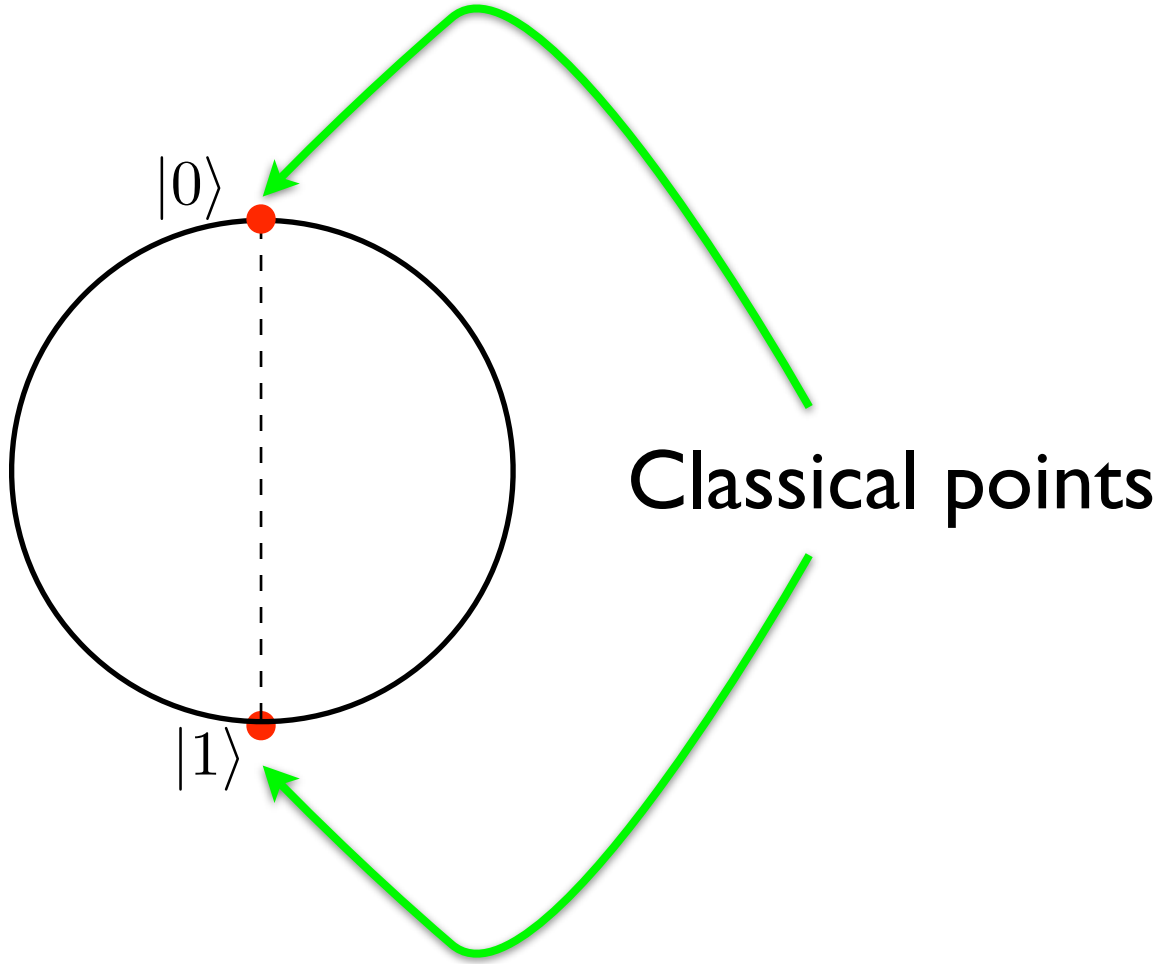
A: In Hilbert spaces, $\Lambda(\psi)$ is unitary iff $|\psi\rangle$ is unbiased w.r.t. the basis copied by δ.

Prop:
1. the unbiased points for (δ, ϵ) form an abelian group w.r.t. to $\odot$;
2. the arrows generated by the unbiased points form an abelian group w.r.t. composition.

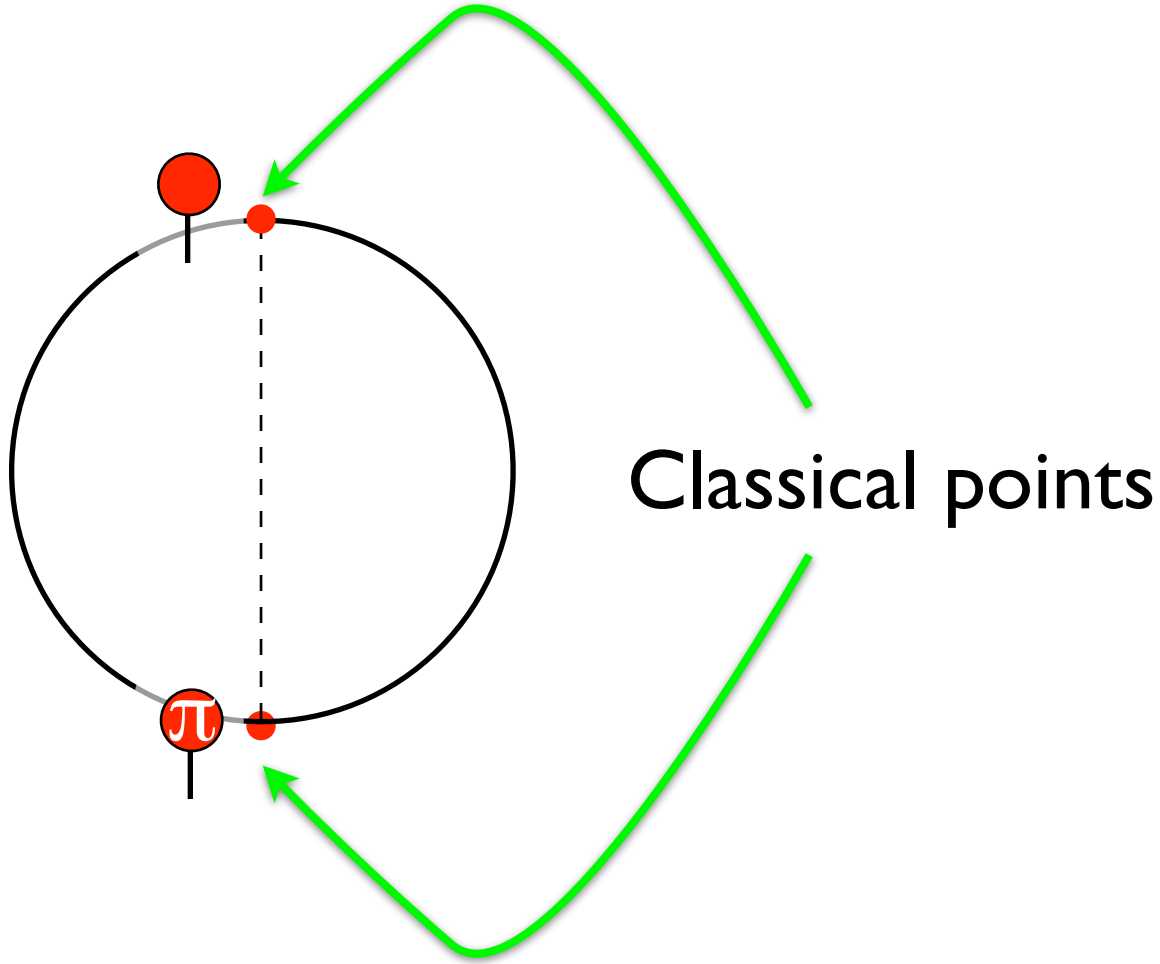
Classical and Unbiased Points



Classical and Unbiased Points



Classical and Unbiased Points



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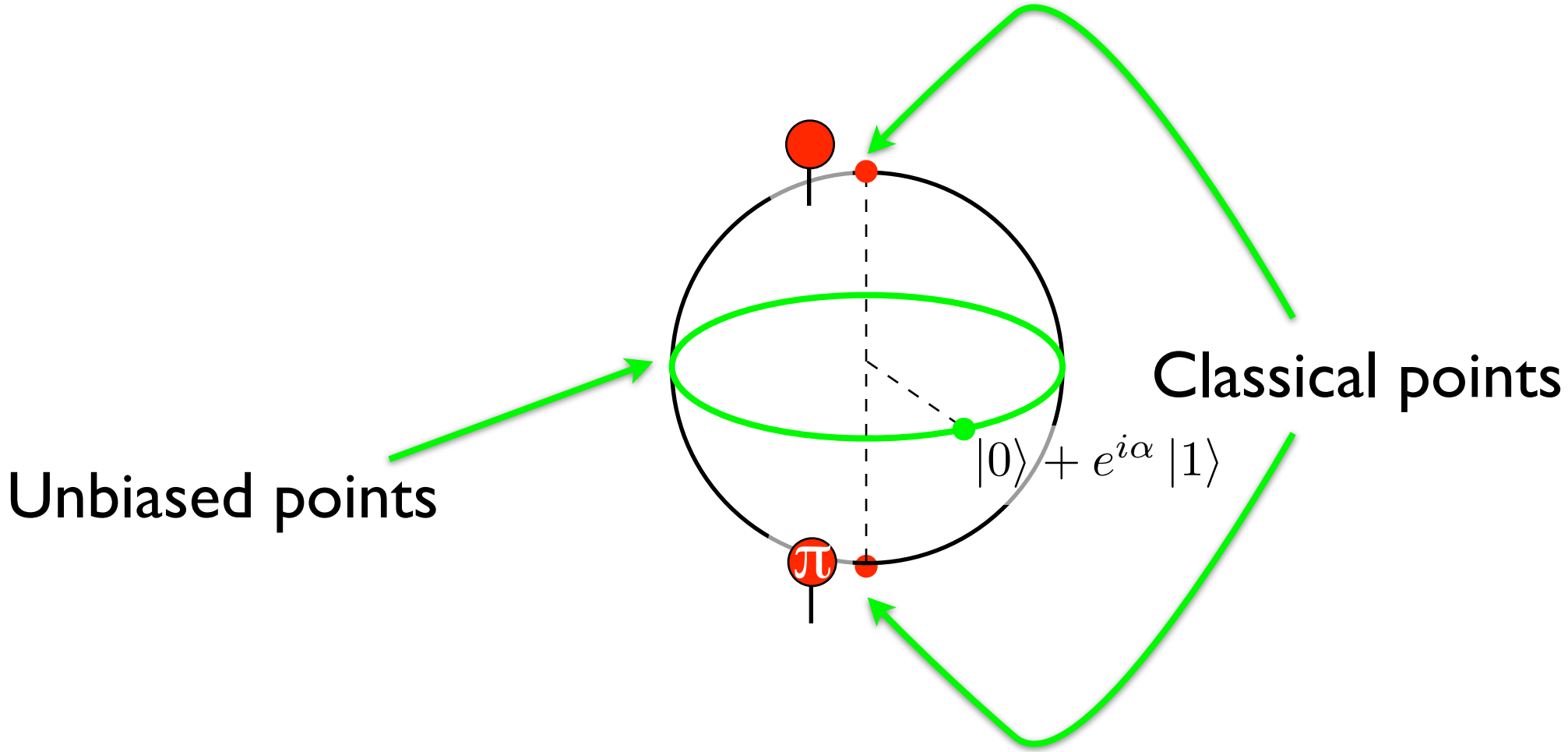

