Interacting Observables in Categorical Quantum Mechanics

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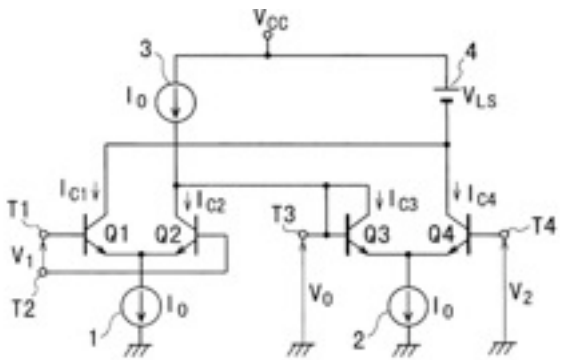
Université Libre de Bruxelles

Background and Motivations

"I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more."

--John von Neumann, letter to G. Birkhoff, 1935

Motivations



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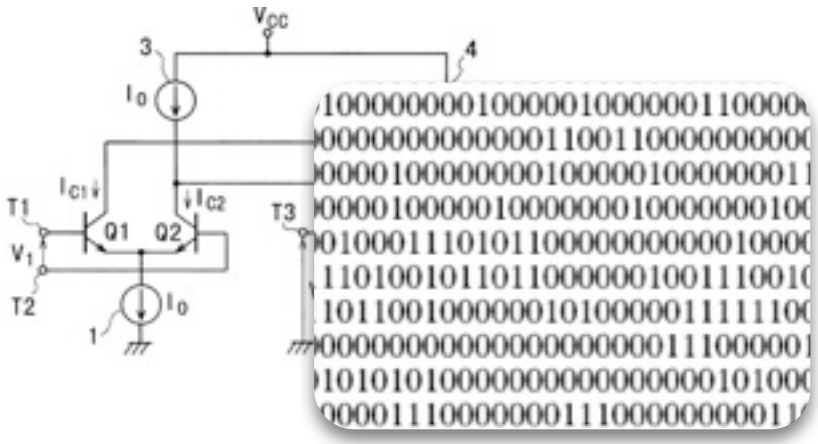

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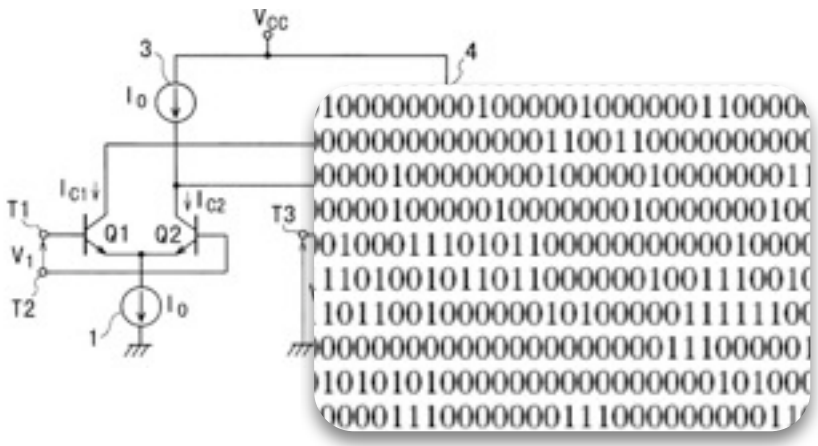
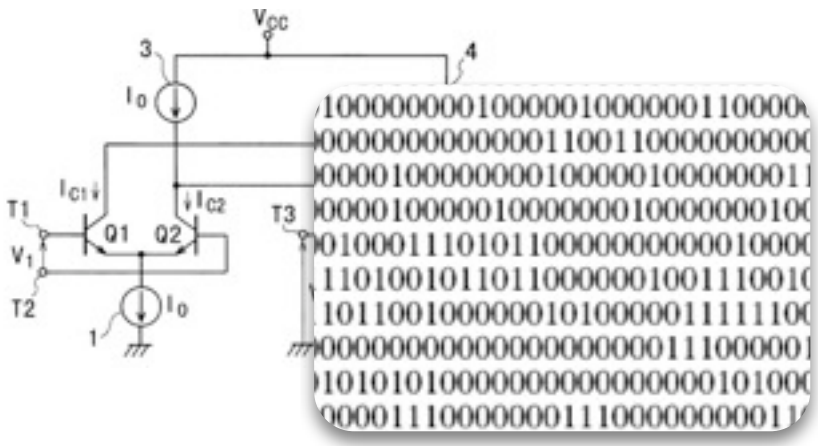



```
fun fact 0 = 1
  | fact x = x * fact (x-1)
```

Motivations



$$\lambda x.\lambda y.\lambda z.xz(yz)$$

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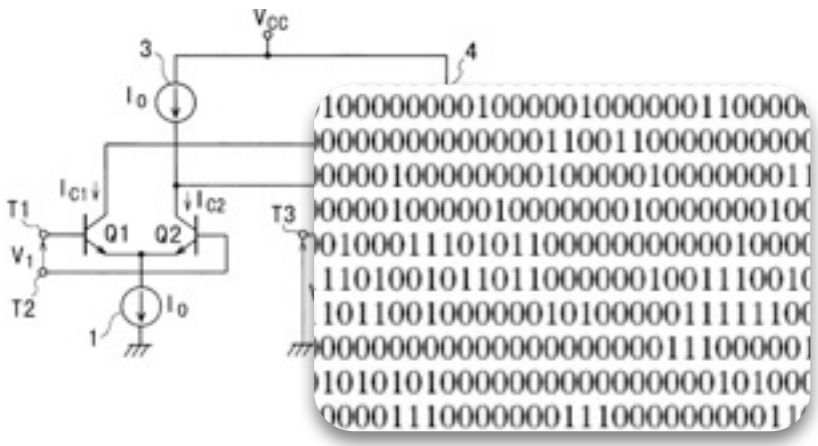

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Hilbert space,
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self-adjoint
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?

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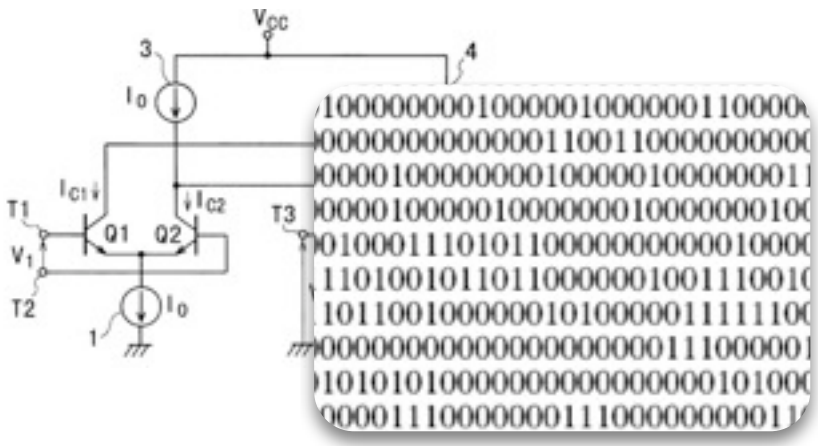

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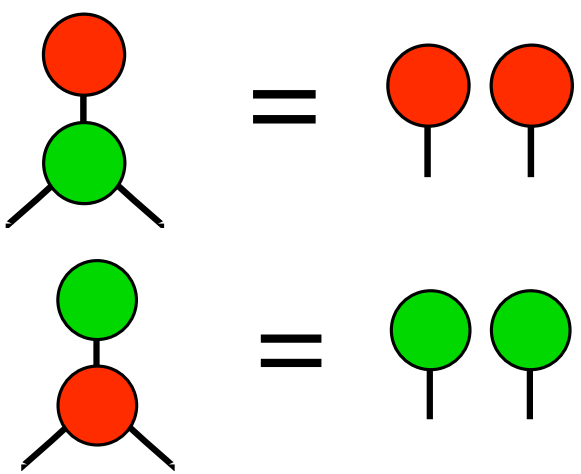
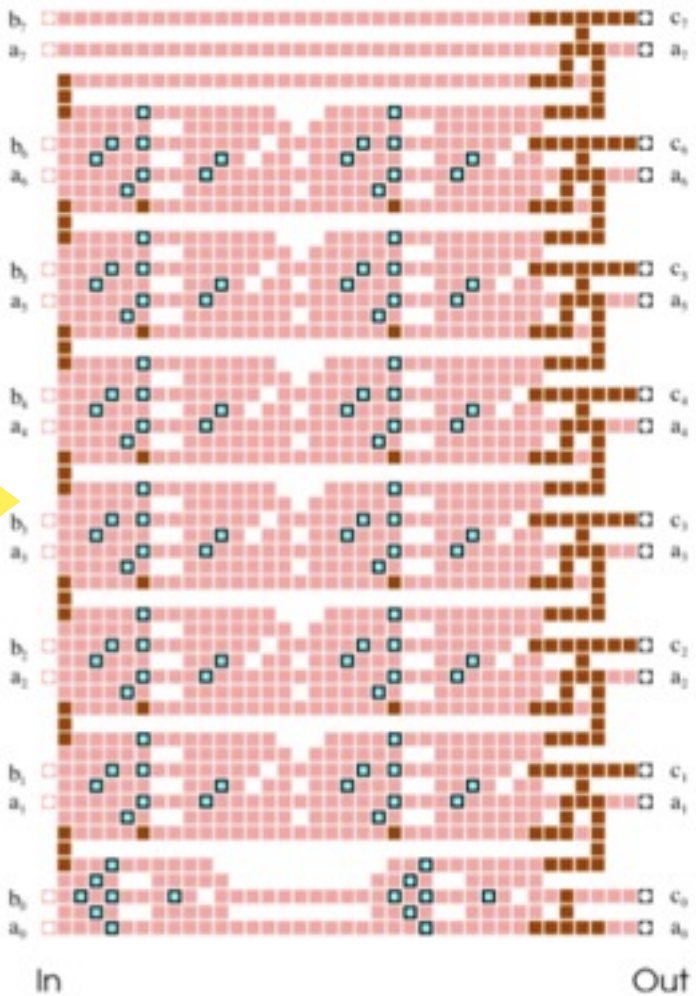


The need for abstraction:

This is an
8-bit adder



$D \sim 2^{1764}$

Motivation & Context

New **structural** axiomatisation of QM:

- Hilbert space formalism not a perfect fit for QM
- Can also study alternative quantum-like theories
- emphasis on **composition**

We use **Monoidal Categories**:

- Very general framework, including Hilbert spaces
- Well studied in literature
- **Beautiful diagrammatic presentation**

Motivations

1. Better Theory
   - Discover the mathematical heart of quantum theory
   - Are there other models which have some of these properties?
   - Clarify the relationship between quantum and classical computation
2. Better Tools
   - Reason more easily about quantum processes
     - formal tools!
   - Synthesise quantum programs more easily
   - Type systems for quantum programs?

Overview

Hilbert Space QM

Categorical QM

Overview

Hilbert Space QM

Hilbert space

Categorical QM

Overview

Hilbert Space QM

Hilbert space

Categorical QM

†-symmetric
monoidal category

Overview

Hilbert Space QM

Hilbert space

Observables

Categorical QM

†-symmetric
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Overview

Hilbert Space QM	Categorical QM
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Observables	special commutative †-Frobenius algebras

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Qubits		

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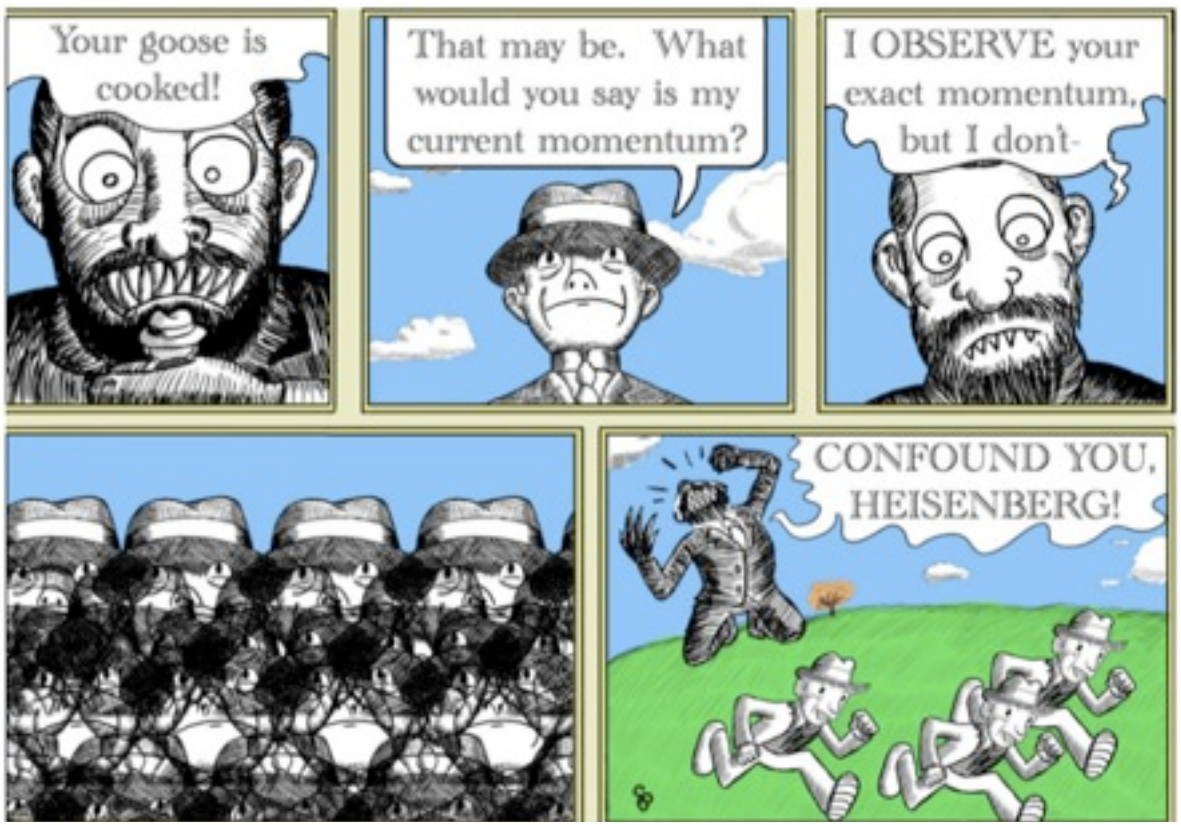
Hilbert Space QM	Categorical QM
Hilbert space	†-symmetric monoidal category
Observables	special commutative †-Frobenius algebras
Unbiasedness & relative phase	Phase group
Complementarity	Bialgebras & Hopf algebras
Qubits	ZX-calculus

DISCLAIMER

There are going to be a **LOT** of definitions in this talk.

sorry.