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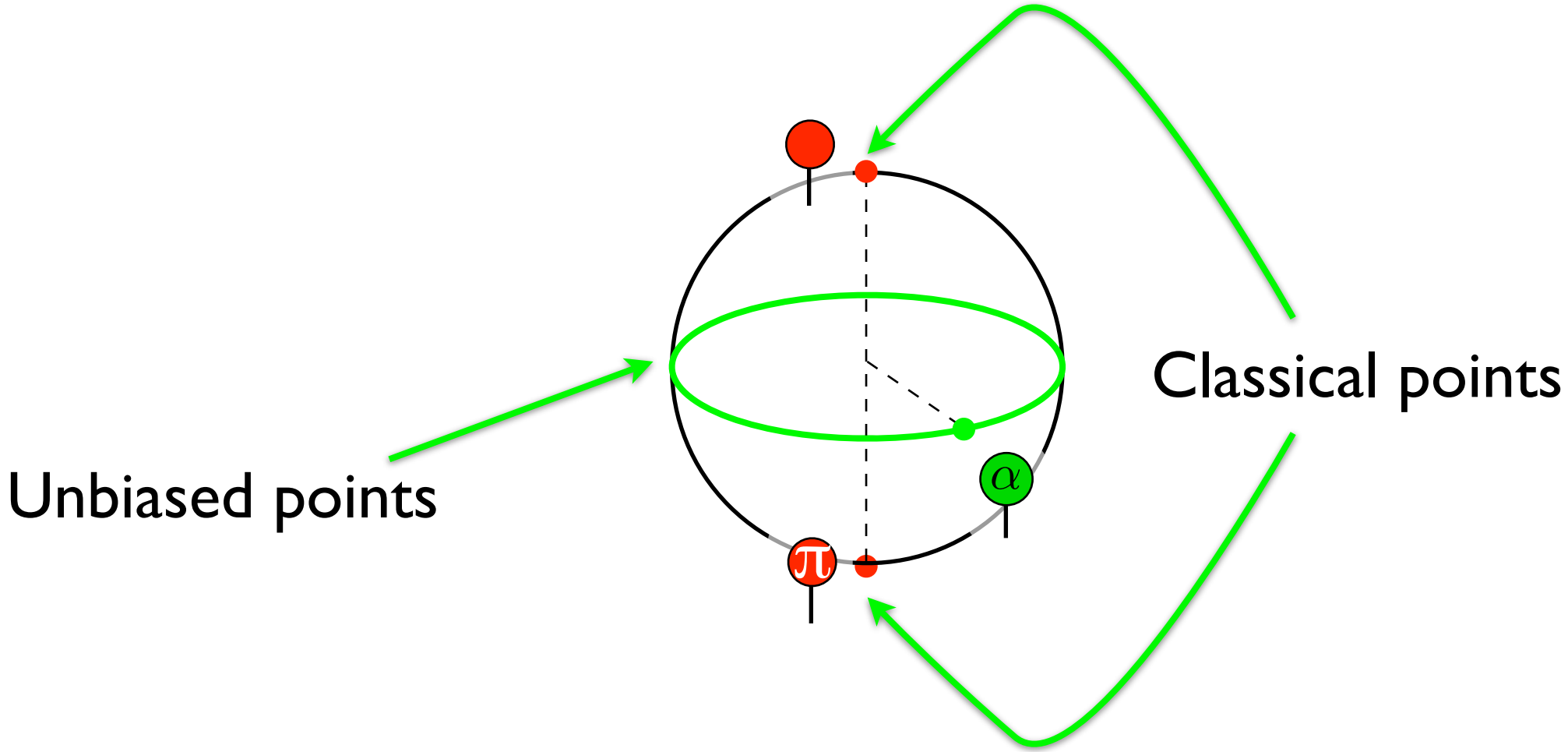
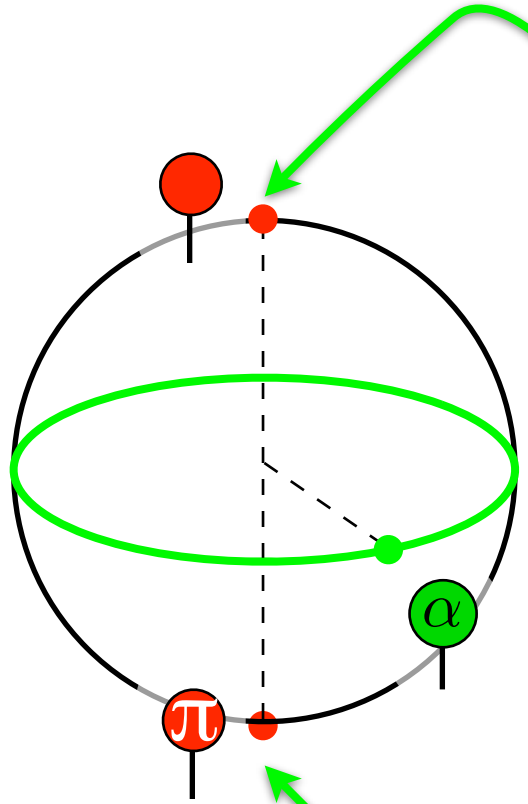


Classical and Unbiased Points

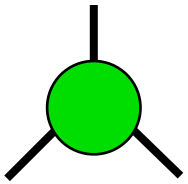
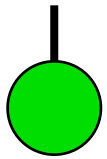
$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



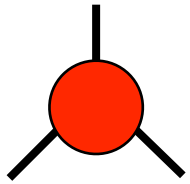
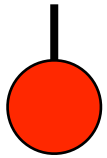
Unbiased points

Classical points

Complementary Classical Structures

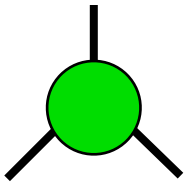
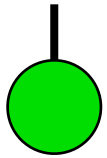
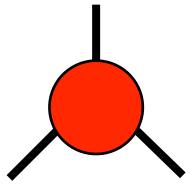
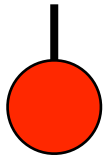
$\delta_Z =$  $\epsilon_Z =$ 

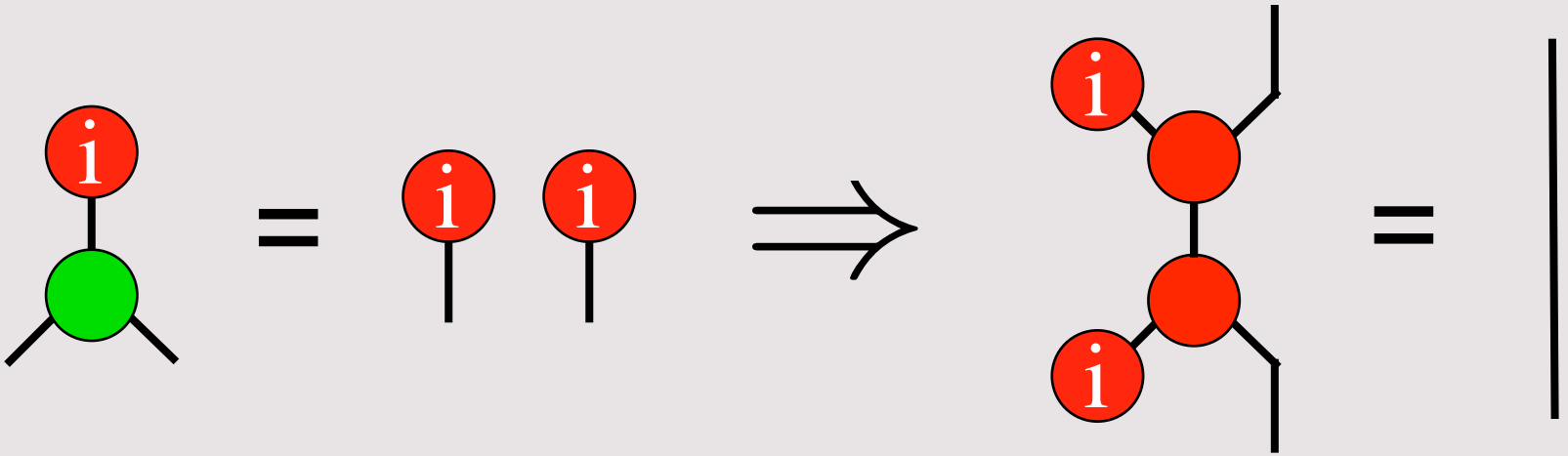
$|i\rangle \mapsto |ii\rangle$ $|+\rangle \mapsto 1$

$\delta_X =$  $\epsilon_X =$ 

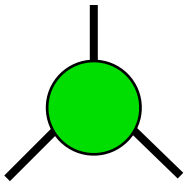
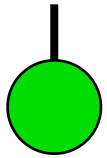
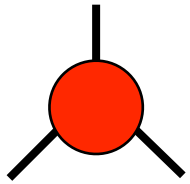
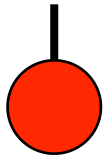
$|\pm\rangle \mapsto |\pm\pm\rangle$ $|0\rangle \mapsto 1$

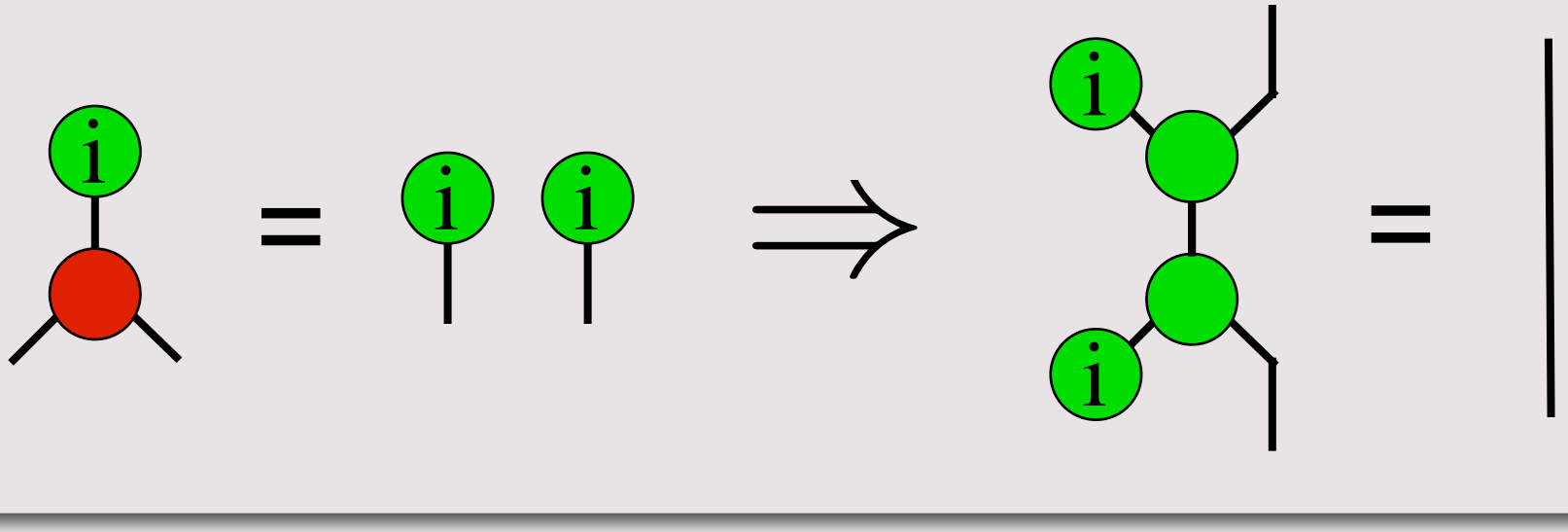
Complementary Classical Structures

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 



Complementary Classical Structures

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 



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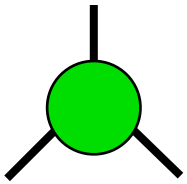
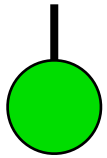
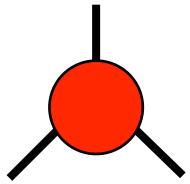
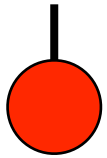
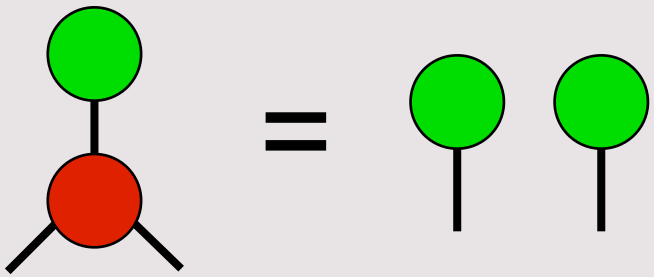
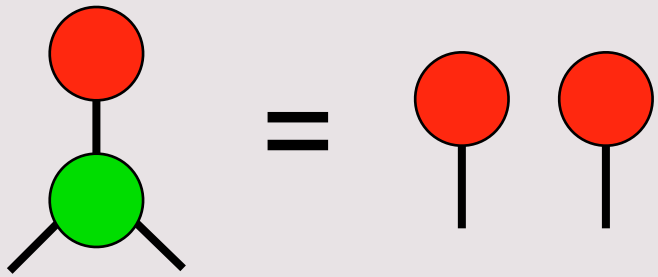
$\delta_Z =$  $\epsilon_Z =$ 