Quantum Observables



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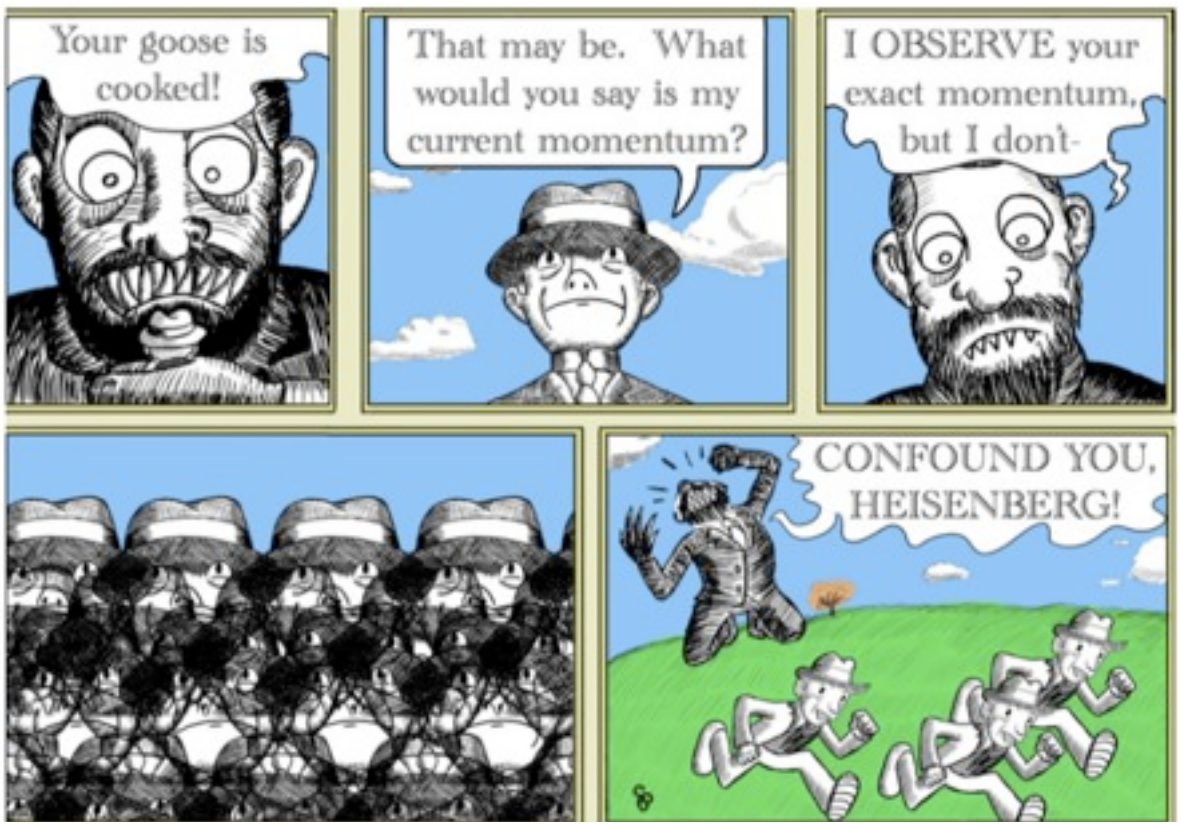


http://dresdencodak.com

Quantum Observables

Each observable quantity O is represented as self-adjoint operator $\hat{O} = \sum_i \lambda_i \left| e_i \right\rangle \left\langle e_i \right|$:

- The possible values of O are the eigenvalues λ_i of $\hat{O}$
- When we observe O for a system in state $|\psi\rangle$, there is probability $\langle e_i \mid \psi \rangle^2$ of observing λ_i .
- If λ_i is the outcome of the measurement, the system is then in state $|e_i\rangle$.

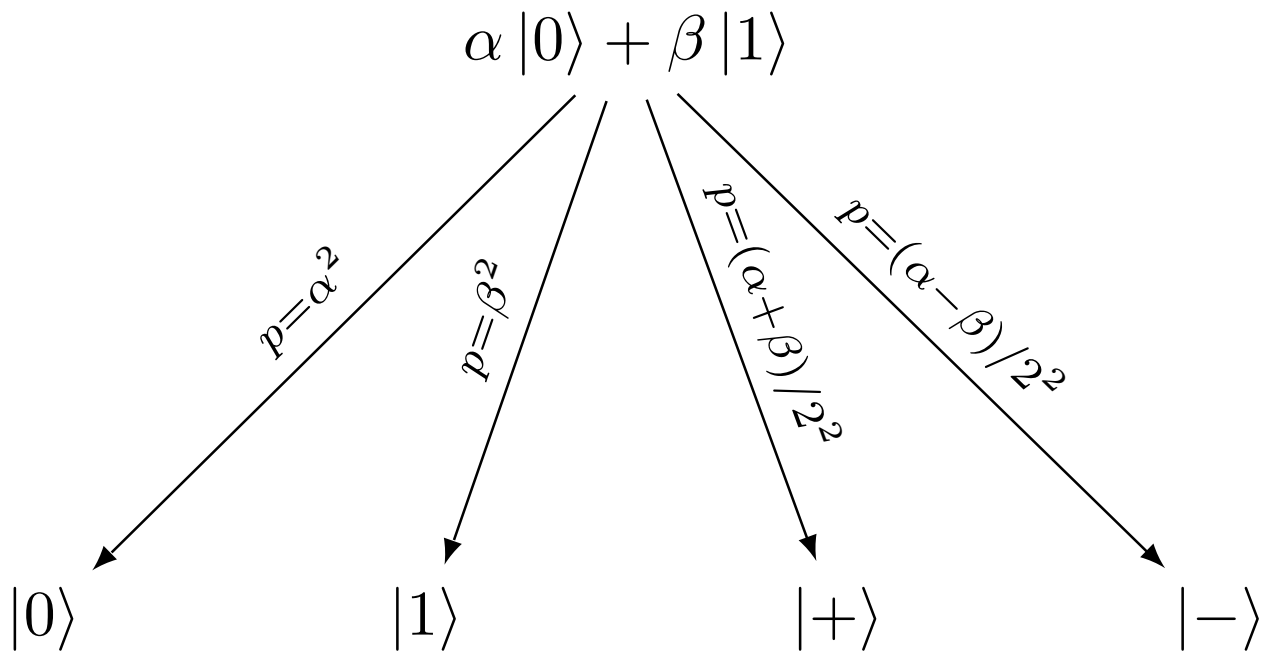
In general, measuring O_1 then O_2 will give a different answer than measuring O_2 first!

Not every quantum observable is well defined at the same time.

X and Z Spins

We can measure the spin of qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



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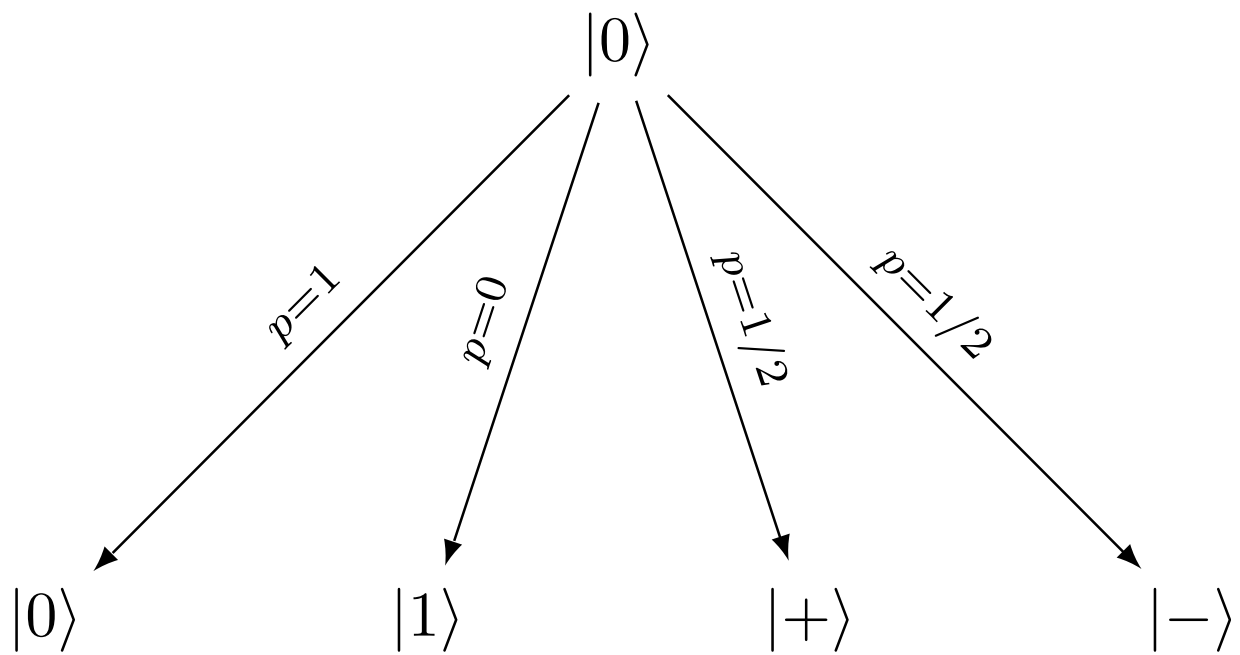

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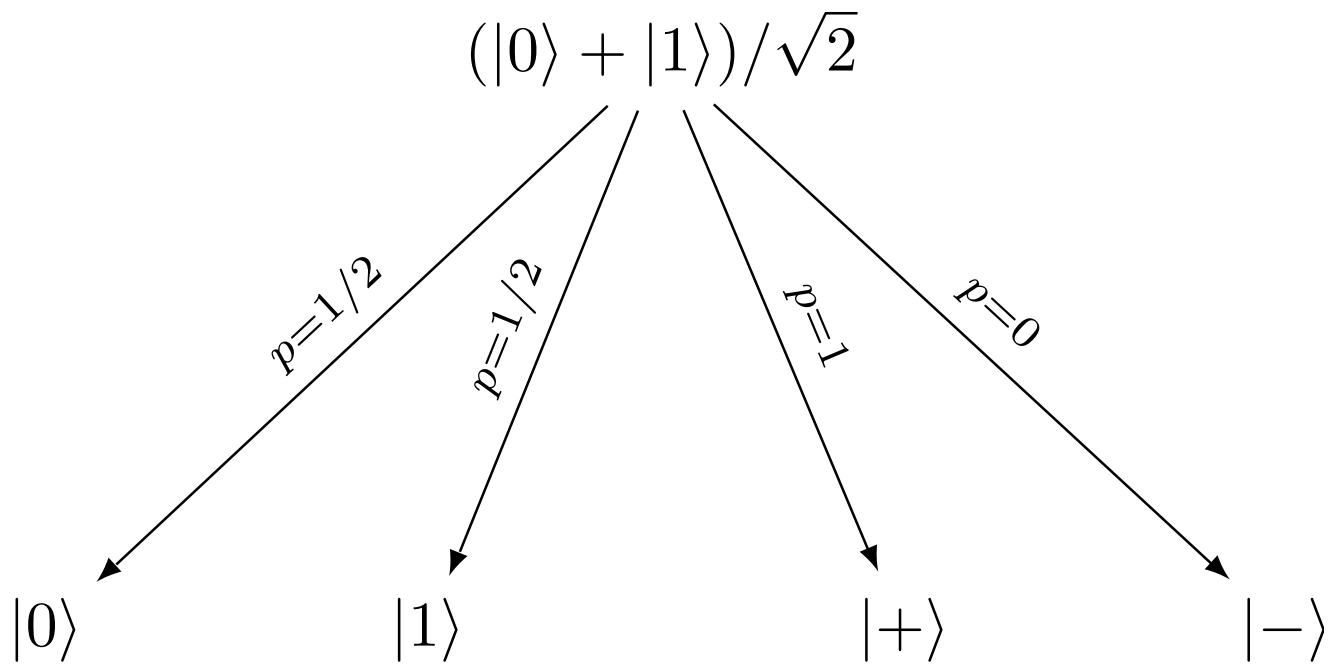
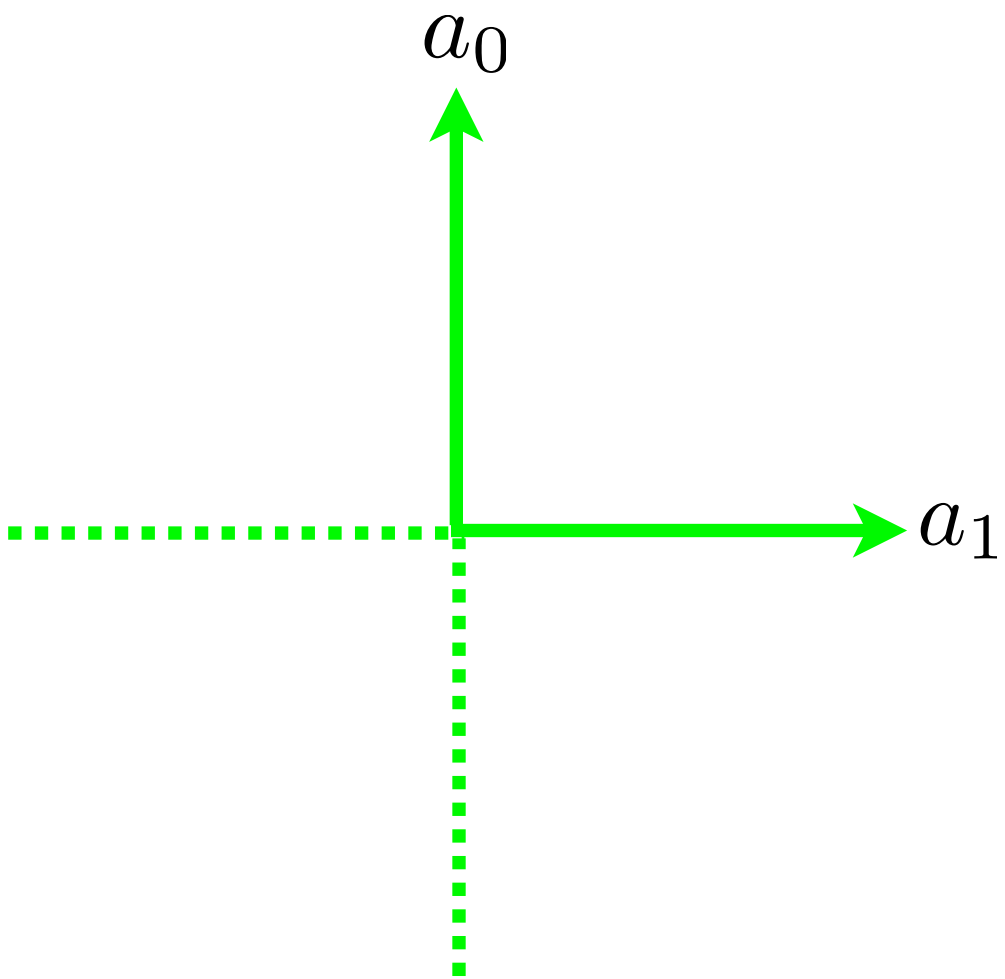


Complementary Observables



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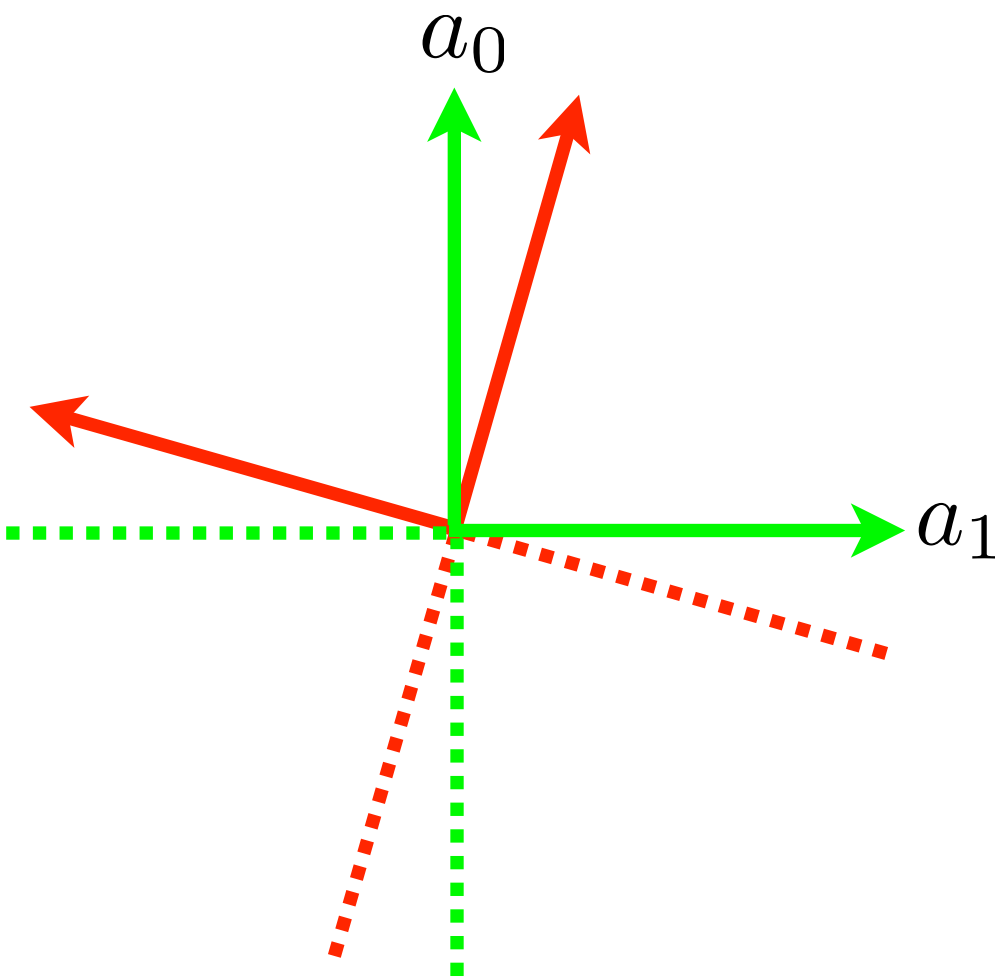

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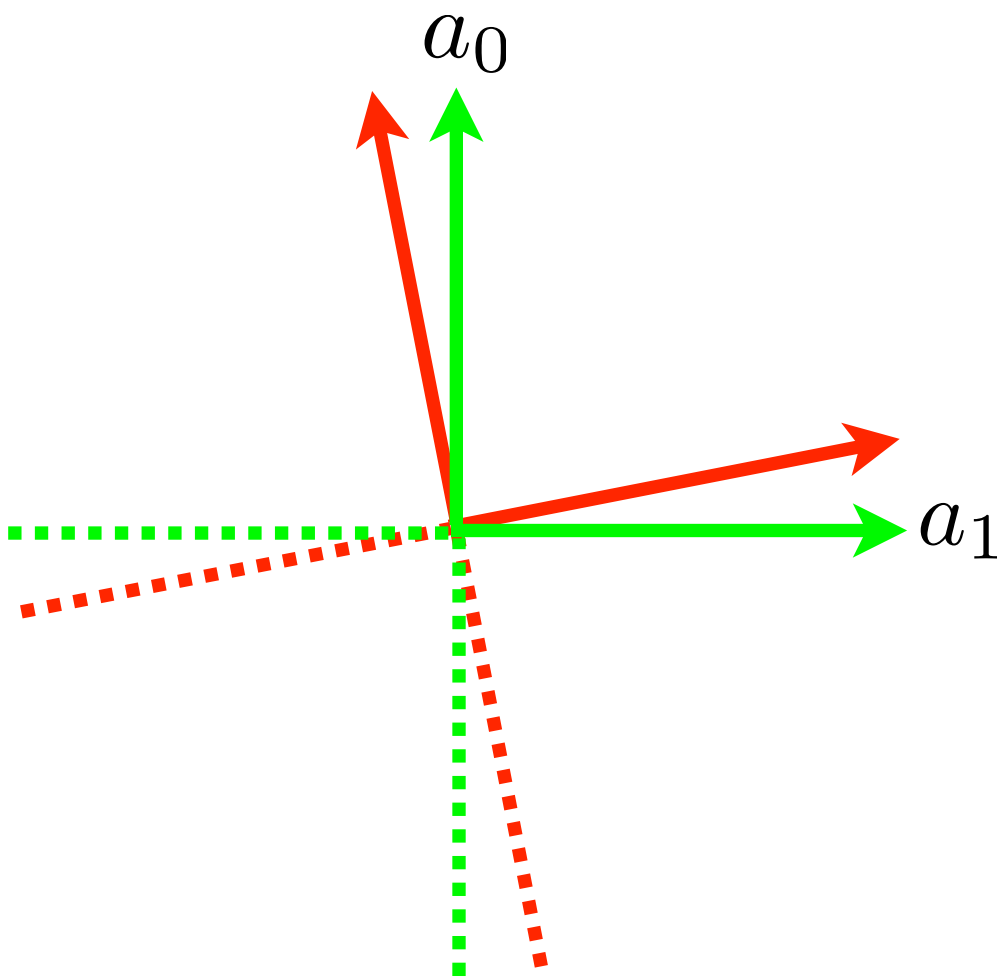

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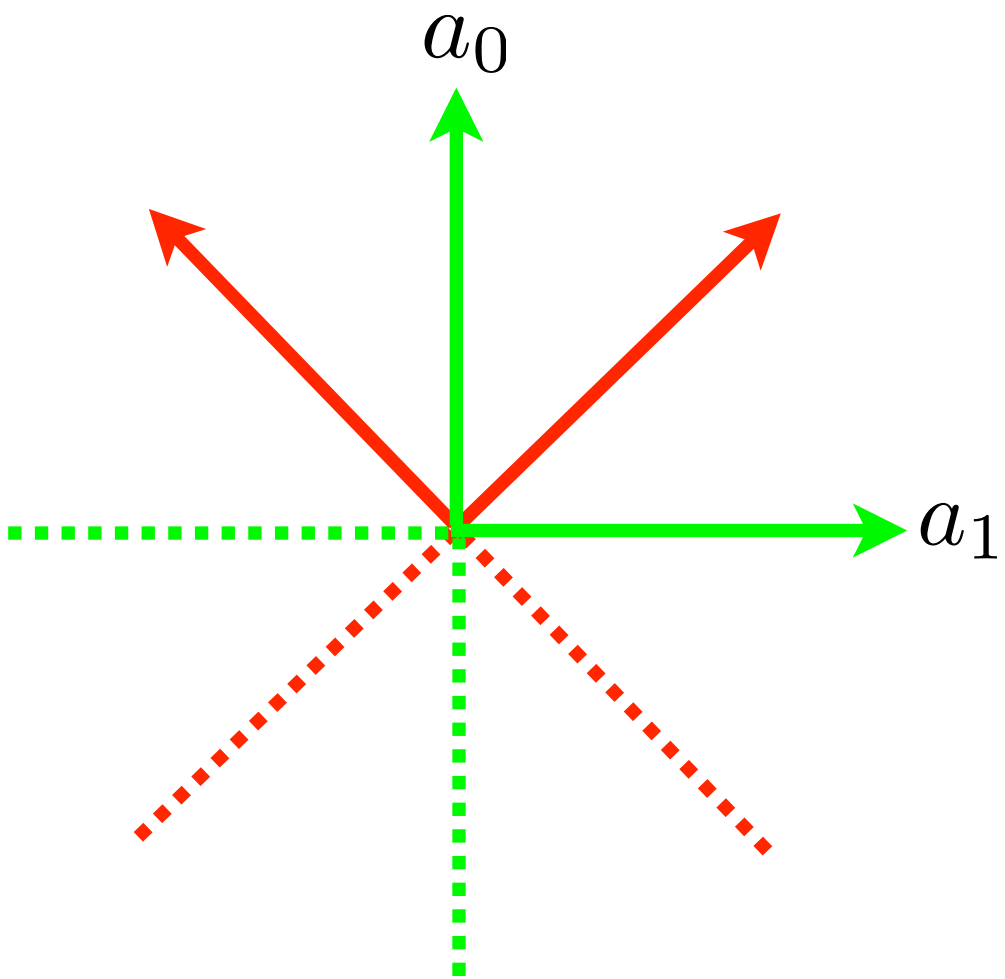

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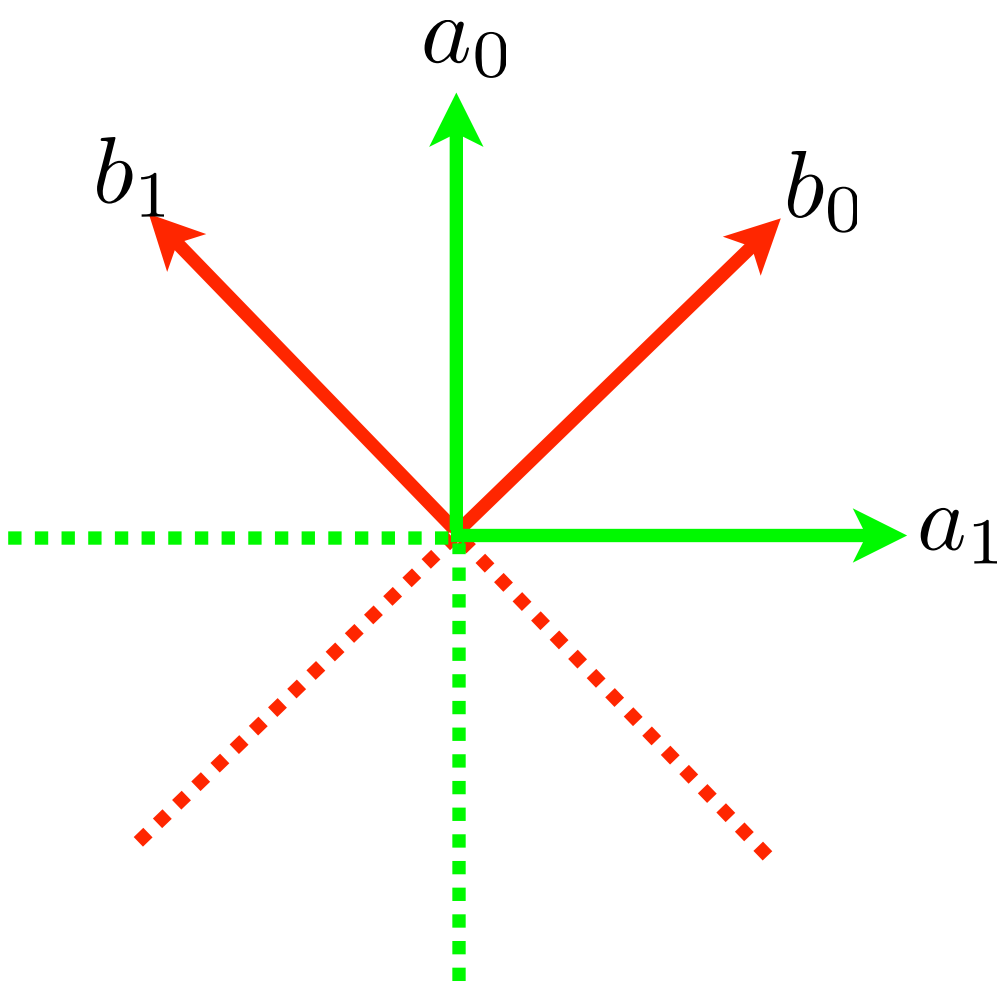

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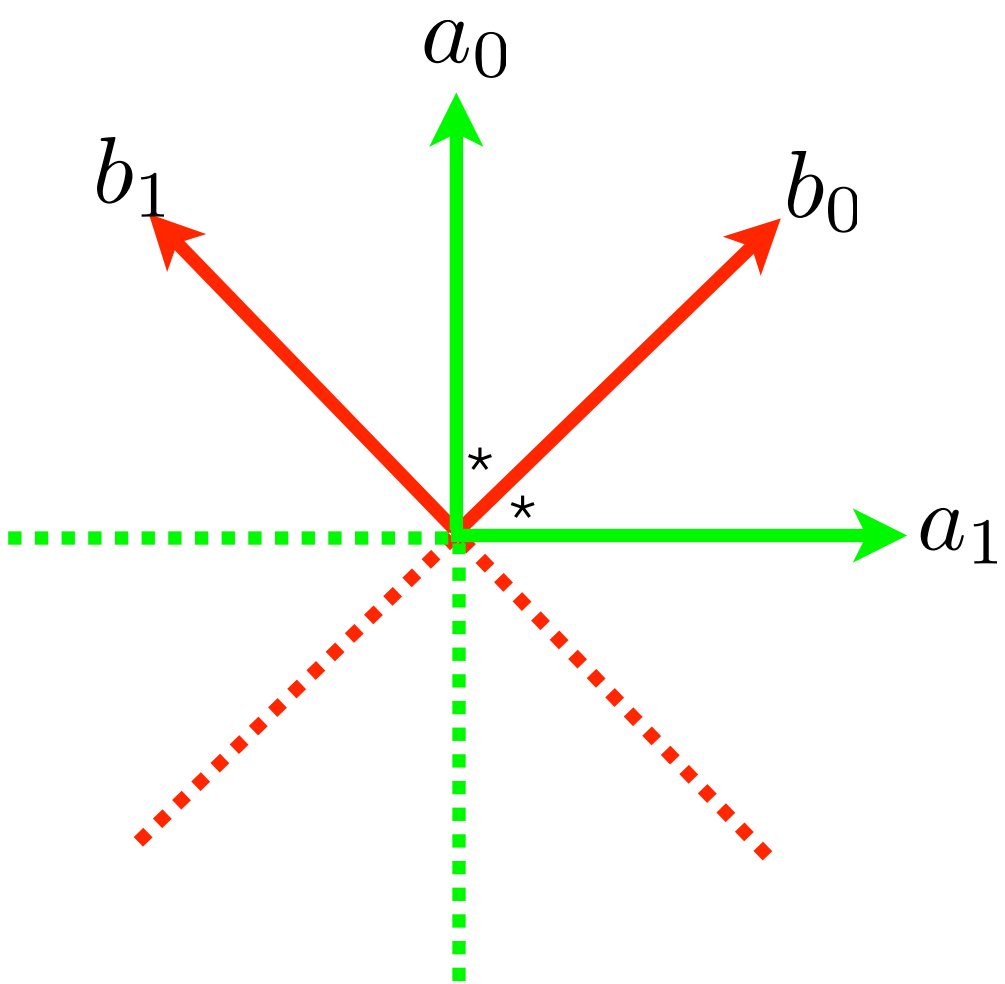