$\delta_X =$  $\epsilon_X =$ 





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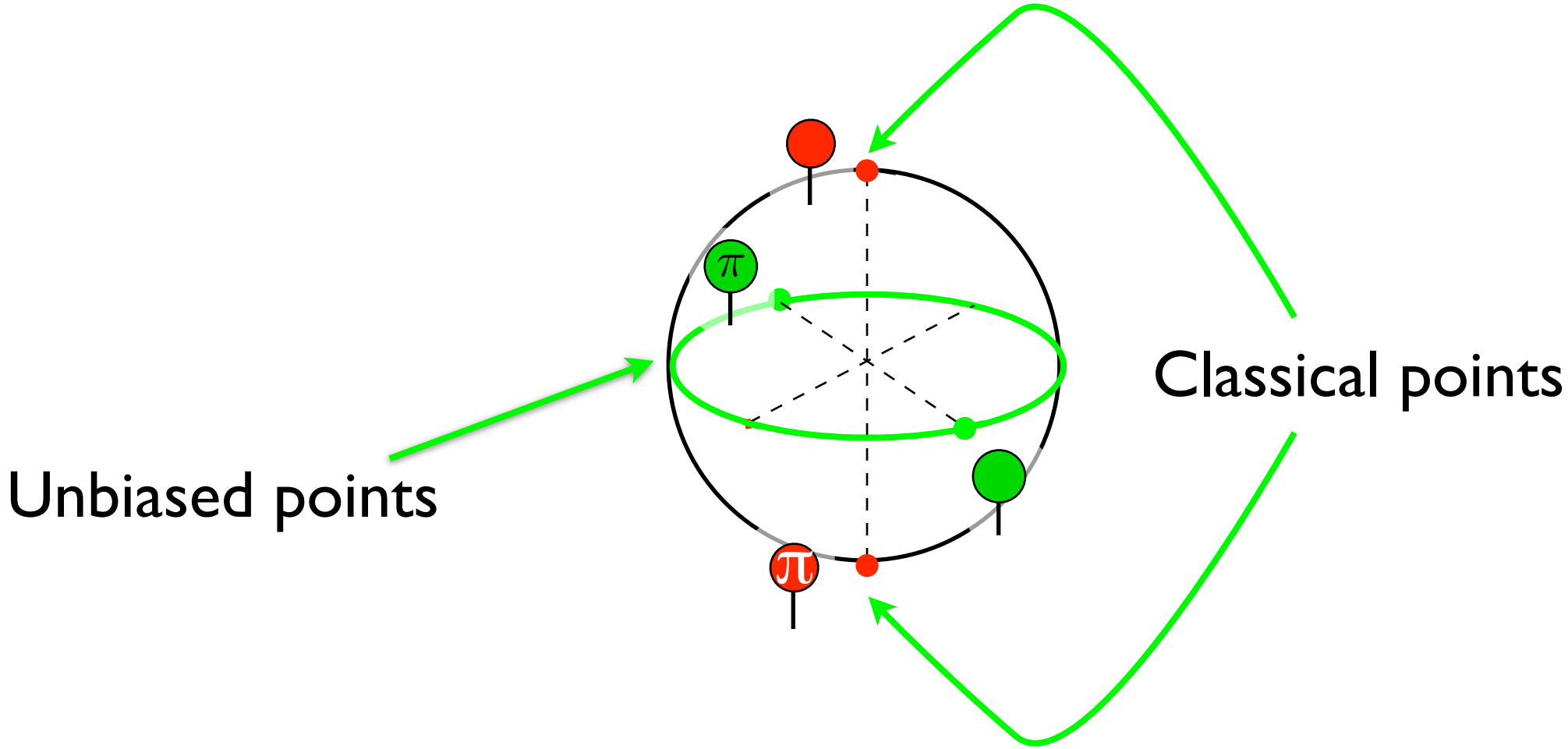

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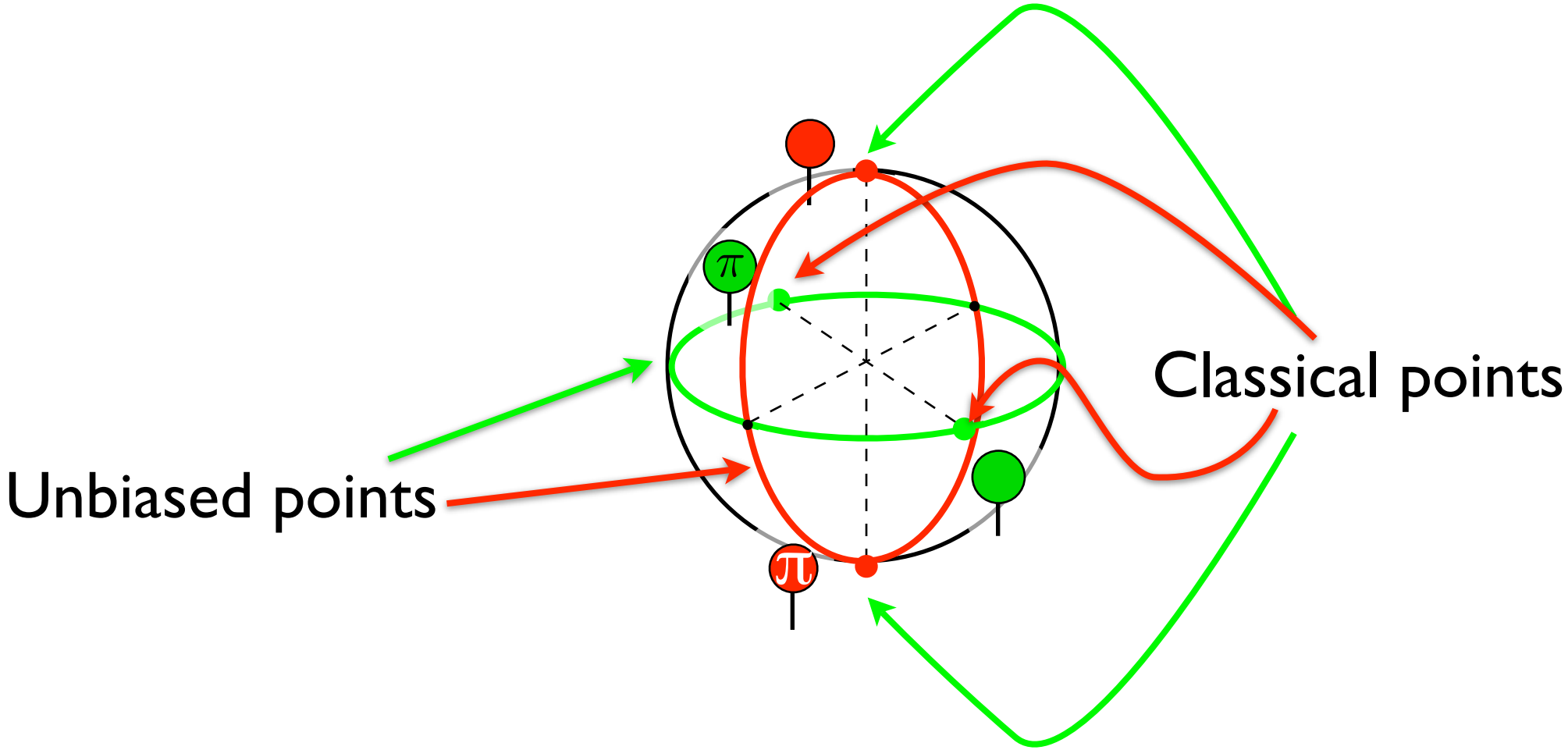
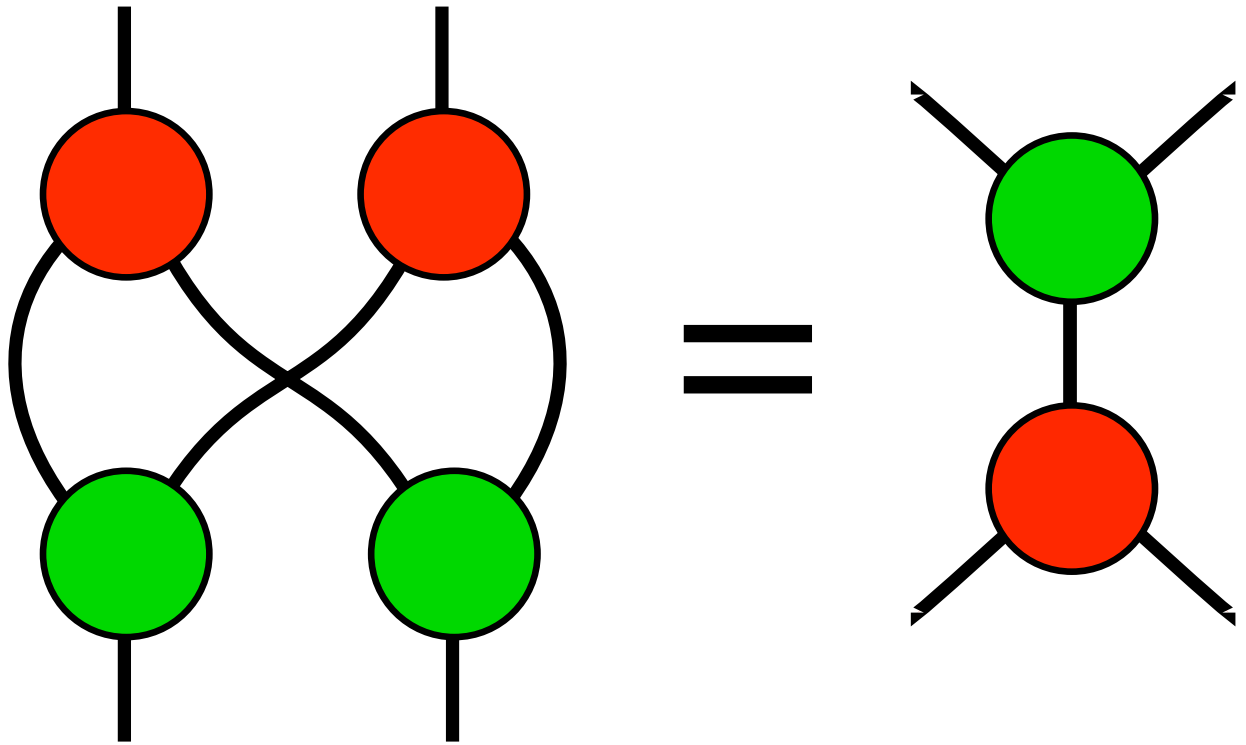


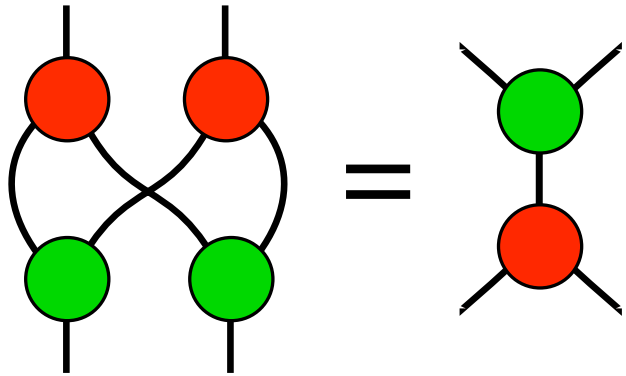
Complementary Classical Structures form a Bialgebra

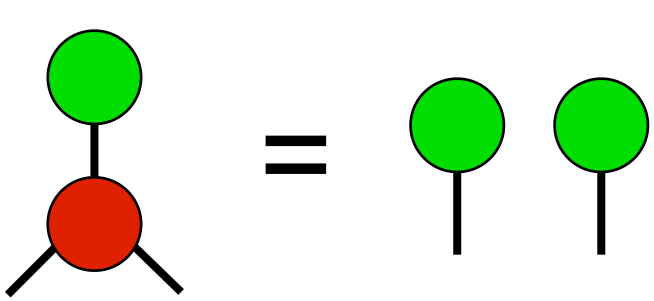
Theorem:

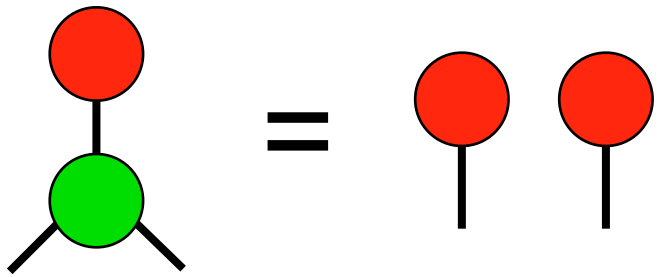


(* if there are enough classical points)

Complementary Classical Structures form a Bialgebra

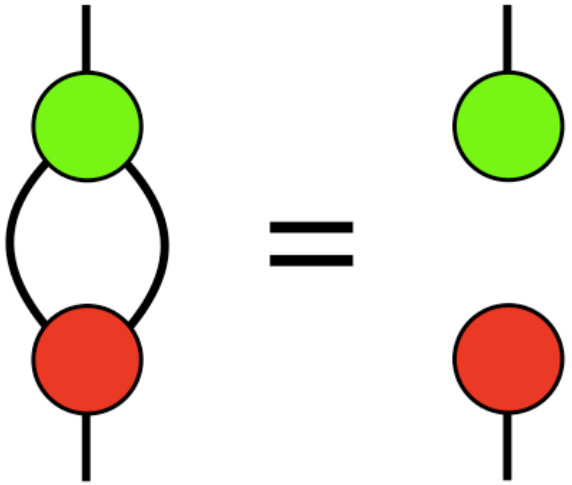






Complementary Classical Structures form a Bialgebra

Corollary: complementary classical structures form a Hopf algebra:



Complementary Classical Structures form a Bialgebra

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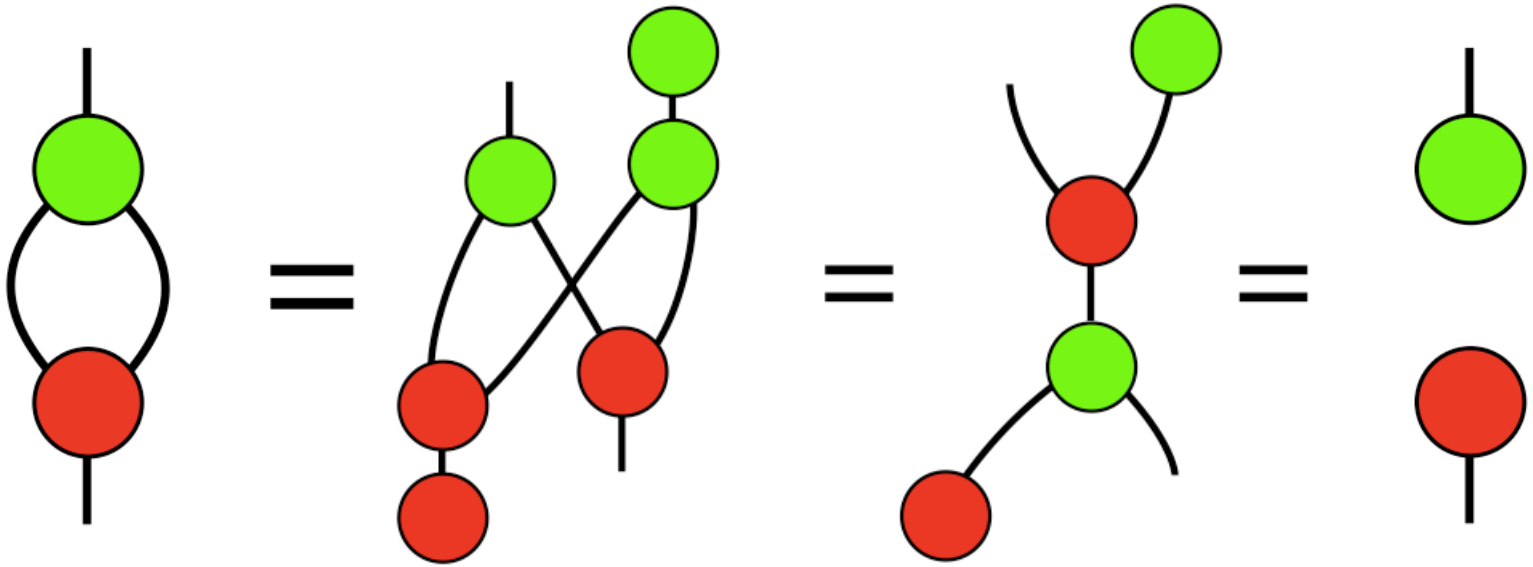
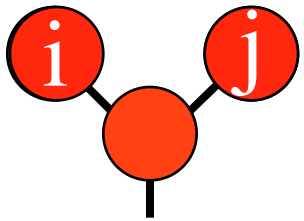


Interference Action

Lemma: The classical points form a subgroup:



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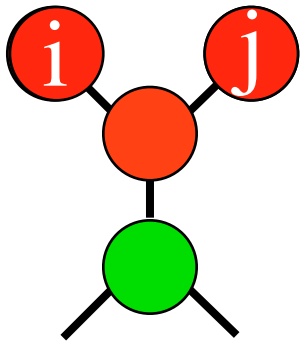

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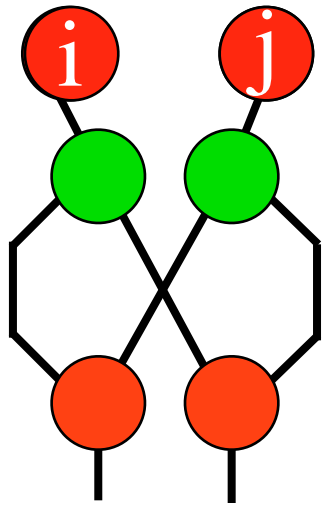

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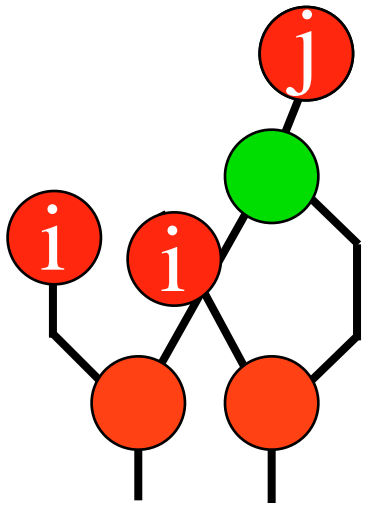

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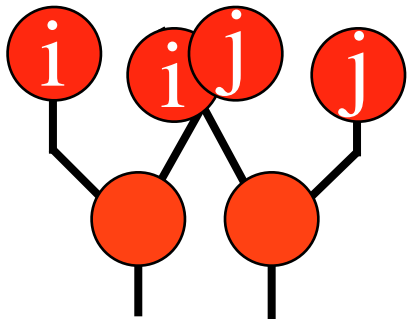

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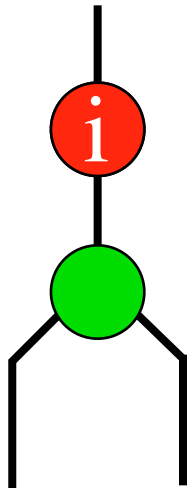
Lemma: The classical maps are comonoid homomorphisms:



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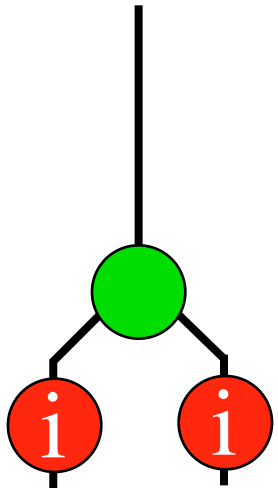
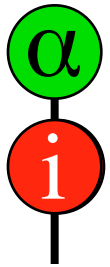


Interference Action

Theorem: The classical maps are group automorphisms of the unbiased points:



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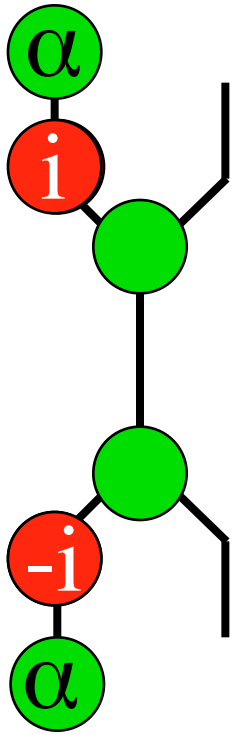

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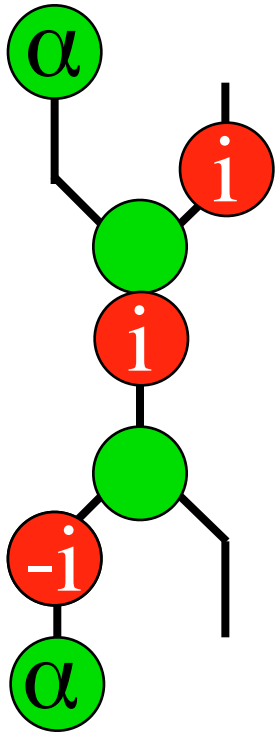

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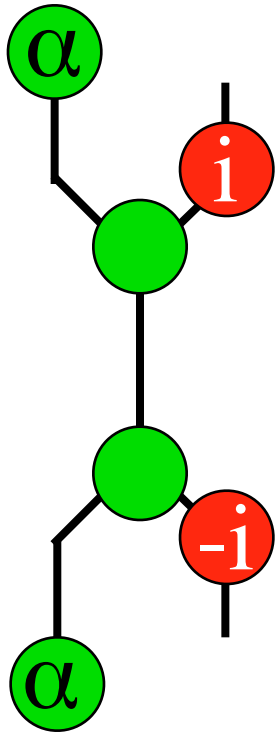

Interference Action

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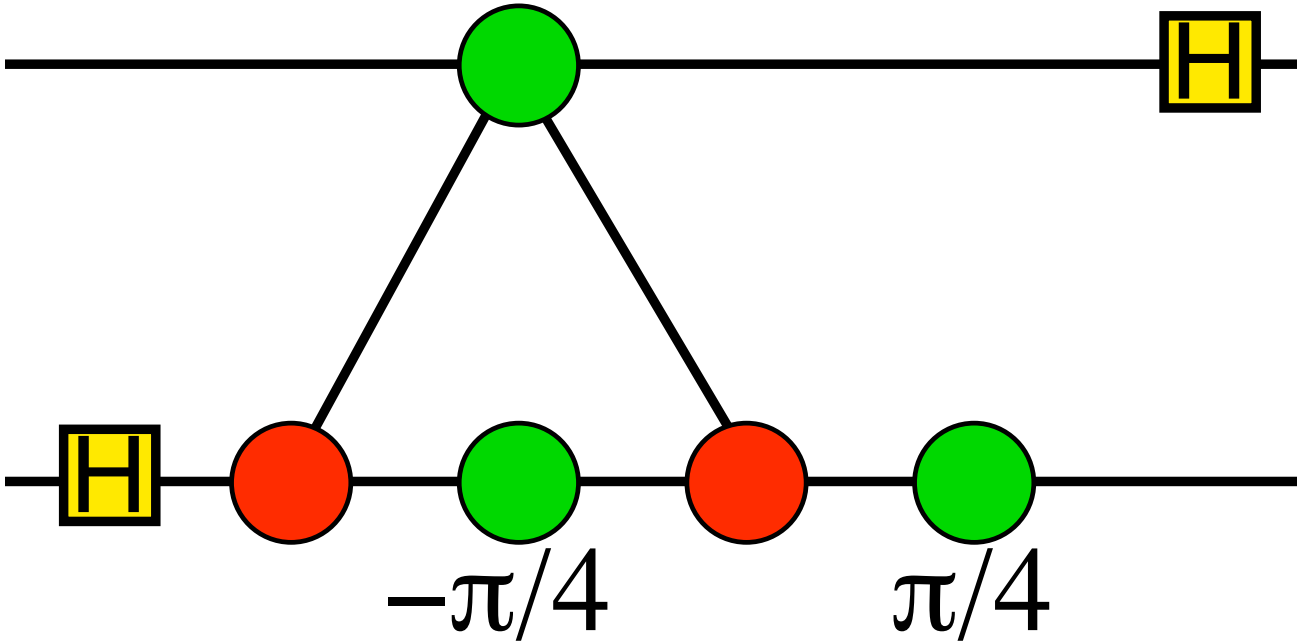
Complementary Qubit Observables



Quantum processes, diagrammatically

The X and Z Spins, again

When we specialise this structure to the case of qubits we get:

- Complementary classical structures defined by the X and Z bases.
- The unbiased group is $[0, 2\pi)$ under addition $\mathrm{mod}\ 2\pi$
- The classical group is just S_2
- The non-trivial action of the classical group is

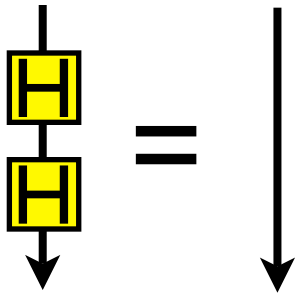
$$\alpha \mapsto -\alpha$$

- *There is a self adjoint map interchanging the colours.*

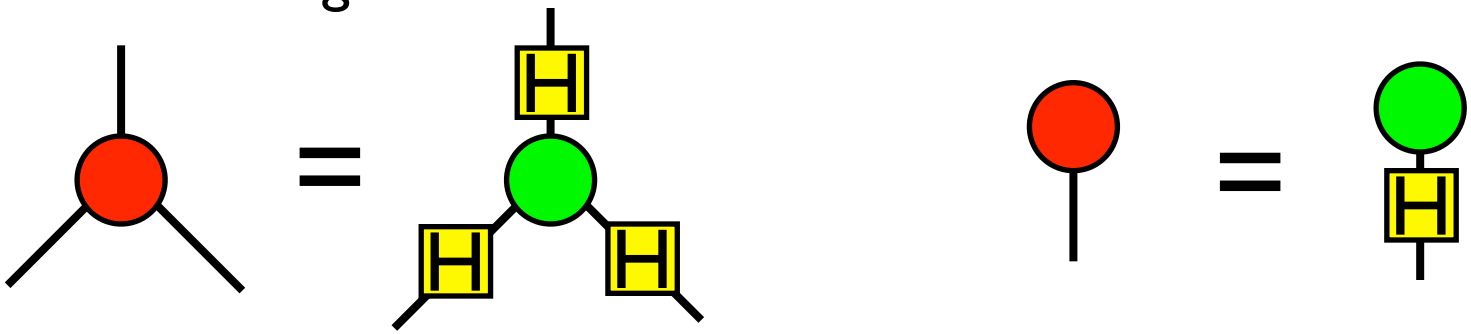
The Hadamard

The Hadamard gate has a number of nice properties:

- Self-adjointness and unitarity:



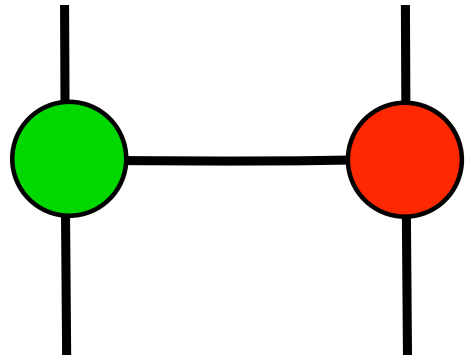
- It exchanges the X and Z bases:



These *colour duality* equations will be used frequently in examples.