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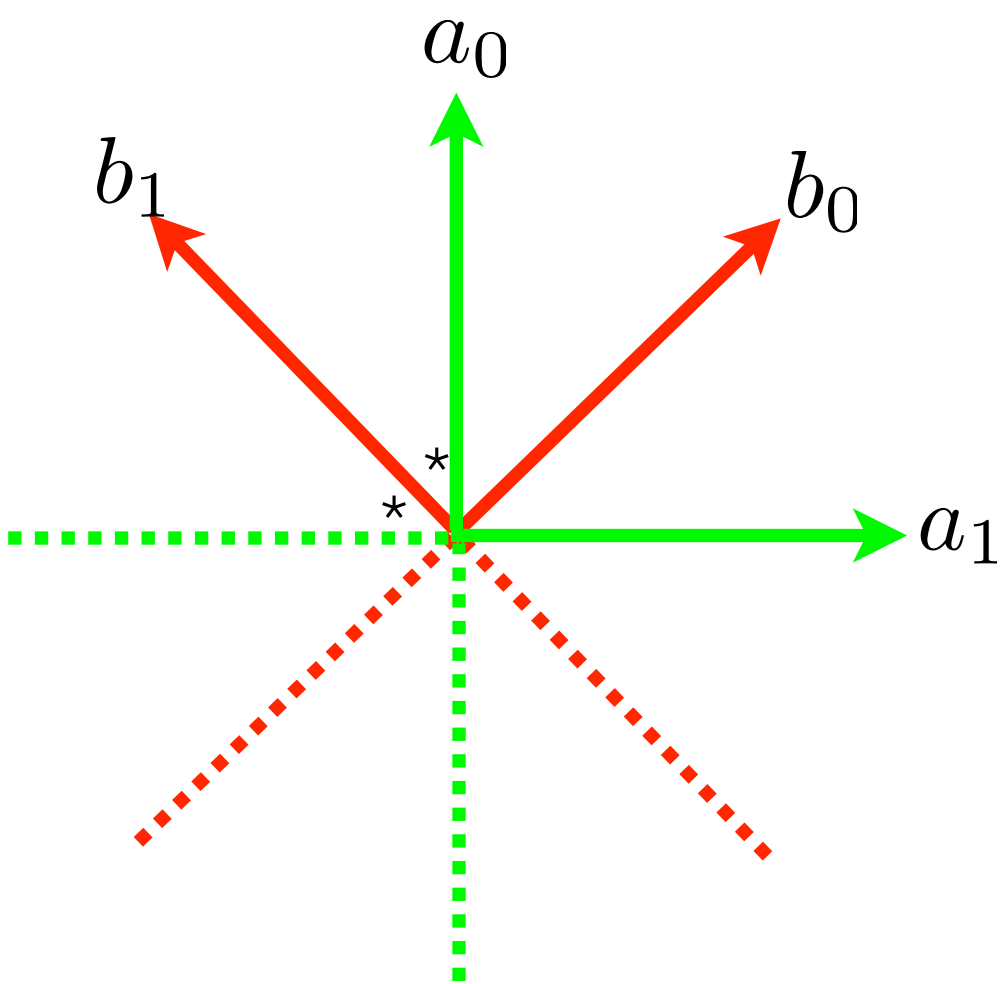

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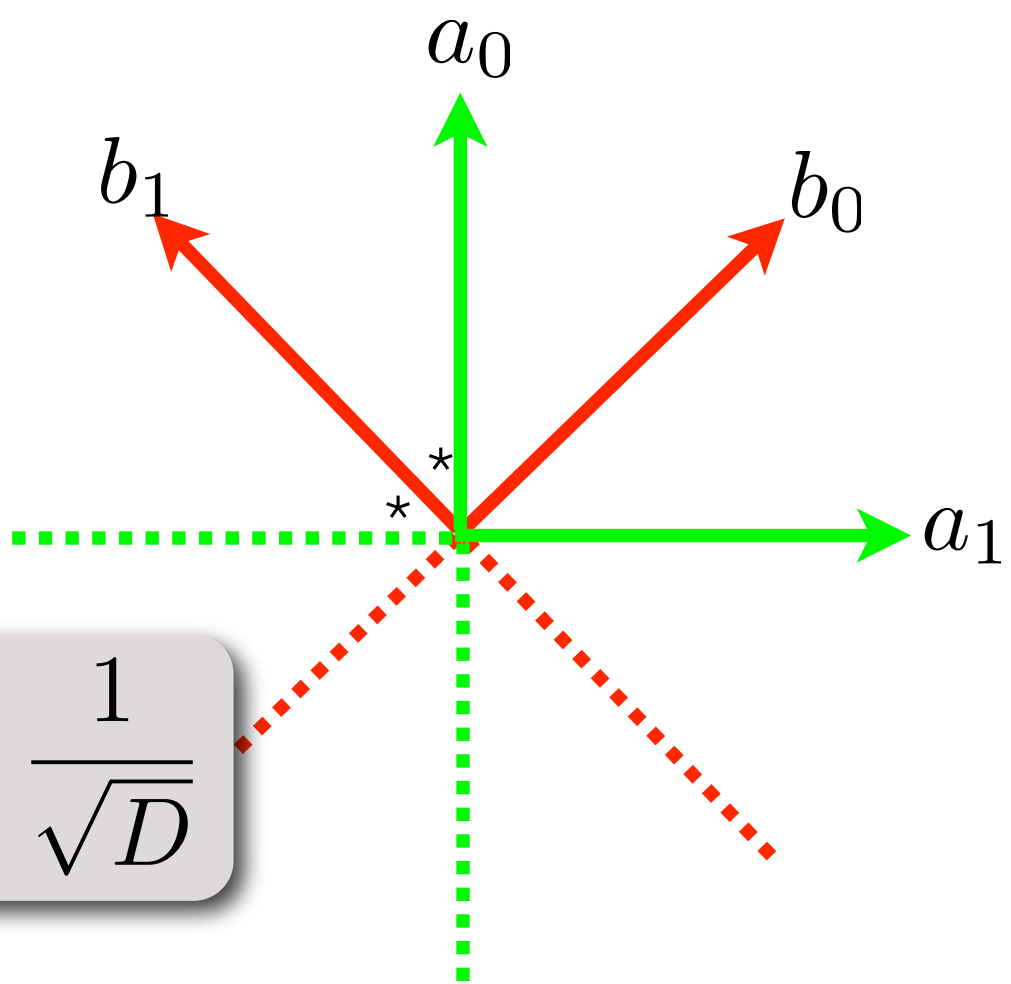


$$|\langle a_i \mid b_j \rangle| = \frac{1}{\sqrt{D}}$$

Monoidal Categories & Diagrams

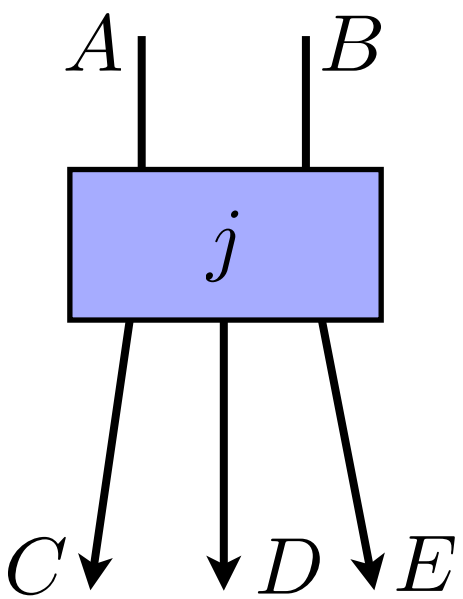
PAST / HEAVEN

FUTURE / HELL

Pessimistic Diagram Convention

Diagrams

a.k.a. Monoidal categories

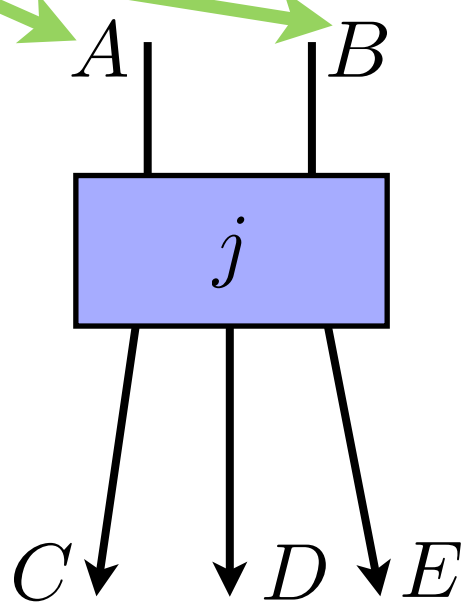


$$j : A \otimes B \to C \otimes D \otimes E$$

Diagrams

a.k.a. Monoidal categories

Input Systems



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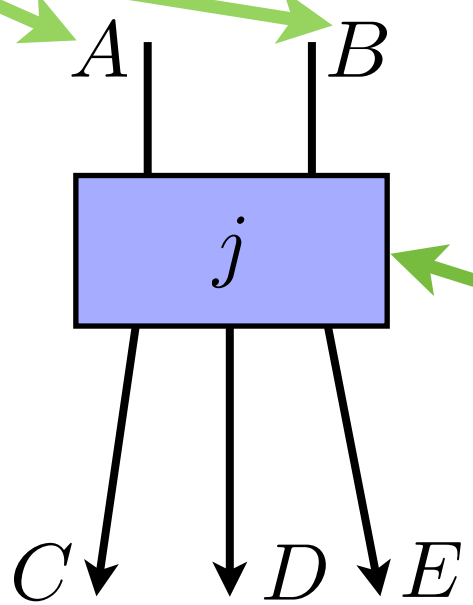

Interaction, or process, or state change, or ...

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Diagrams

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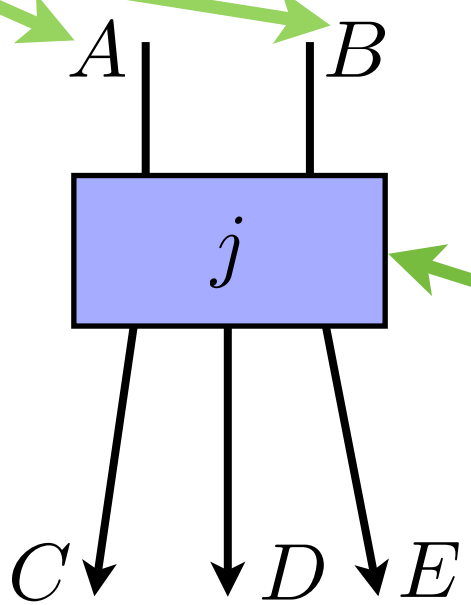

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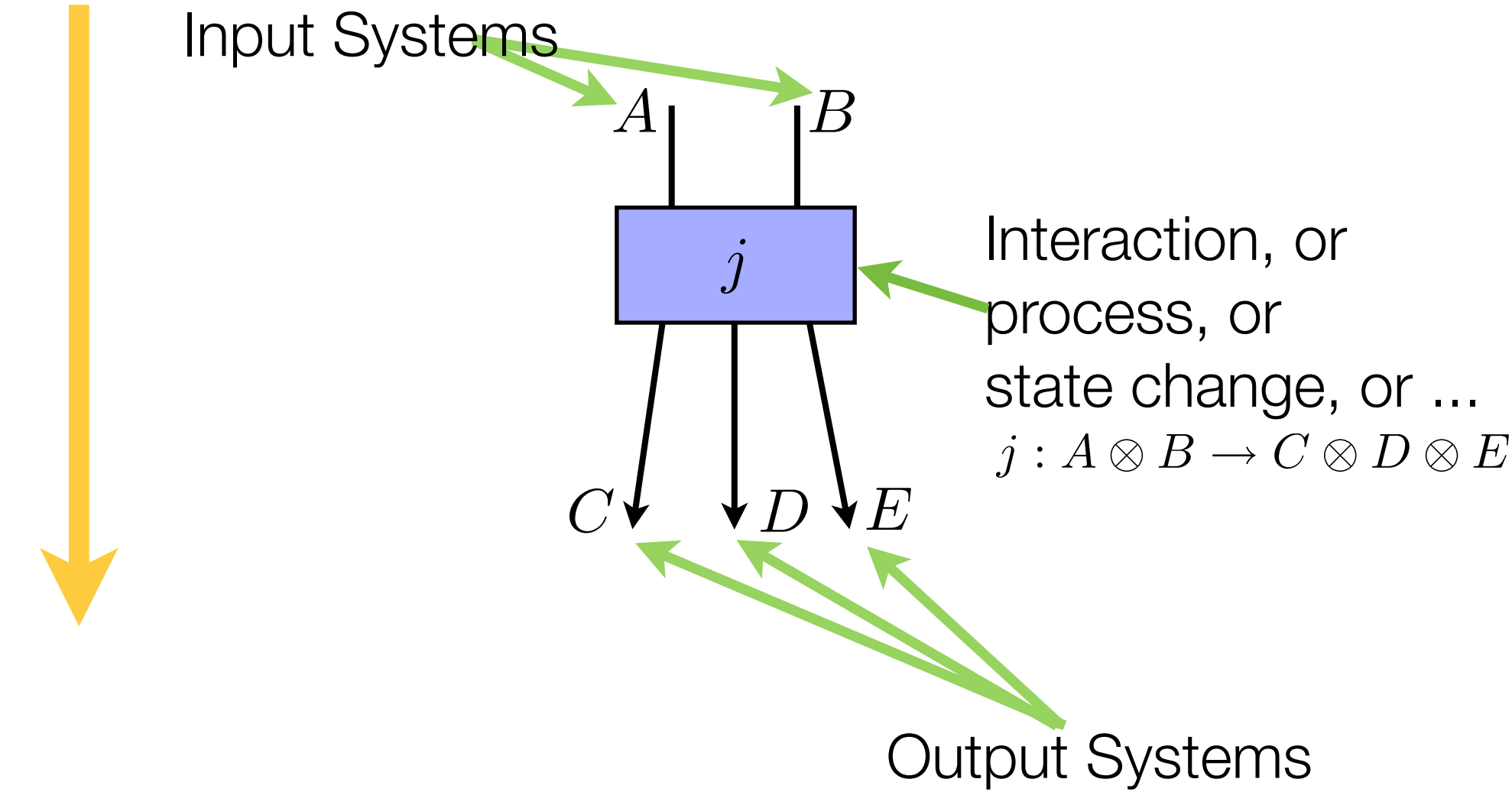

A hierarchy of categories

Categories

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Monoidal Categories

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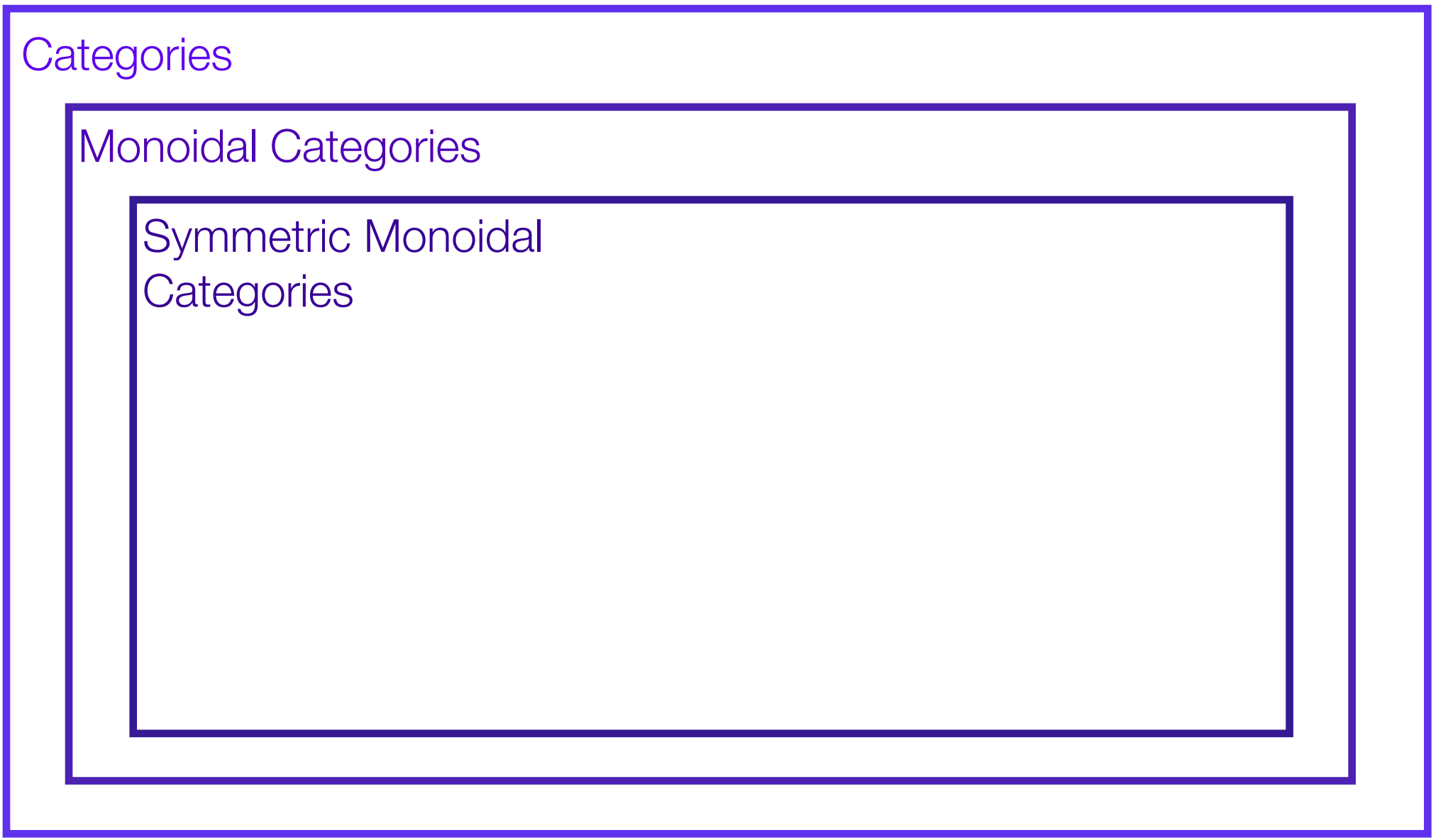

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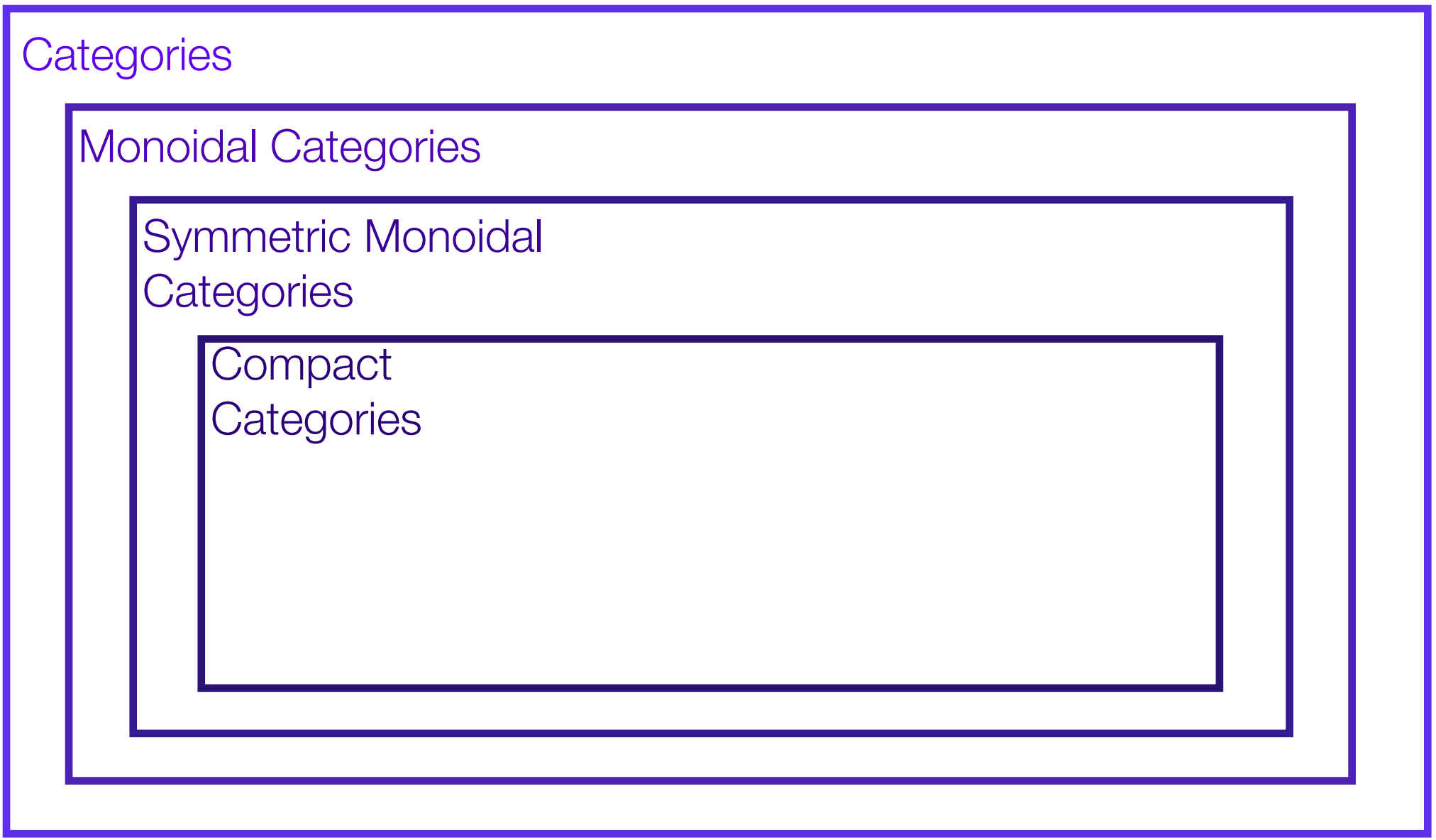

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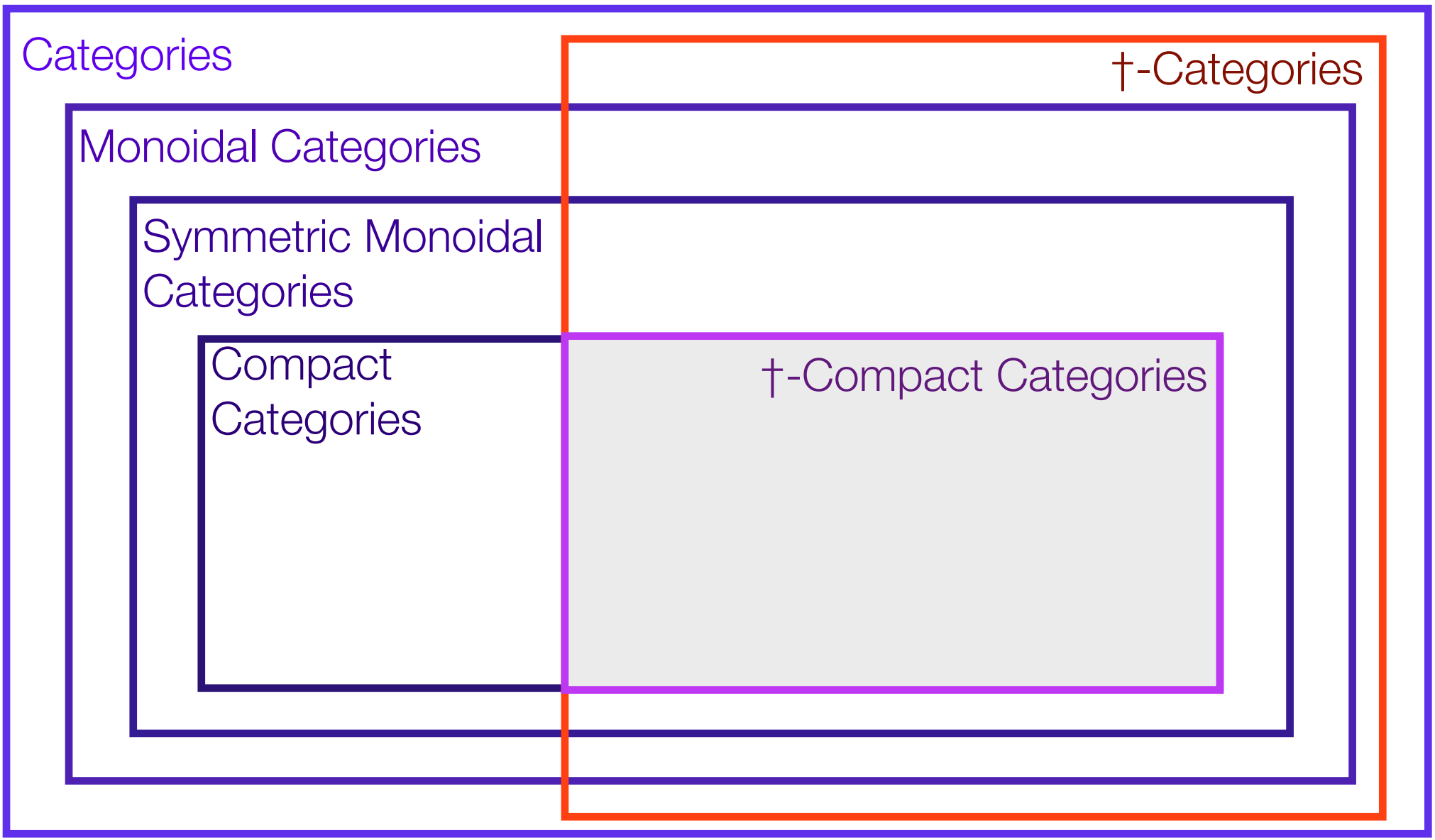

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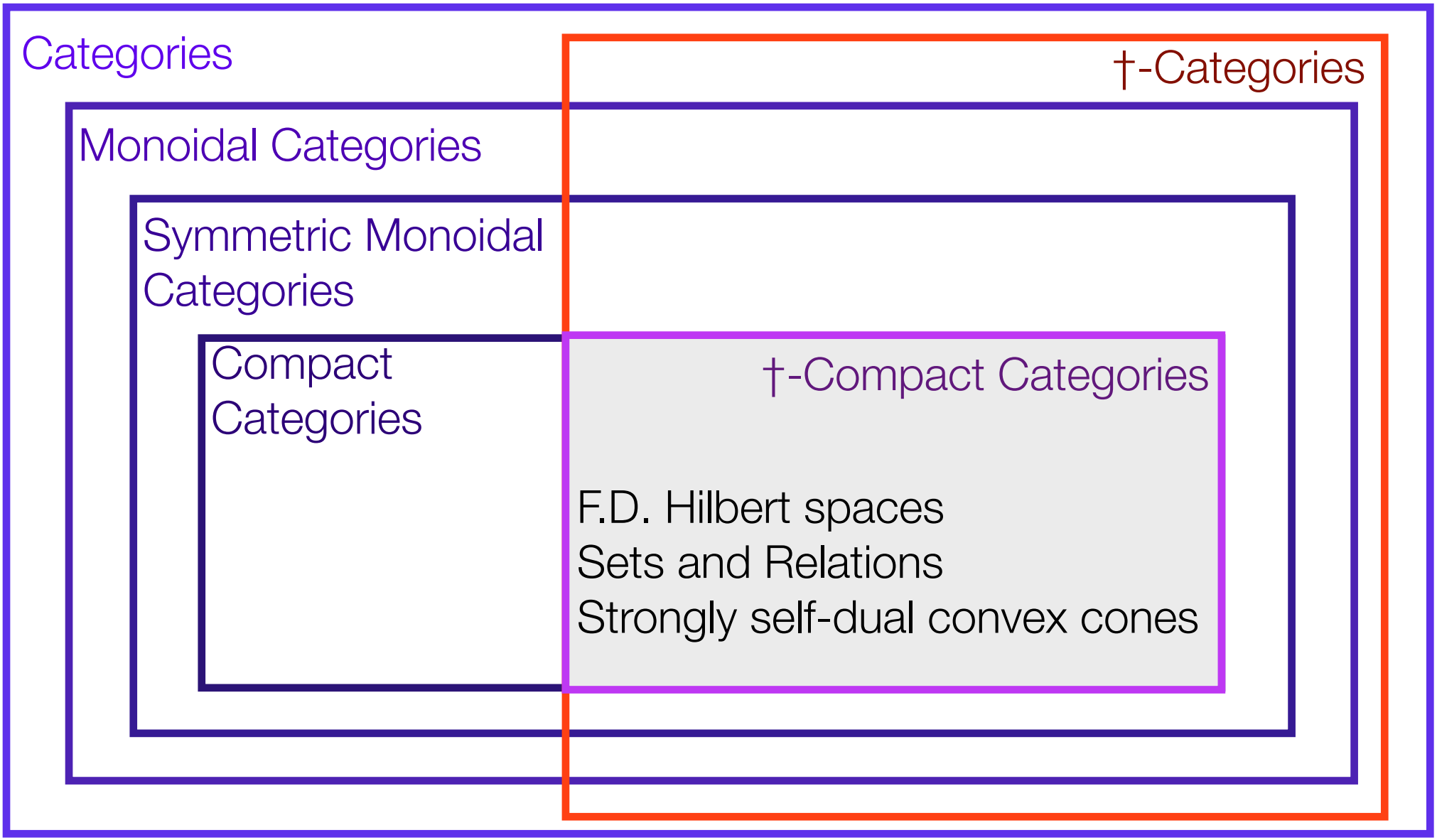
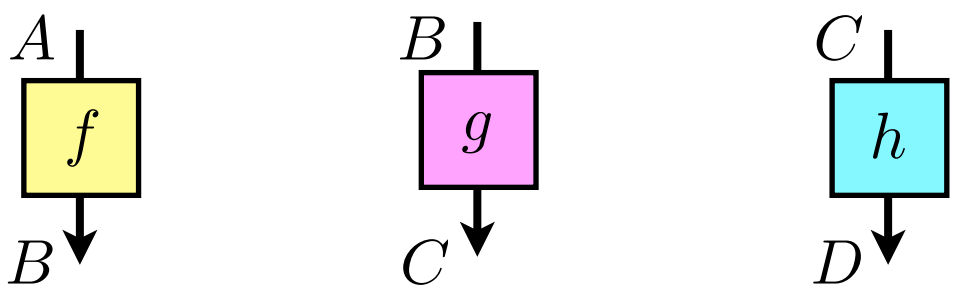