Representing Logic Gates

$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



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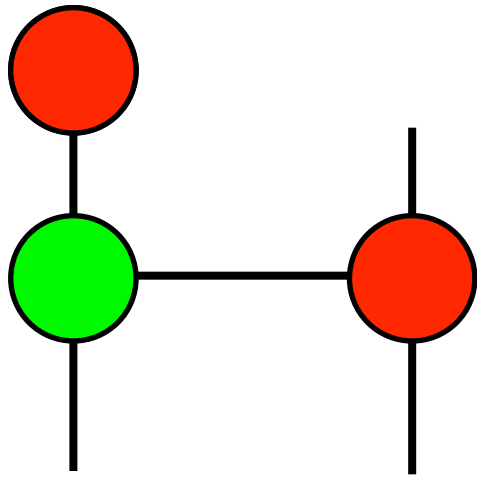

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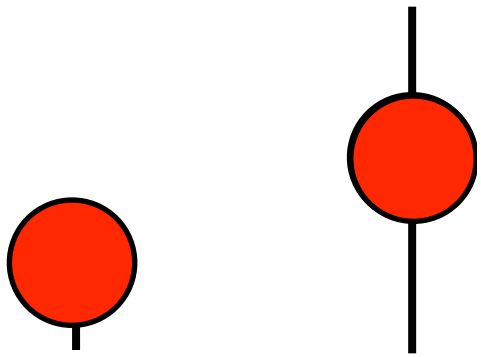

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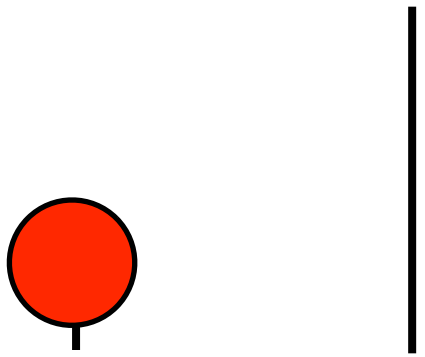

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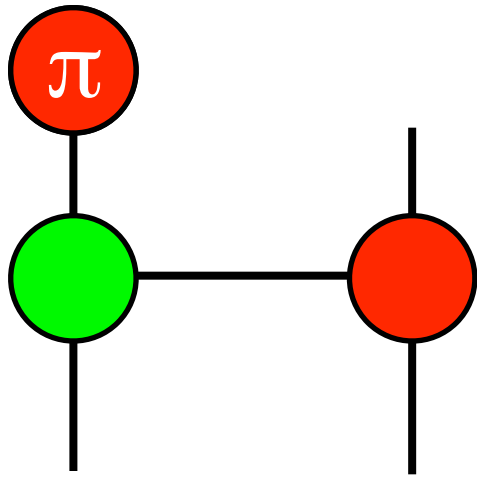

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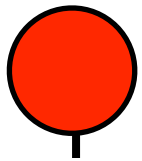
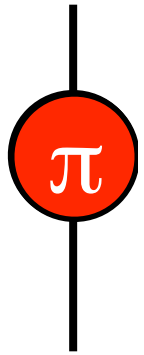
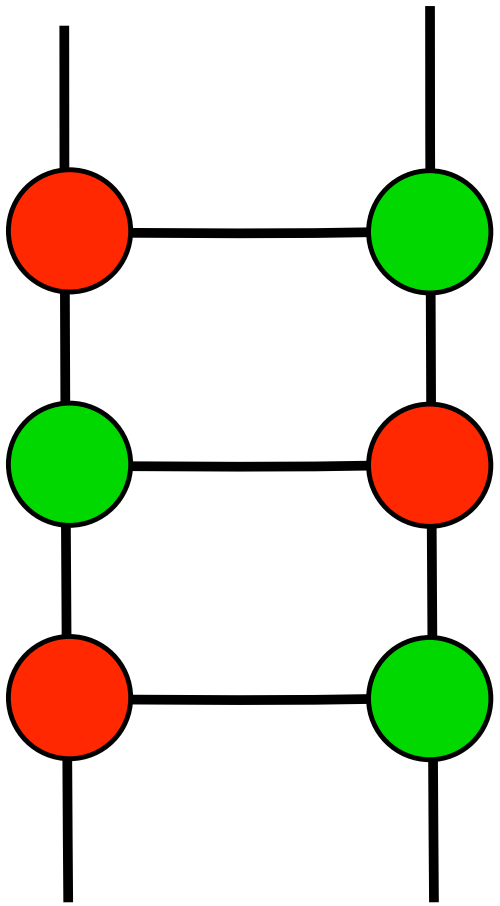




Example: 3 CNOTs



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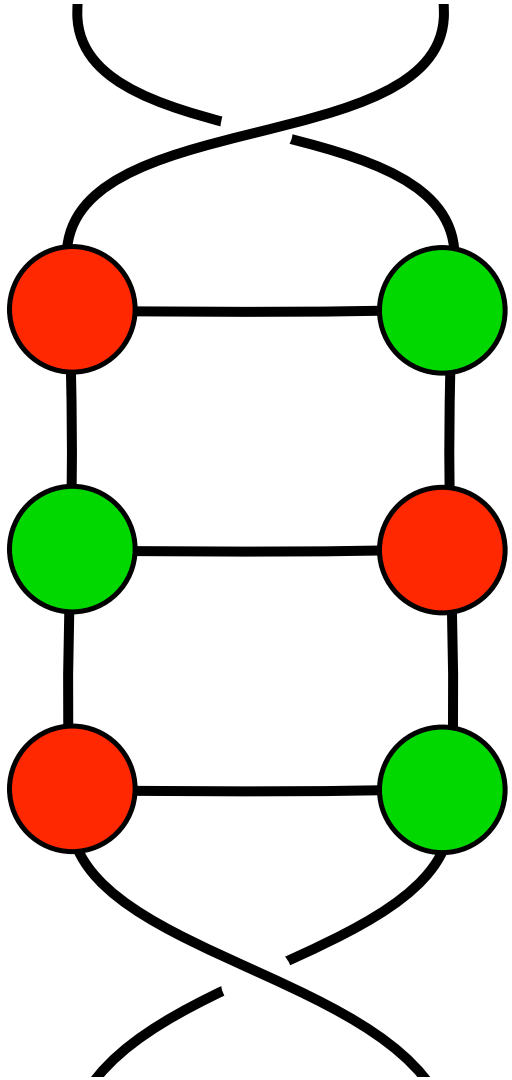

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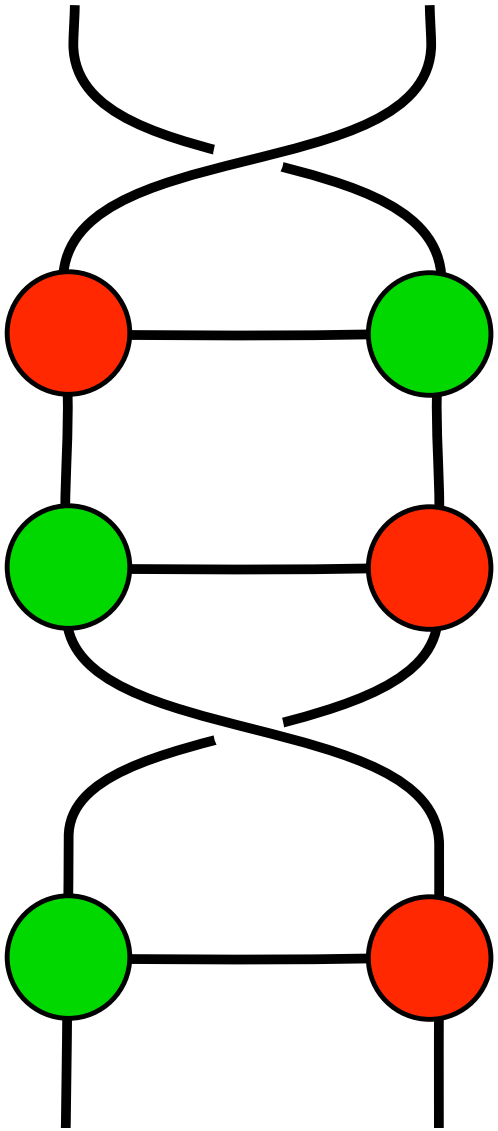

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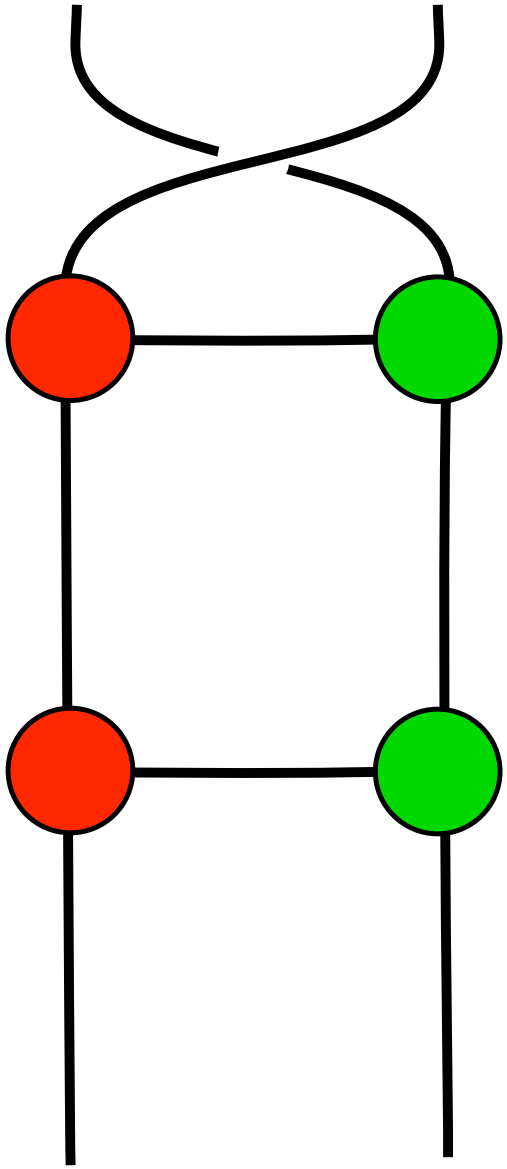

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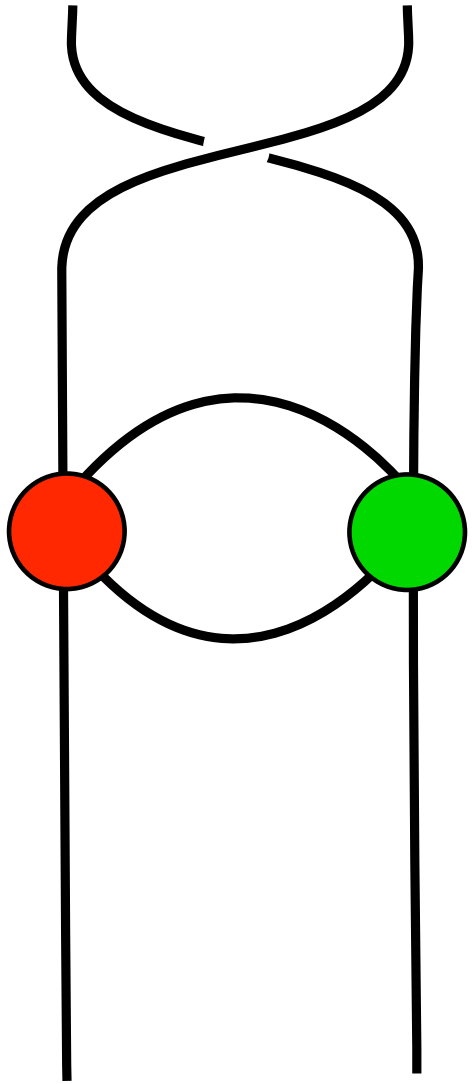

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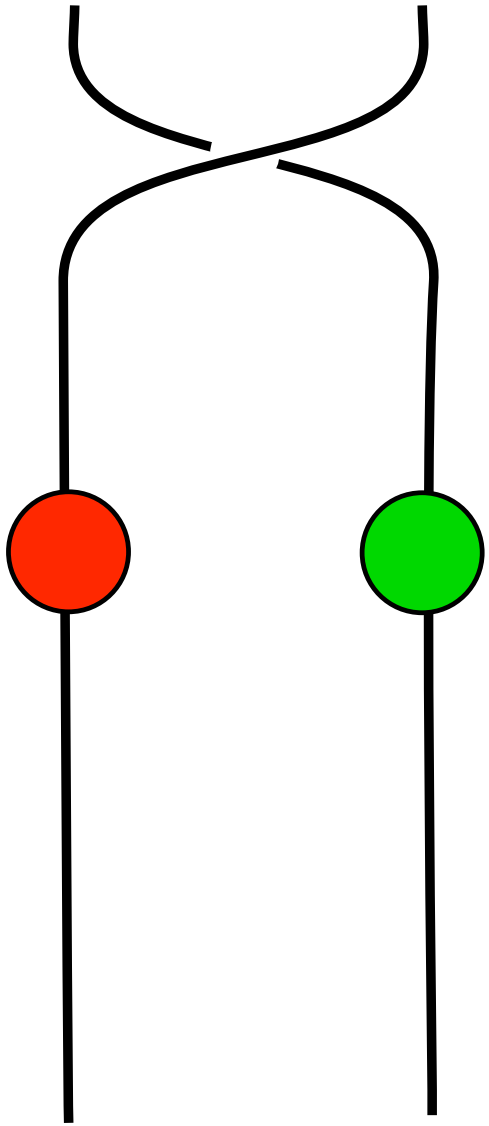