Categories

A *category* consists of *objects* A, B, C, etc, and *arrows* between them:

$$f : A \to B \qquad g : B \to C \qquad h : C \to D$$



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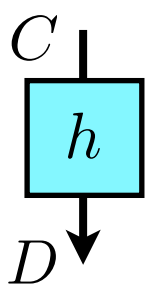
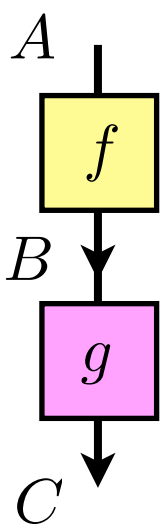


$g \circ f : A \to C$



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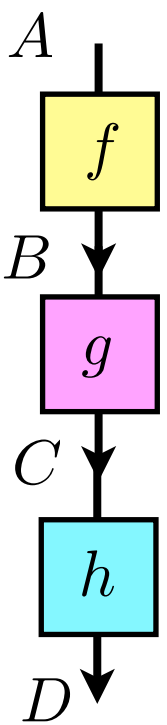

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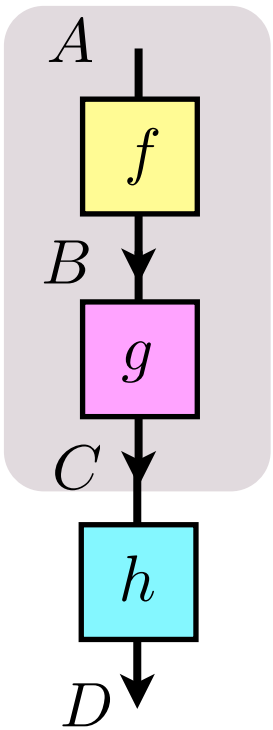

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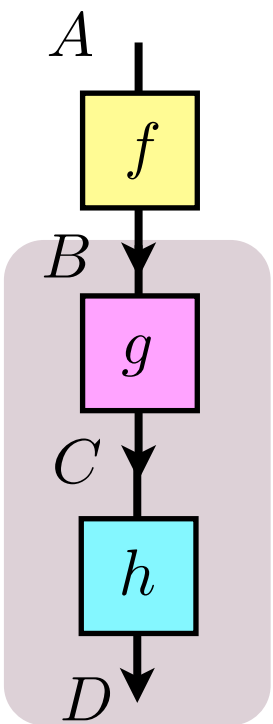

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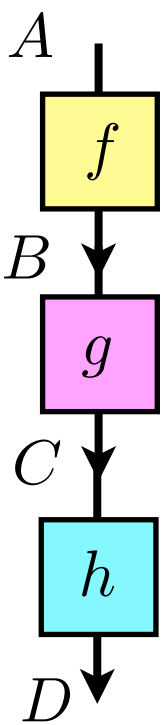
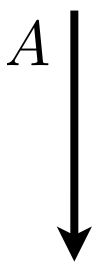


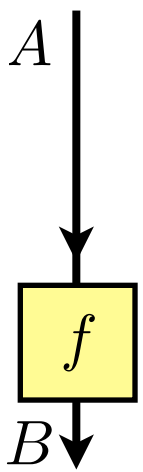
Categories

$$\mathrm{id}_A : A \to A$$



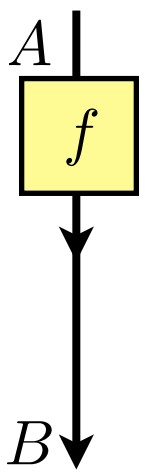
Categories

$$f \circ \mathrm{id}_A : A \to B$$



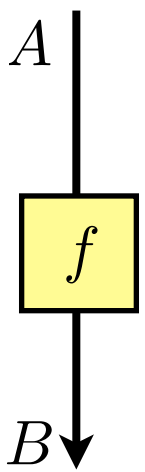
Categories

$$\mathrm{id}_B \circ f : A \to B$$



Categories

$$f : A \to B$$



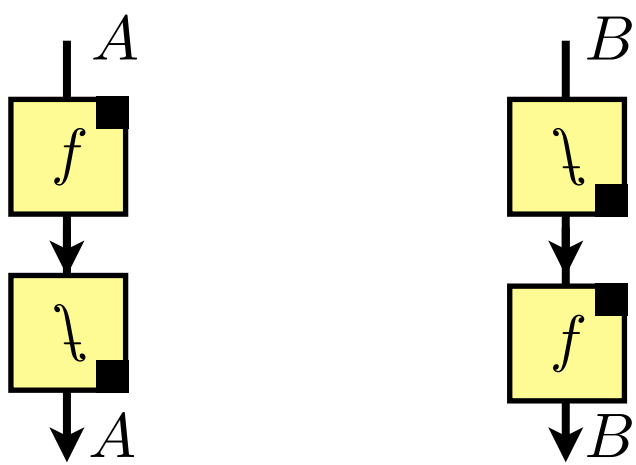
†-Categories

A category is a *†-category* if it is equipped with an involutive functor, $(\cdot)^\dagger$ which reverses the arrows while leaving the objects unchanged.

$$f : A \to B \qquad f^\dagger : B \to A$$

†-Categories

An arrow $f : A \to B$ is called *unitary* when:



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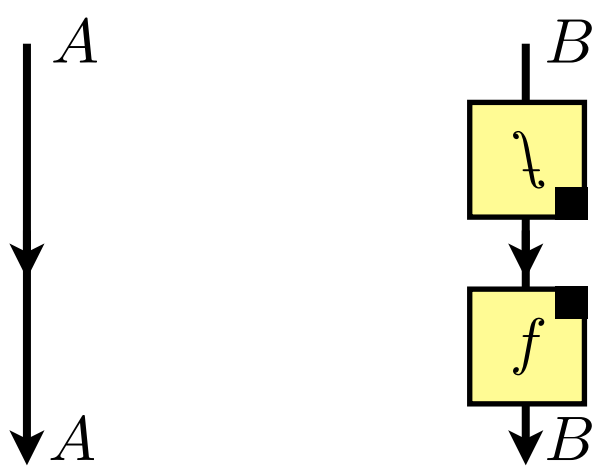

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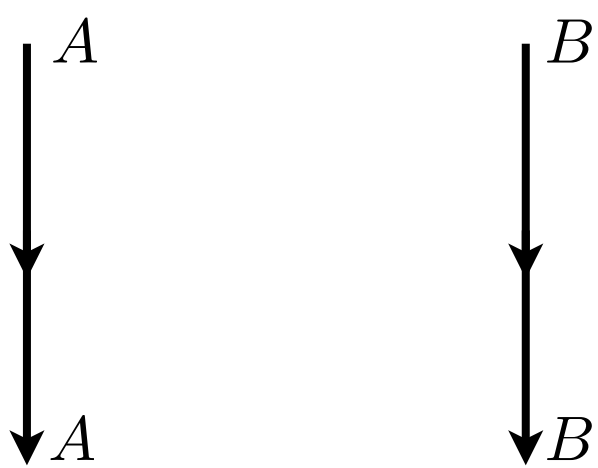
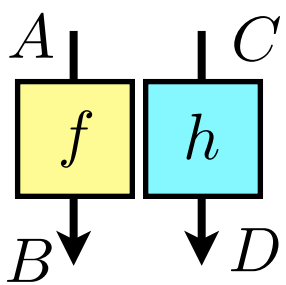


Monoidal Categories

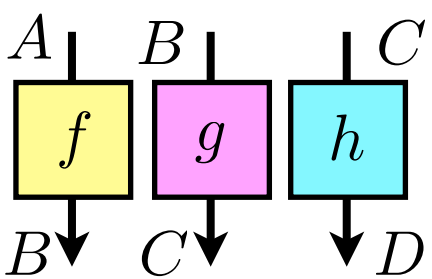
A *strict monoidal category* is a category equipped with a tensor product on both objects and arrows:

$$f \otimes h : A \otimes C \to B \otimes D$$



The tensor is associative:

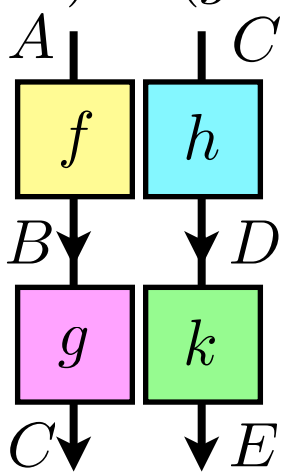
$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$



Monoidal Categories

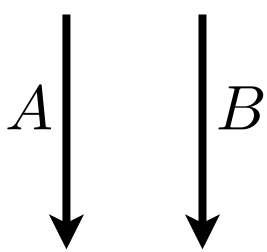
The tensor product is *bifunctorial*, meaning that it preserves composition:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$



and identities:

$$\mathrm{id}_{A \otimes B} = \mathrm{id}_A \otimes \mathrm{id}_B$$



Monoidal Categories

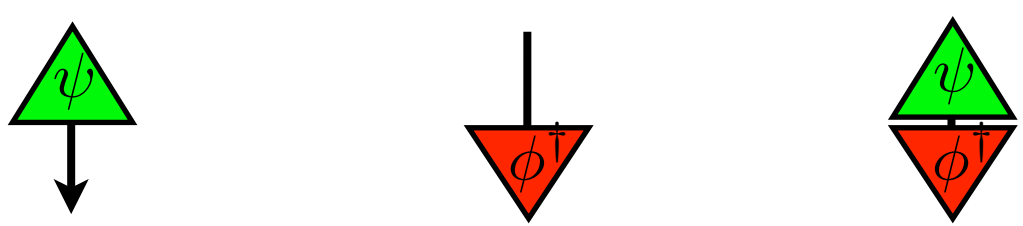
Monoidal categories have a special *unit* object called I which is a left and right identity for the tensor:

$$I \otimes A = A = A \otimes I$$

$$\mathrm{id}_I \otimes f = f = f \otimes \mathrm{id}_I$$

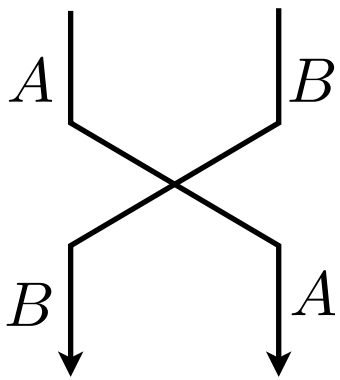
No lines are drawn for I in the graphical notation:

$\psi : I \to A$ $\qquad$ $\phi^\dagger : A \to I$ $\qquad$ $\phi^\dagger \circ \psi : I \to I$



Symmetric Monoidal Categories

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$



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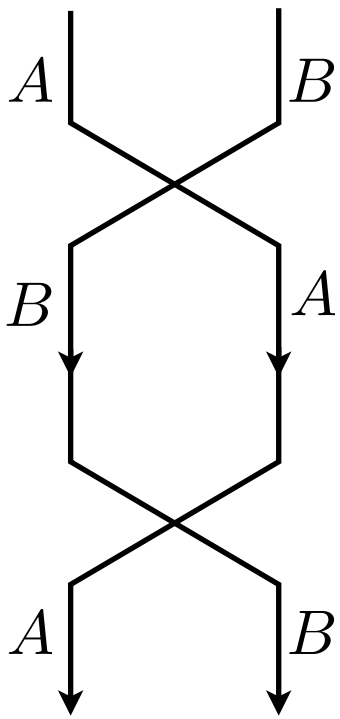

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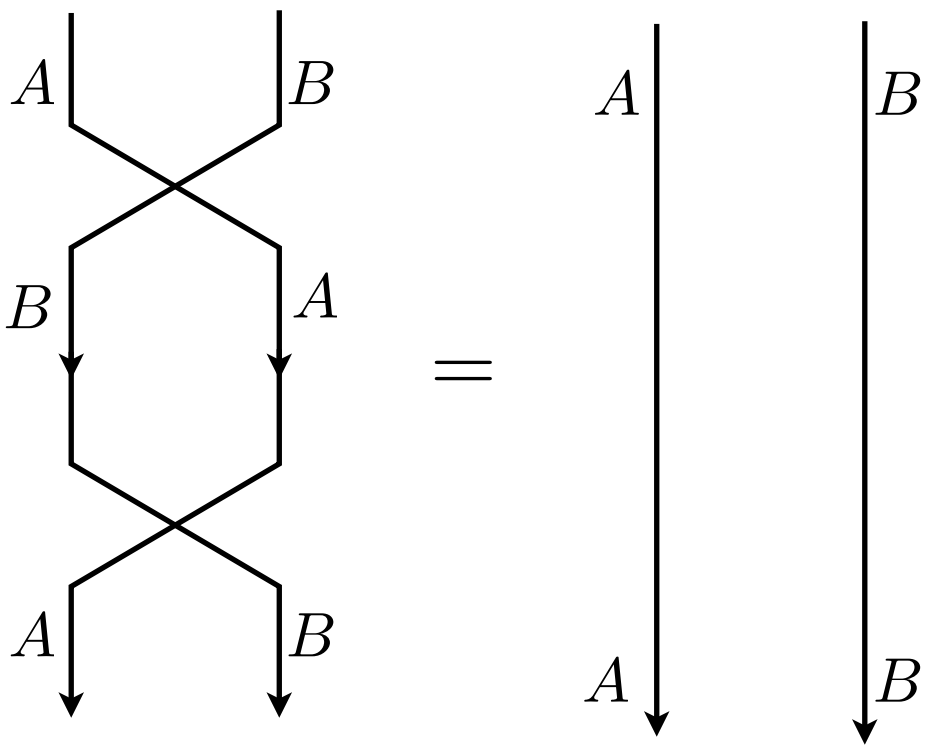