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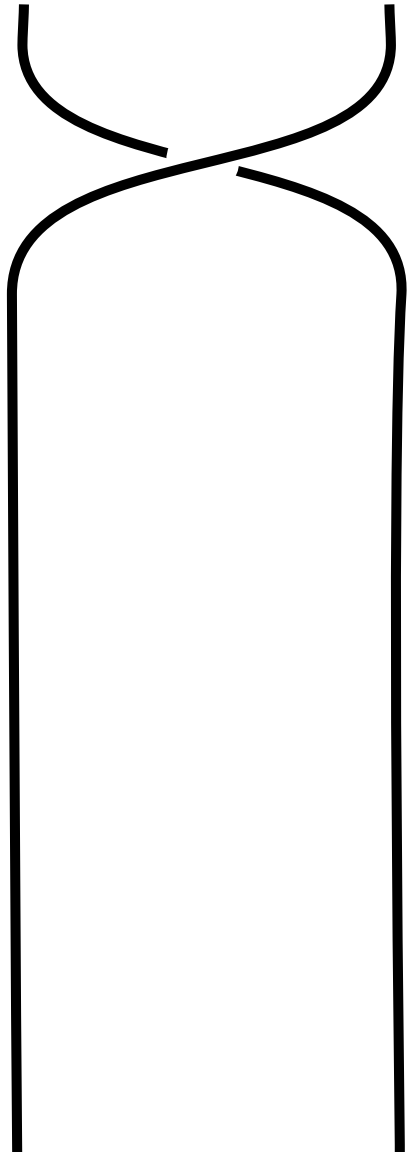
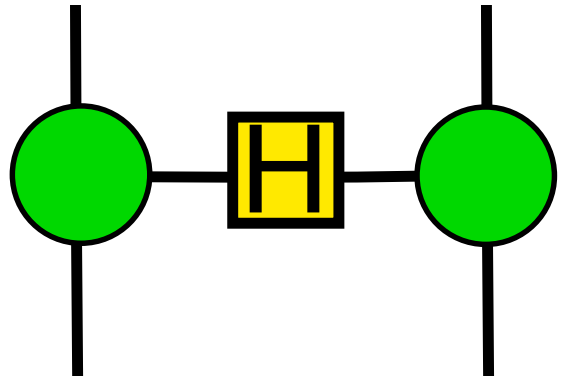
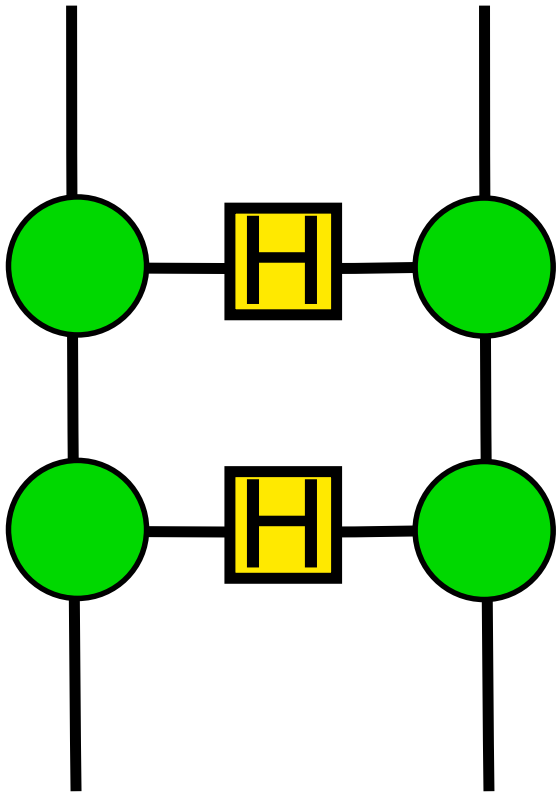


Representing Logic Gates

$$\wedge Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



Example: 2 CZs



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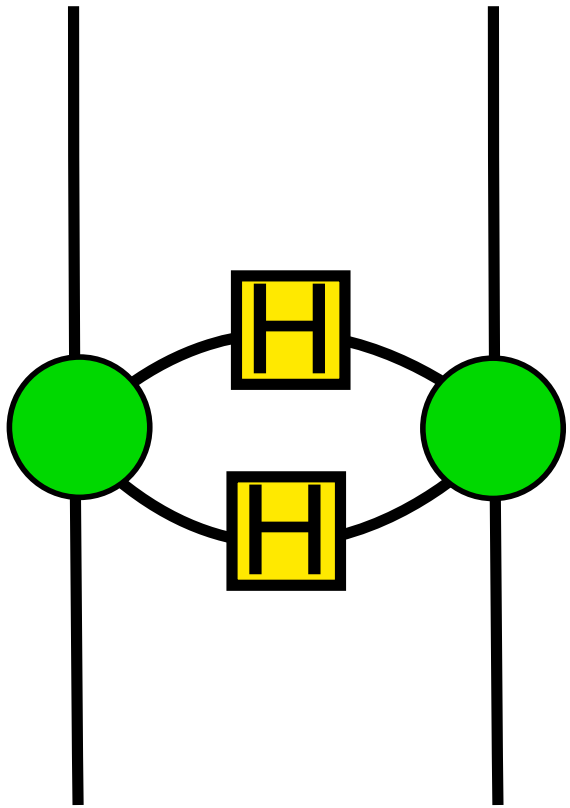

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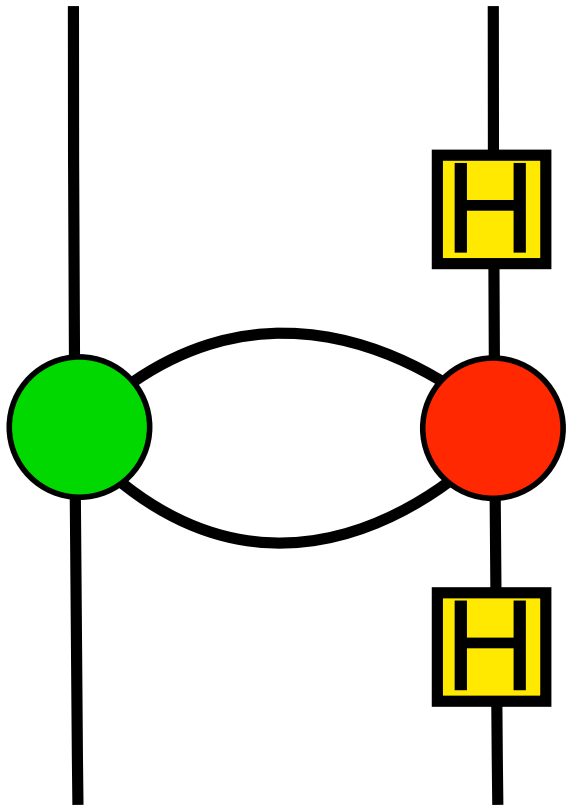

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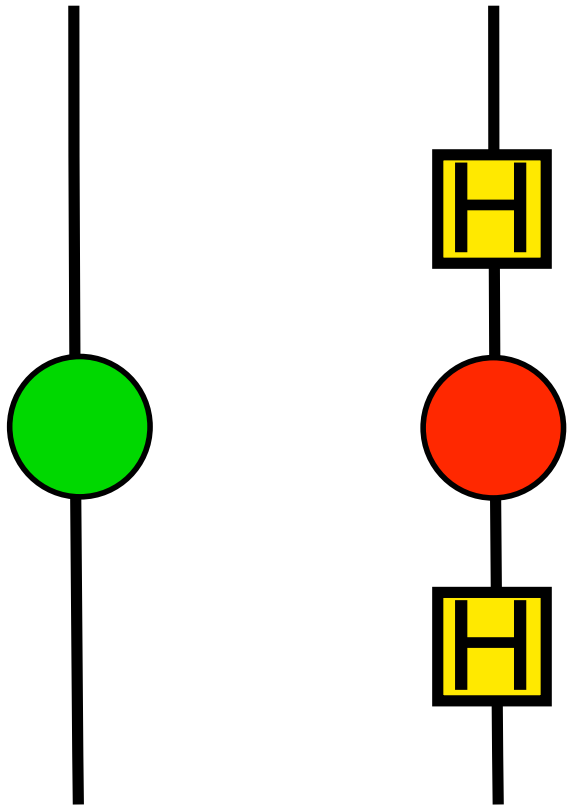

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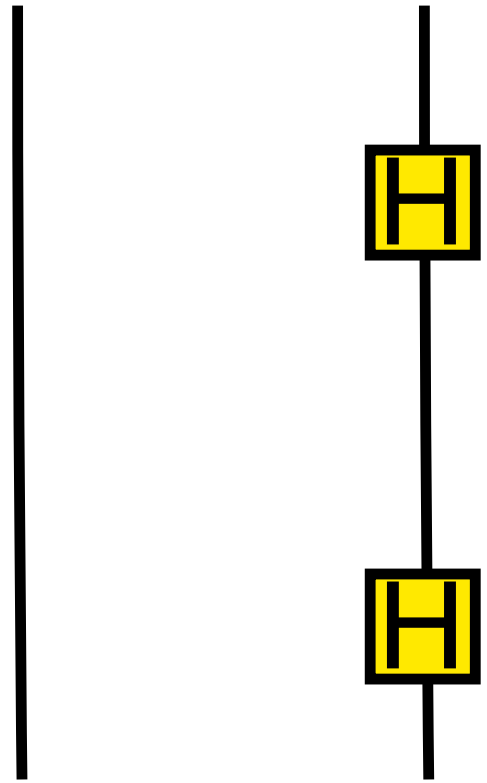

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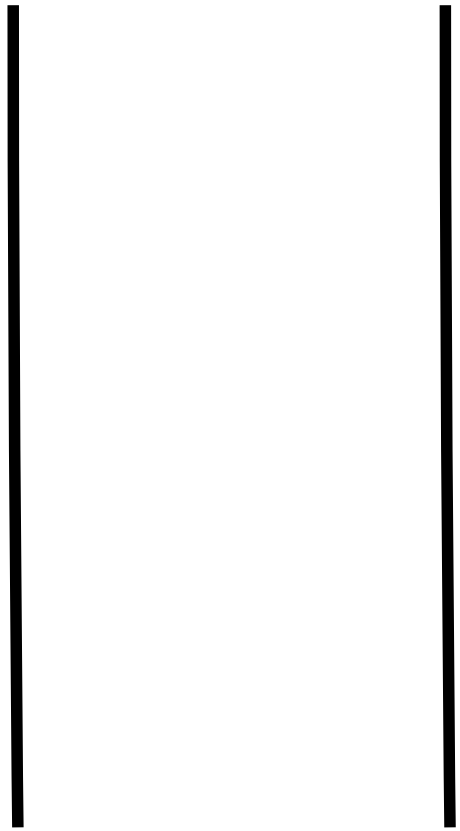
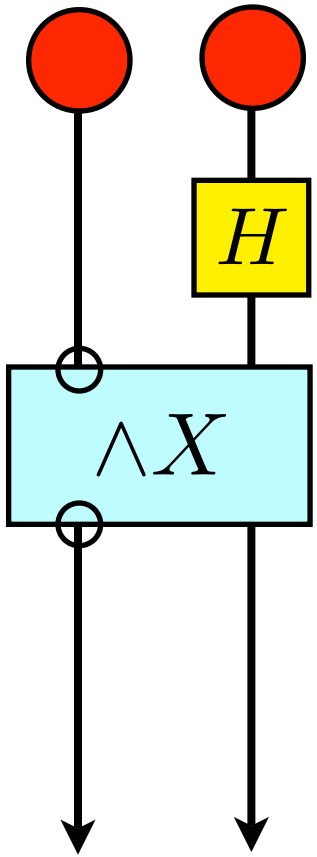


Example: Preparing a Bell state



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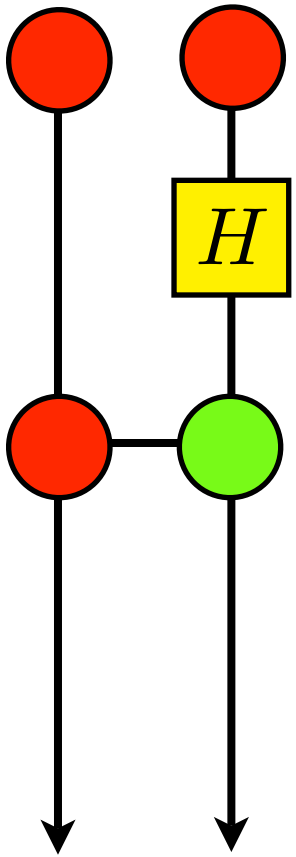

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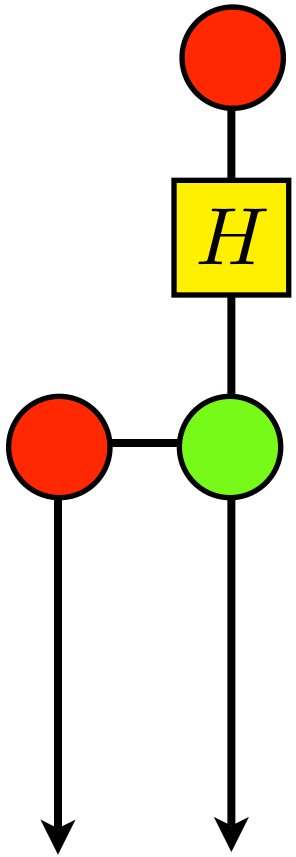

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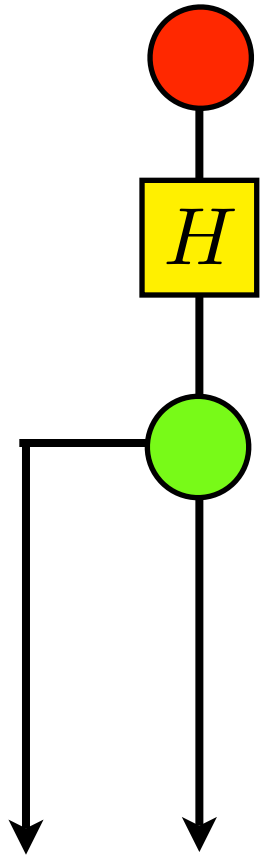

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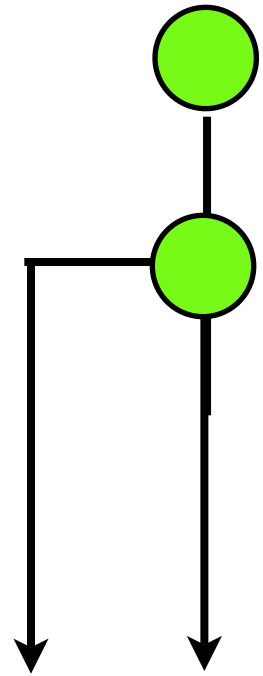

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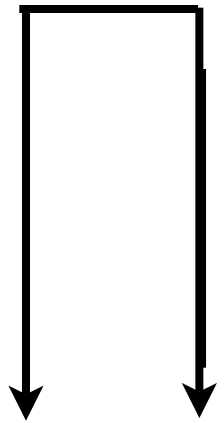
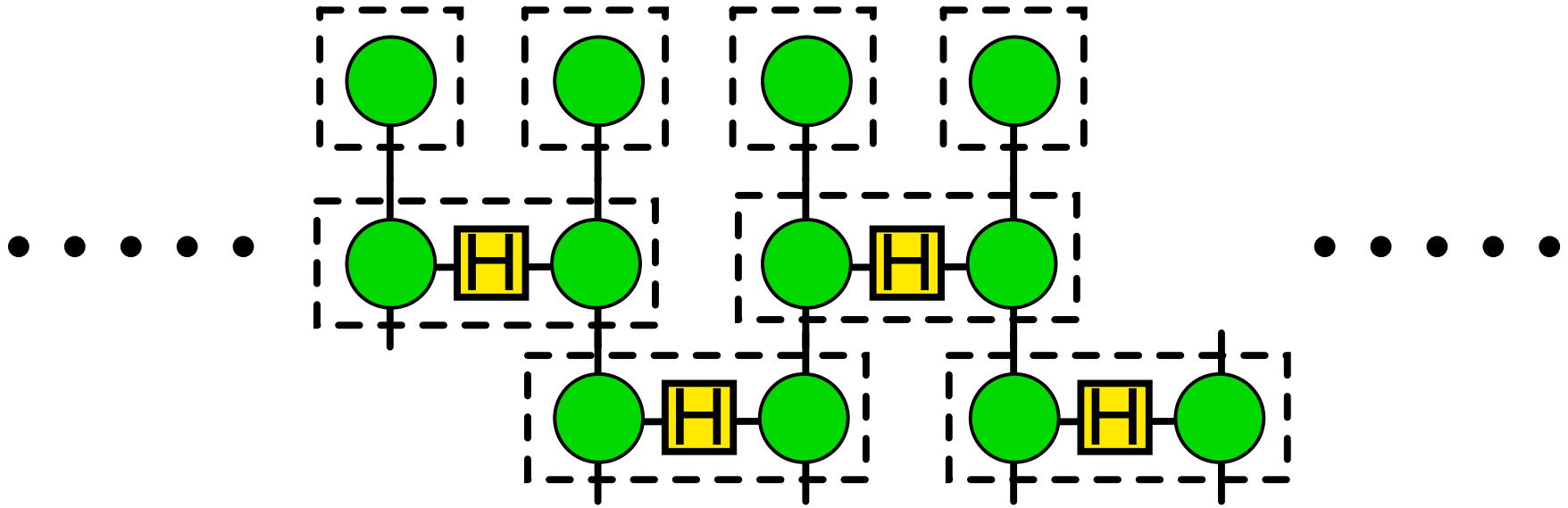


Example: 1D Cluster State



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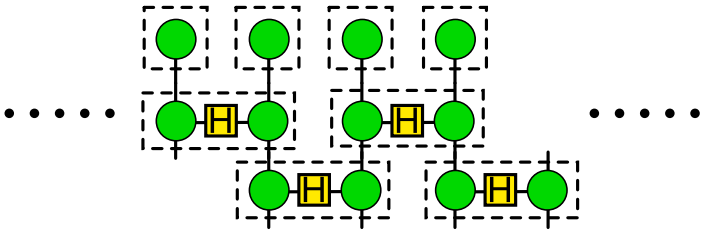

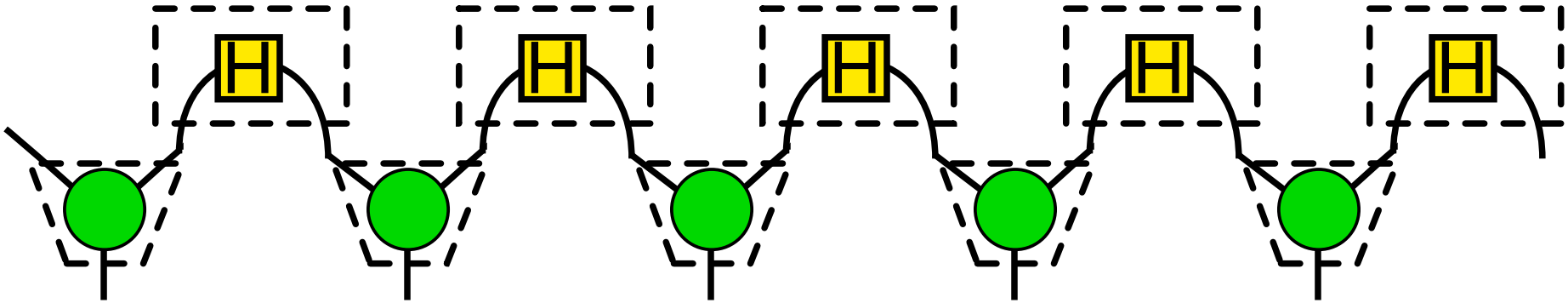

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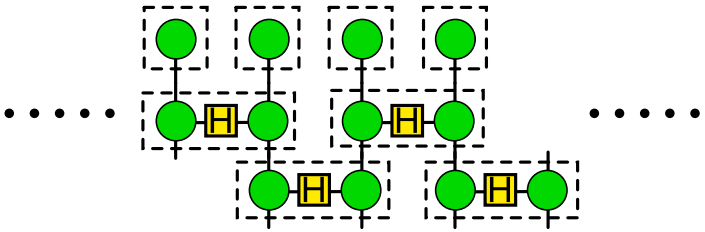
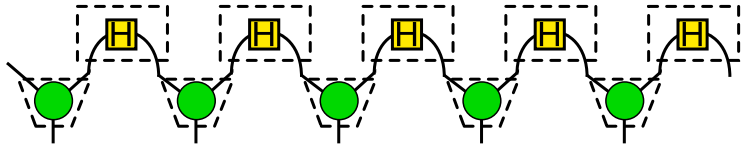
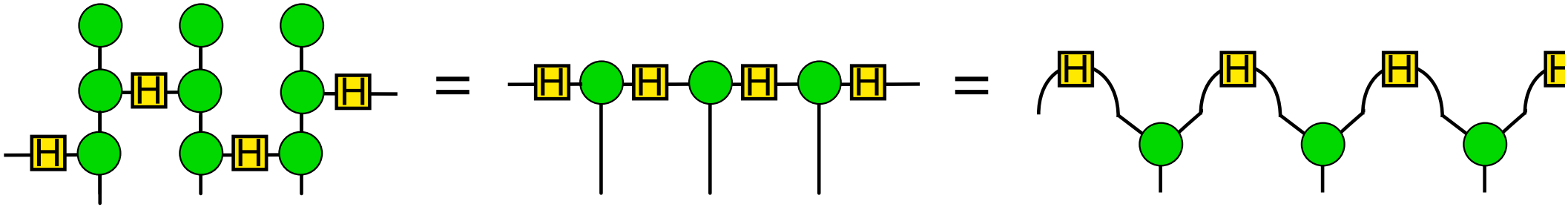
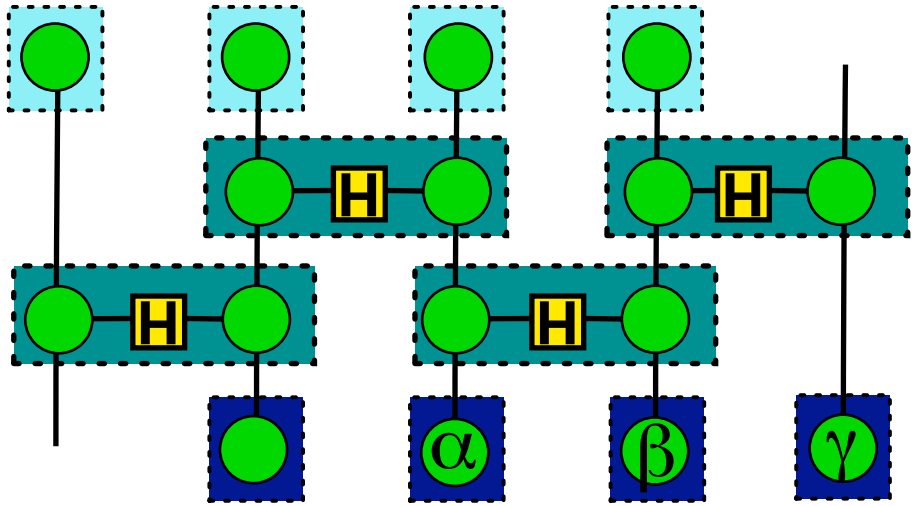






Example: MBQC 1-Qubit Unitary



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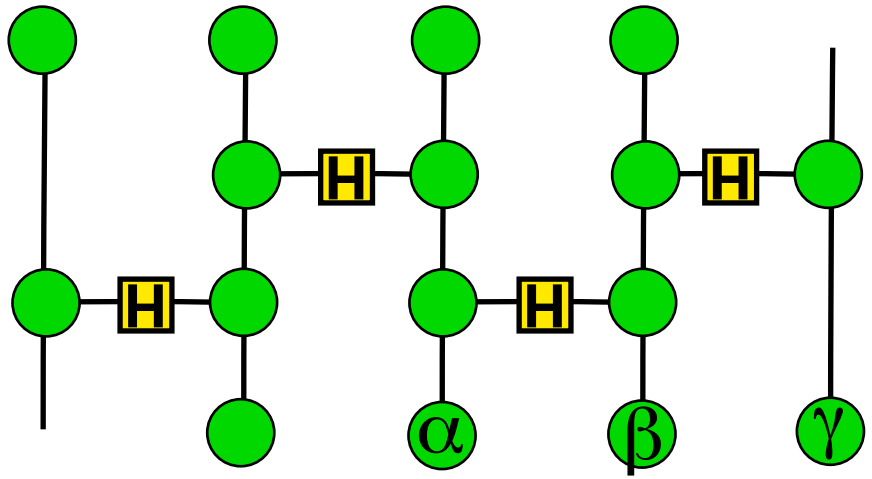

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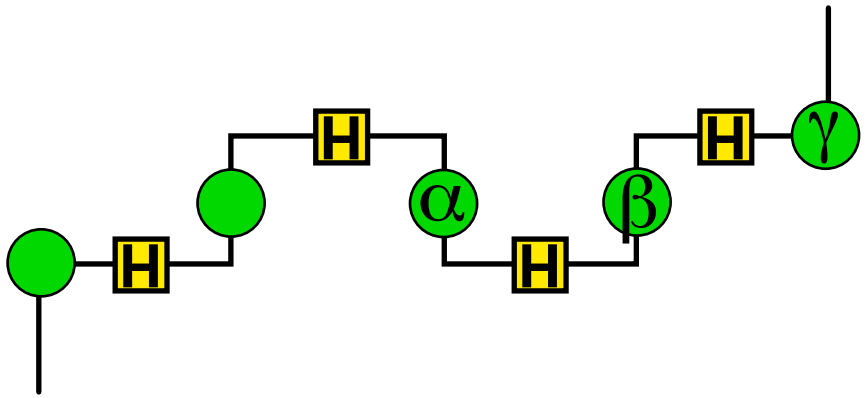

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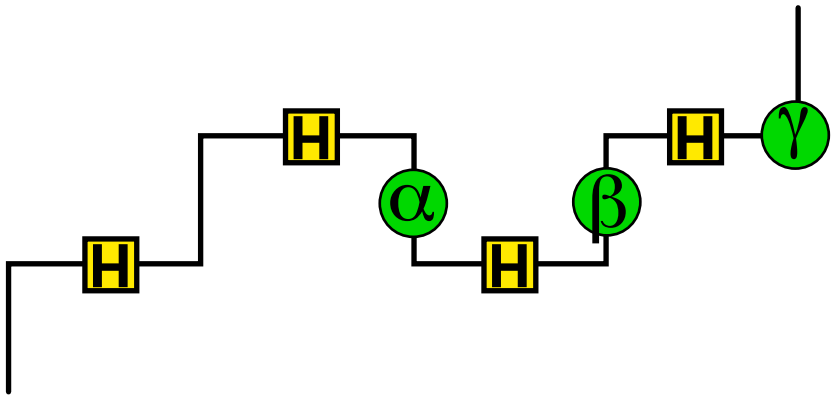