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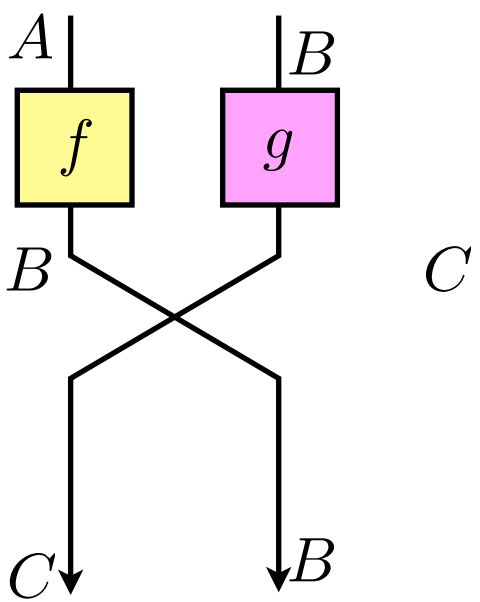

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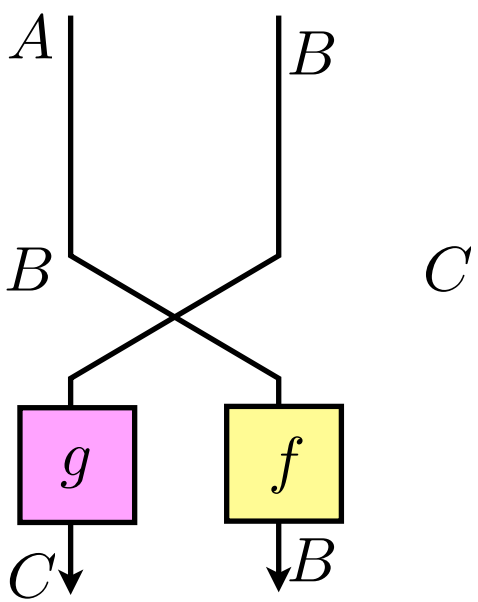
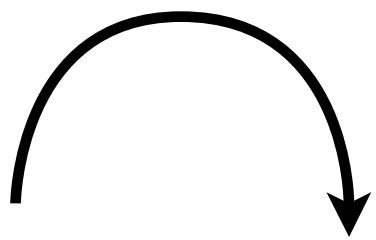
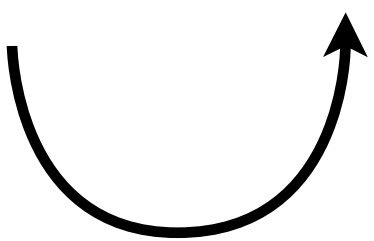


Compact Closure

A symmetric monoidal category is called *compact* if, for every object A, there exists a *dual* object $A*$ and arrows:

$$d : I \to A^* \otimes A \qquad e : A \otimes A^* \to I$$





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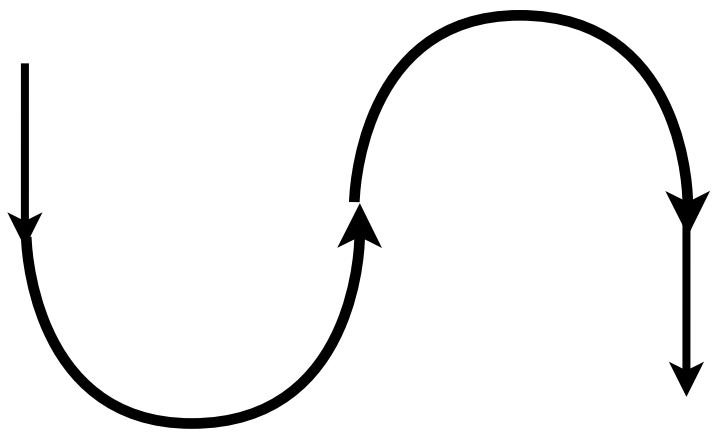

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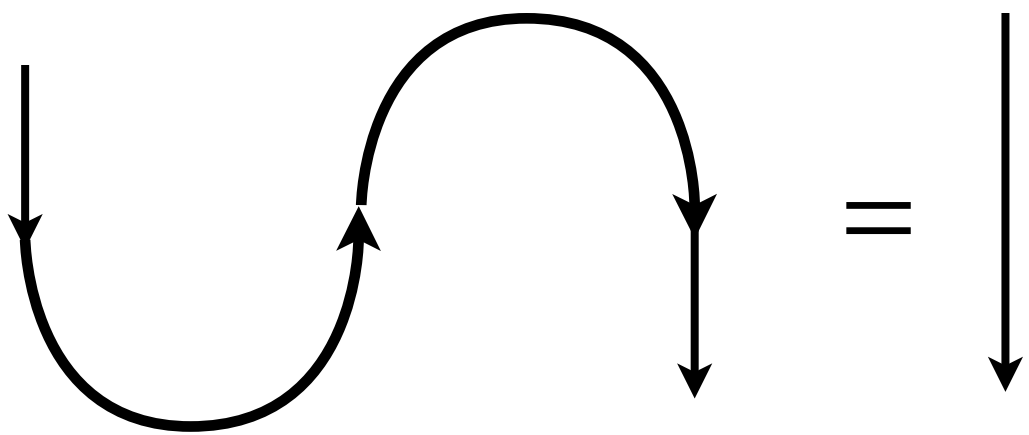
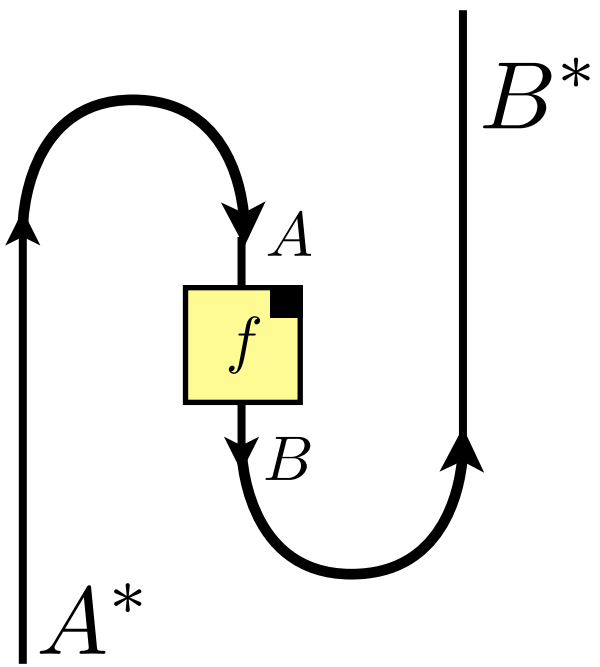


Compact Closure

Every arrow f in a compact category has a dual f^* defined by

$f^* =$



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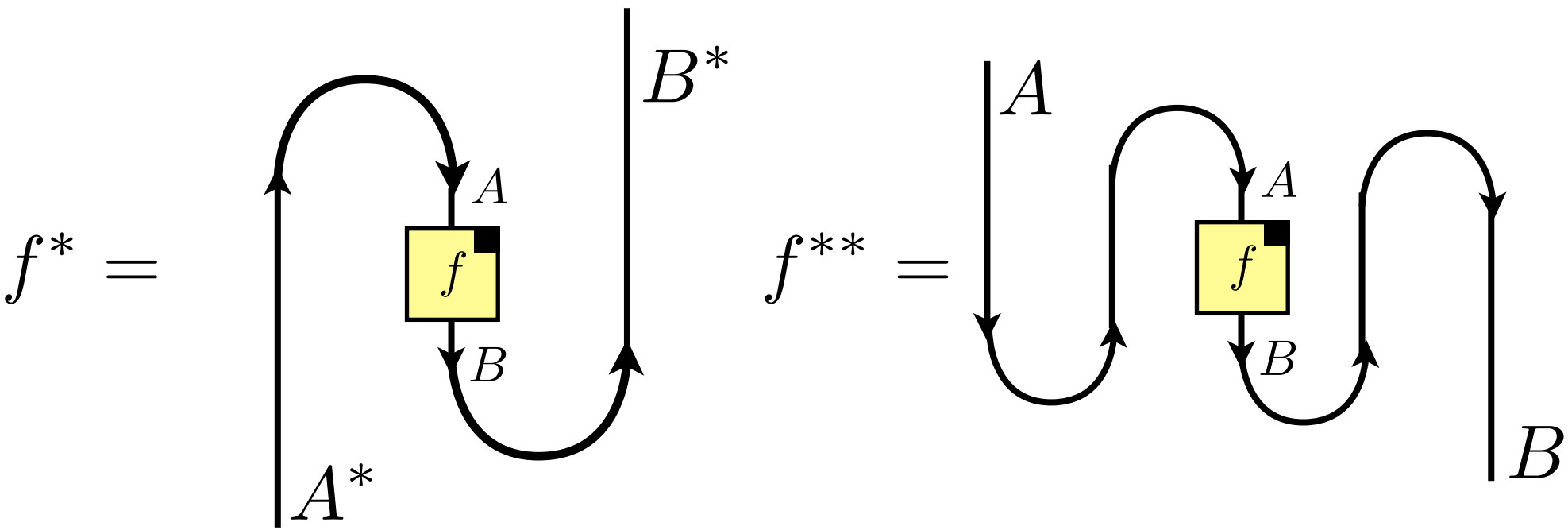

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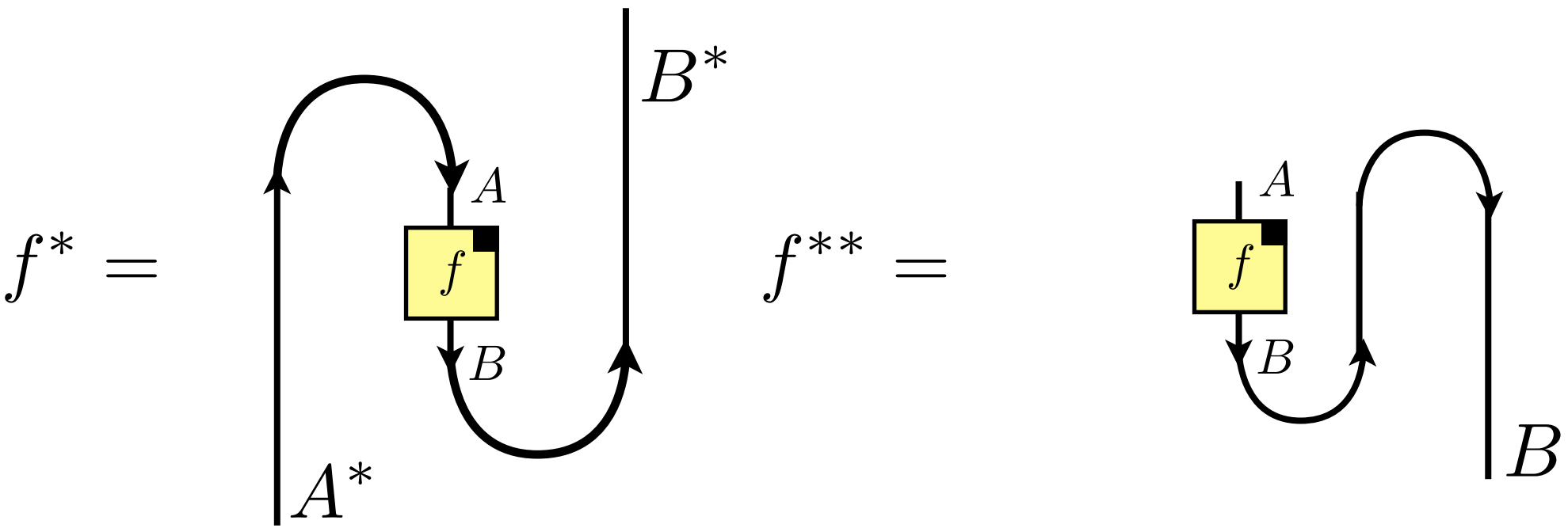

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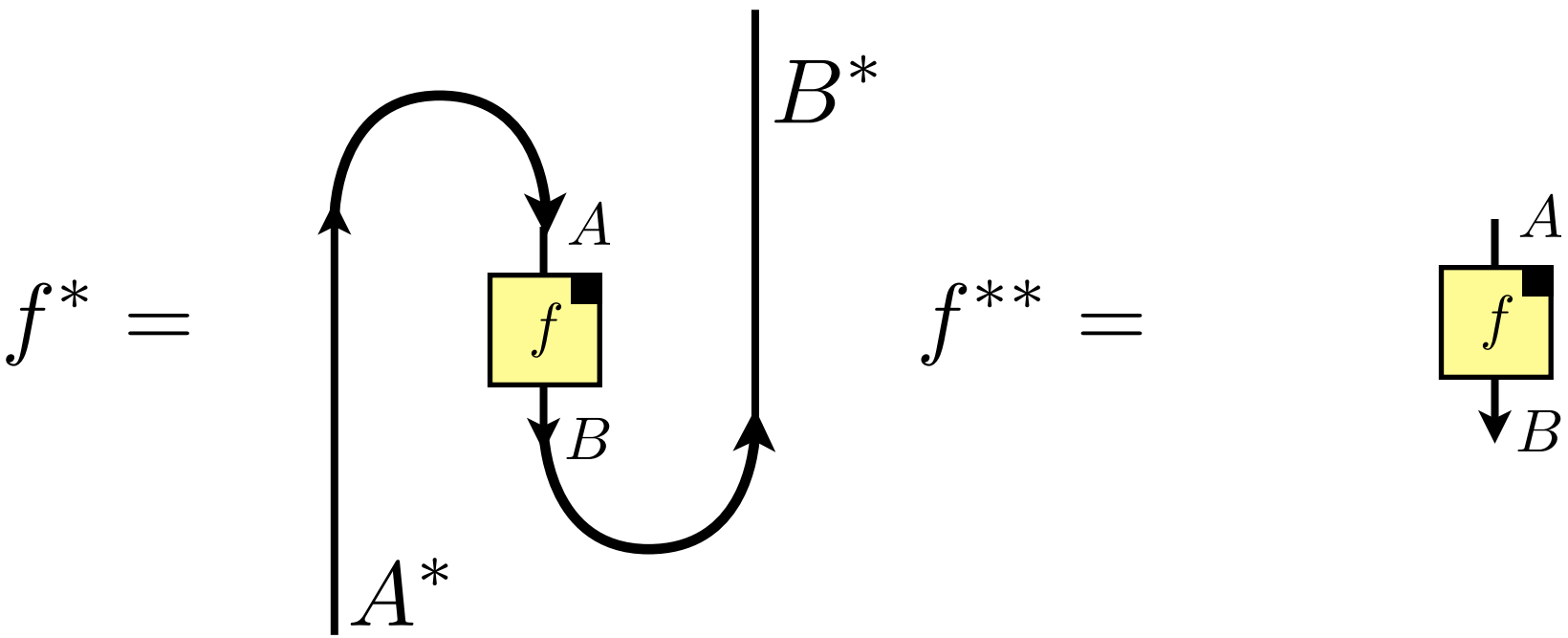