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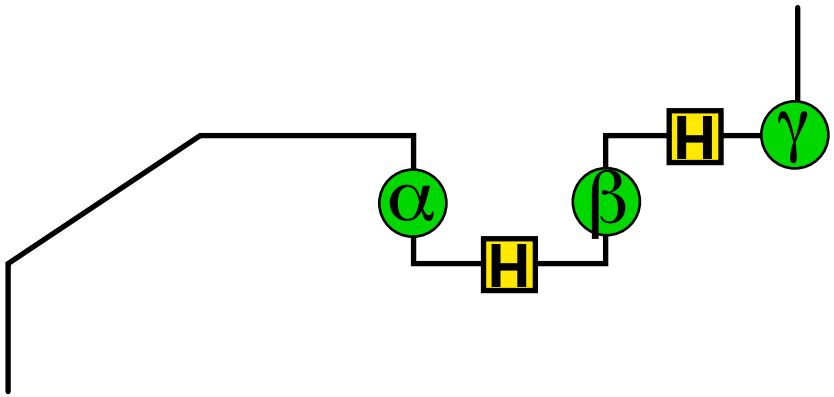

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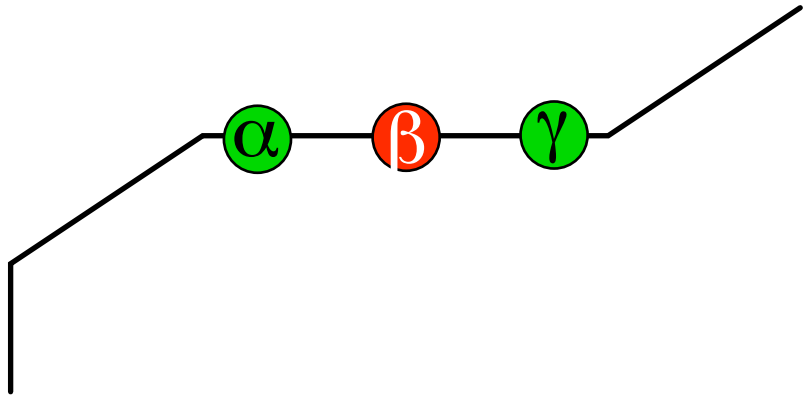
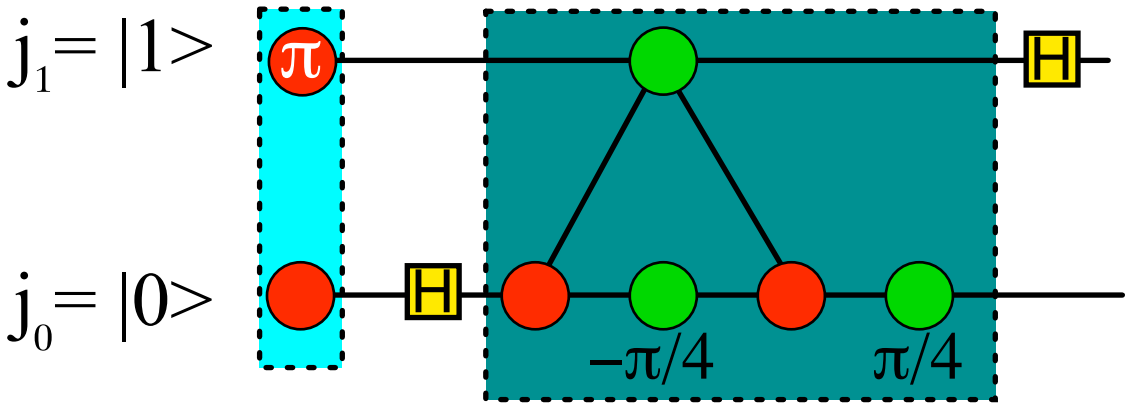


Example: 2 Qubit Quantum Fourier Transform



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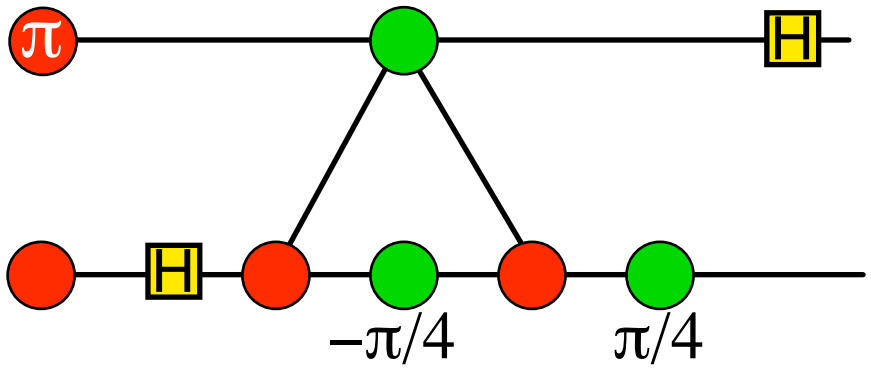

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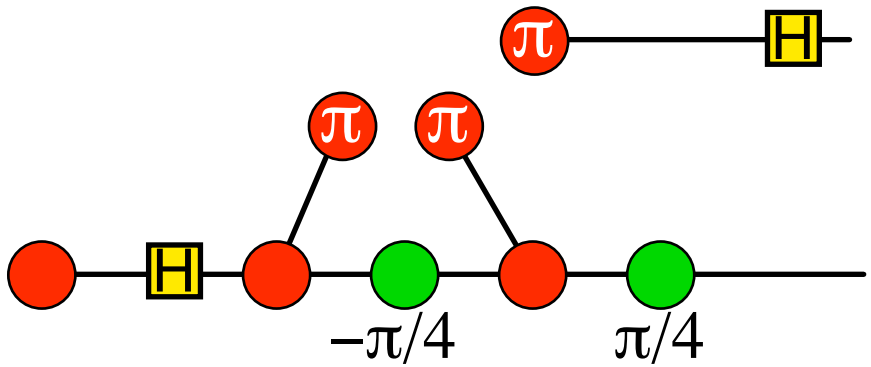

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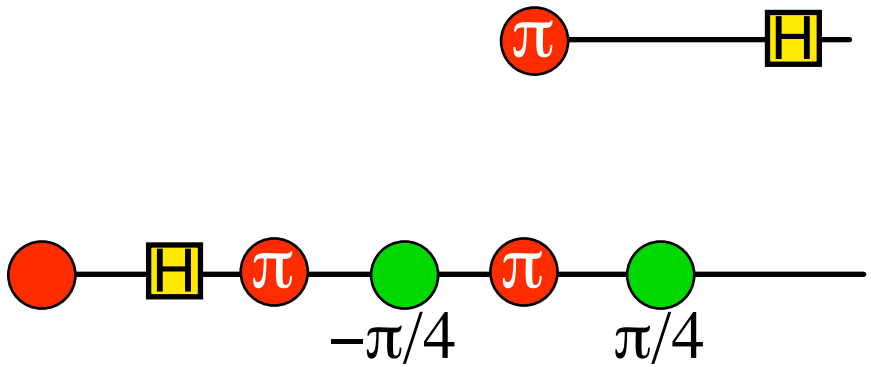

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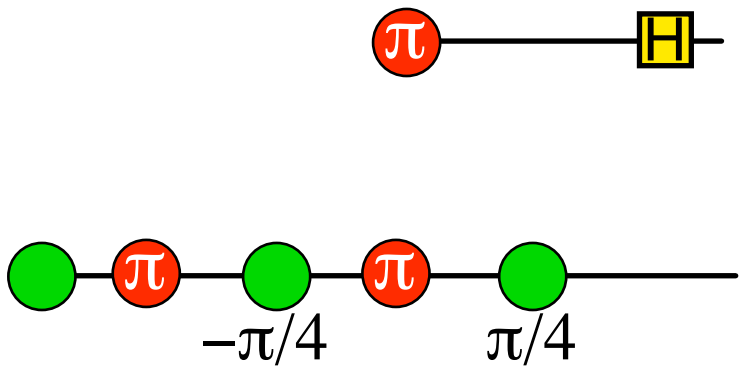

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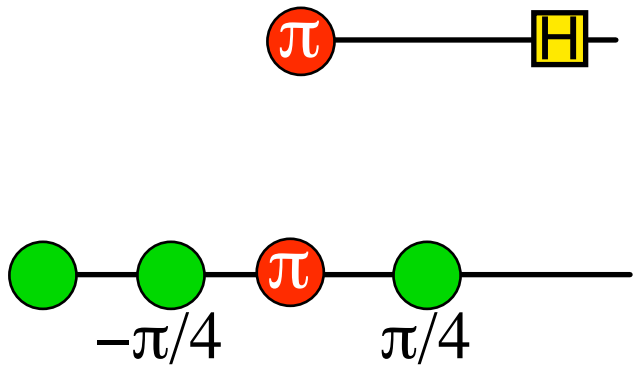

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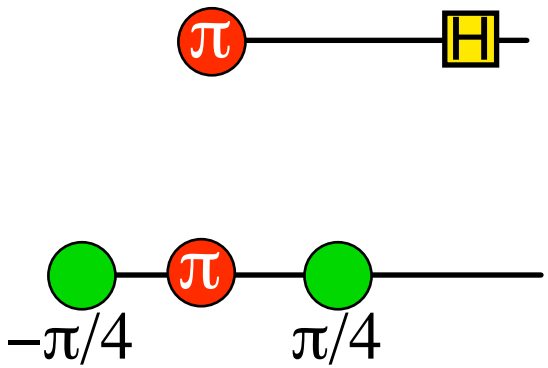

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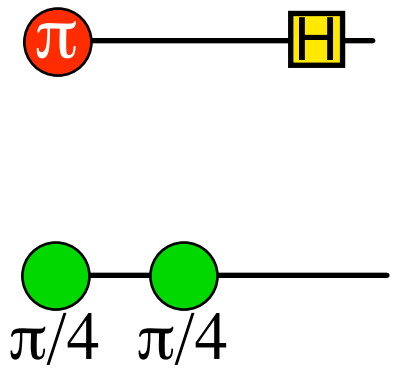

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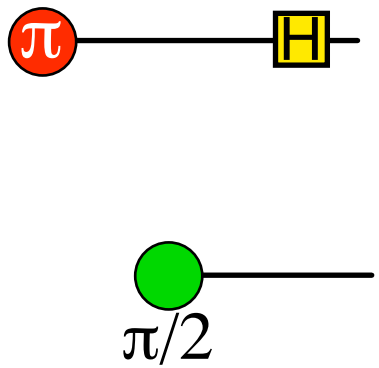

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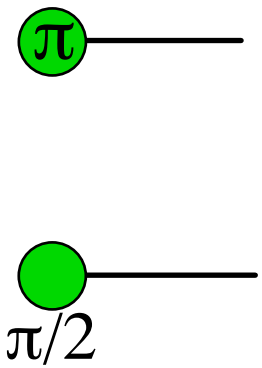


Conclusions

We formalised complementary quantum observables in the language of monoidal categories:

- each pair of complementary observables defines a Hopf algebra (up to scalar factors)
- each observable defines two groups: its classical points and its unbiased points
- Interference between pairs of complementary observables is characterised via a group of automorphisms
- These equations are strong enough to do a lot of quantum computing!

Ongoing work I

Current research directions:

- Investigate connections with multipartite entanglement (with Bill Edwards)
- Algorithmic properties of MBQC (with Simon Perdrix)
- Study toy models of QM based on the automorphism approach.
- Qubits: get rid of explicit Hadamard and study the concrete theory of general qudits.

Ongoing work 2

Collaborating with Lucas Dixon (Edinburgh) and Aleks Kissinger (Oxford) to automate this calculus using a graphical rewriting system / interactive theorem prover:

Rewriting properties e.g. normal forms?

Pattern languages: ellipses and indexed containers?

Ongoing work 2

Collaborating with Lucas Dixon (Edinburgh) and Aleks Kissinger (Oxford) to automate this calculus using a graphical rewriting system / interactive theorem prover:

Rewriting properties e.g. normal forms?

Pattern languages: ellipses and indexed containers?

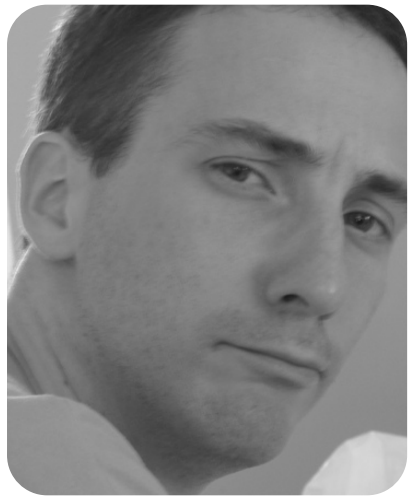
The papers are by these people

Bob Coecke

Samson Abramsky

Eric Paquette

Dusko Pavlovic



Peter Selinger

Simon Perdrix

... and Jamie Vicary

Use it: http://arxiv.org