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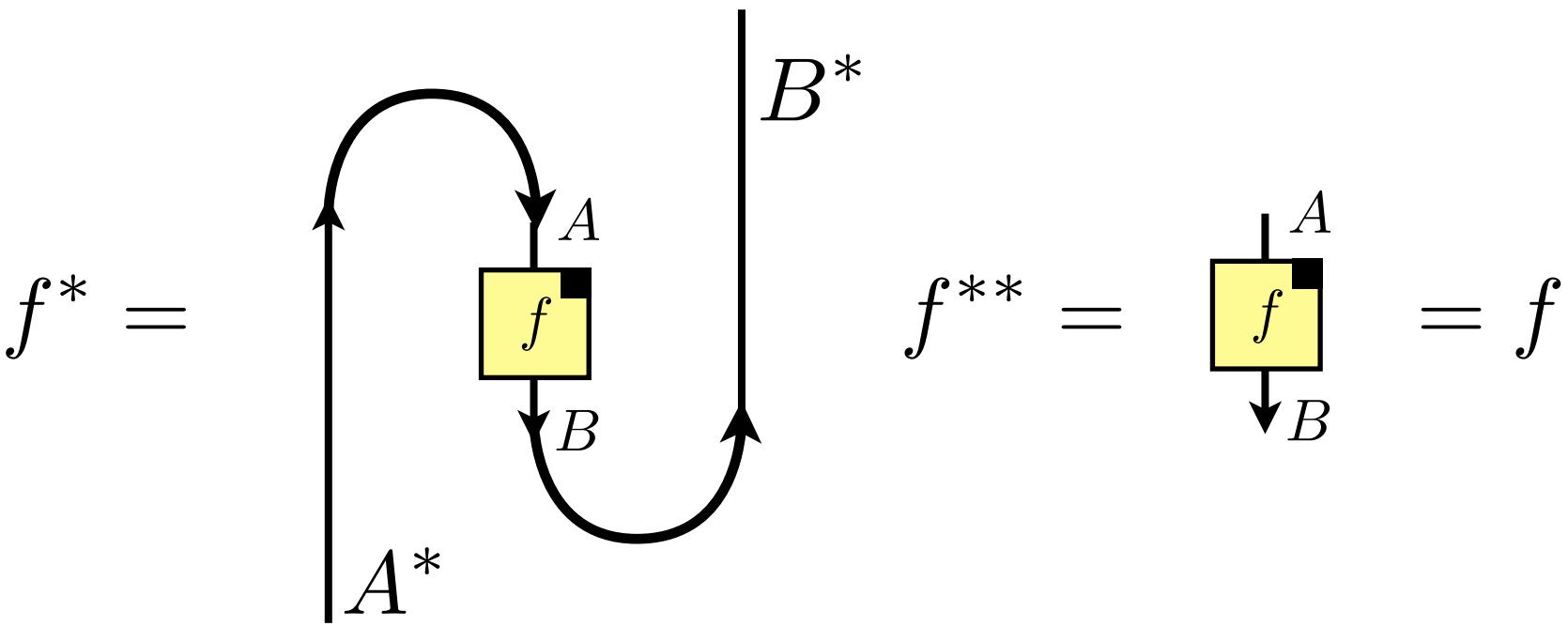

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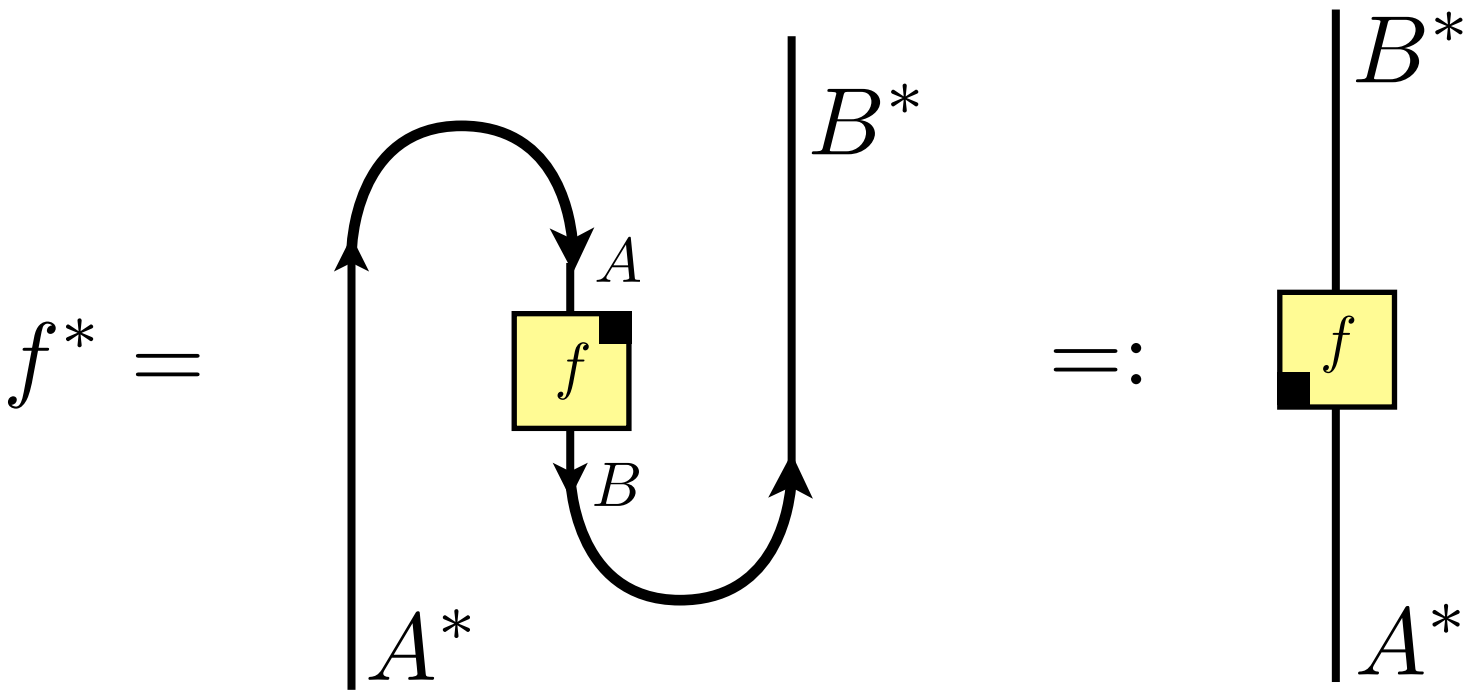

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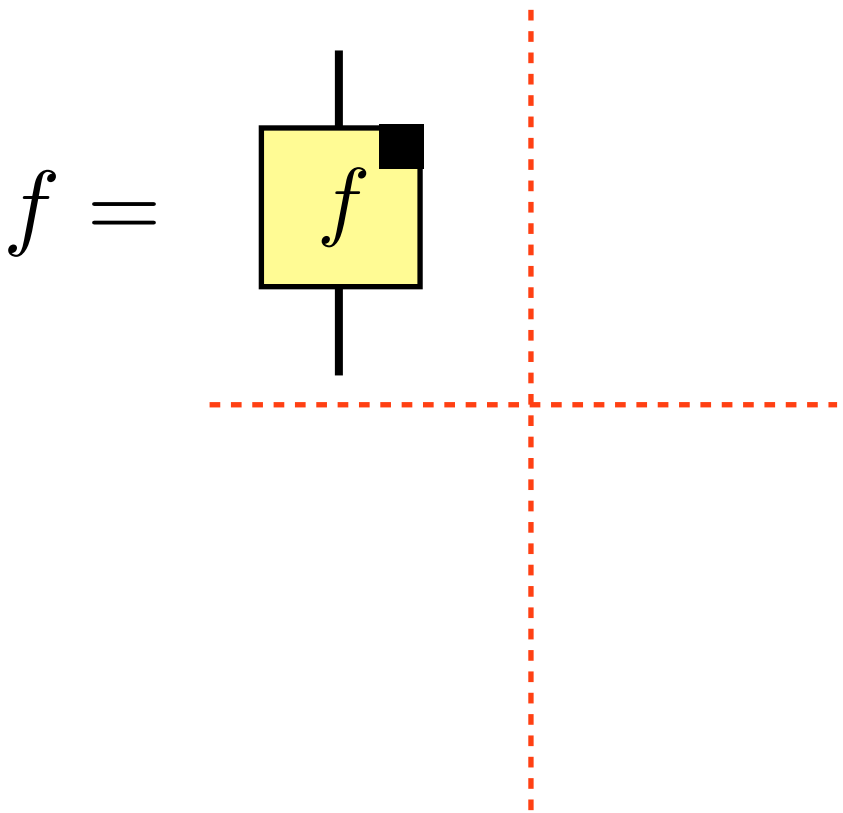

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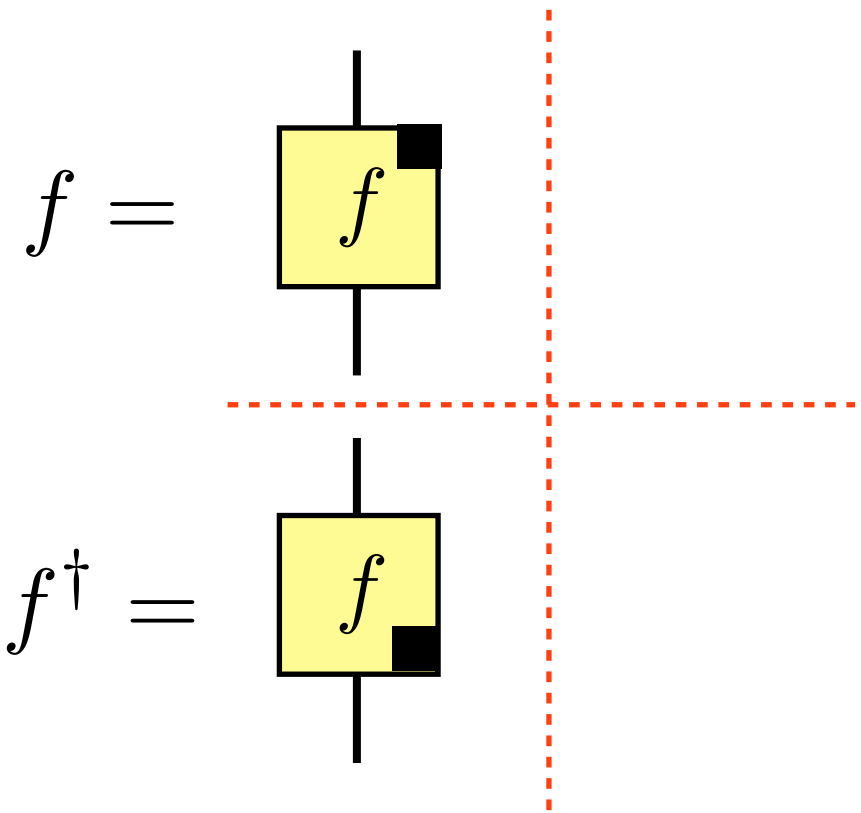

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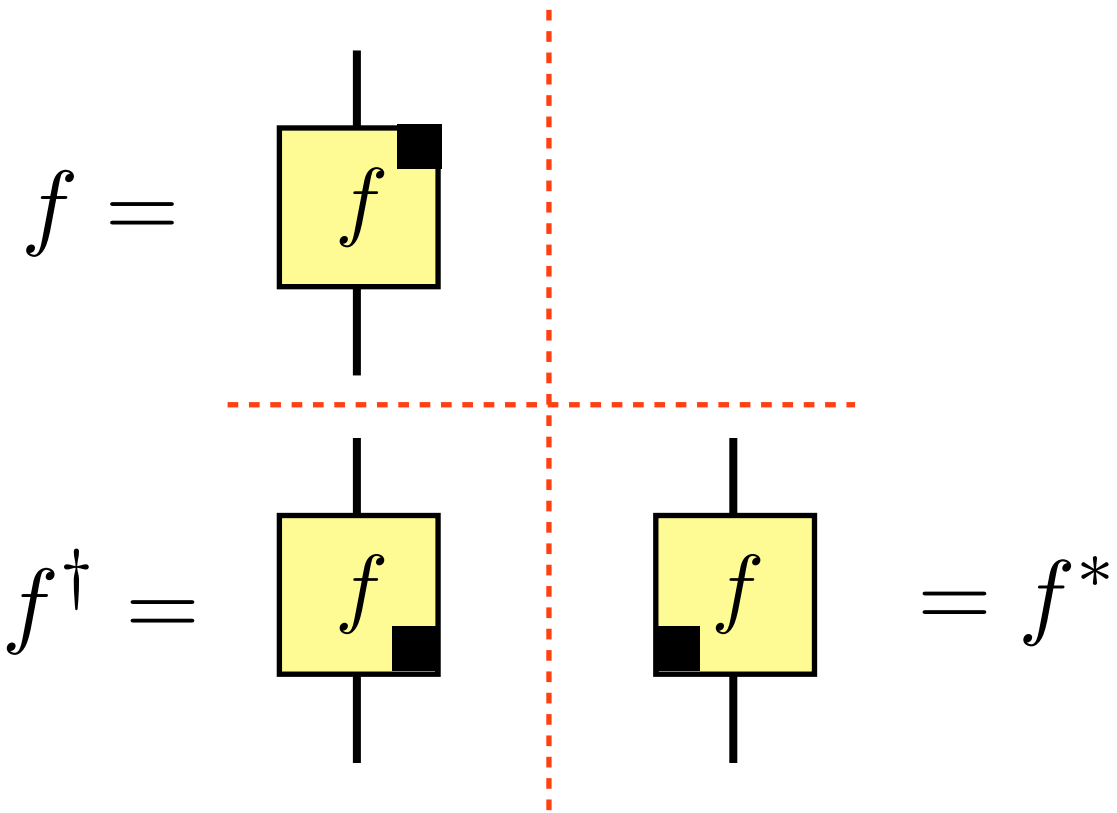
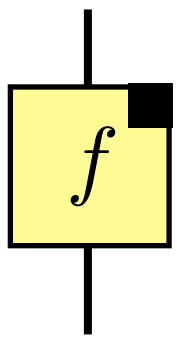
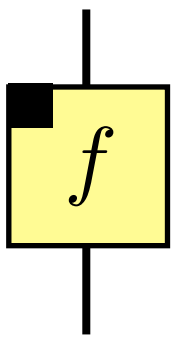
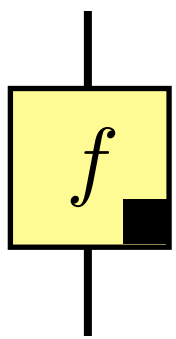
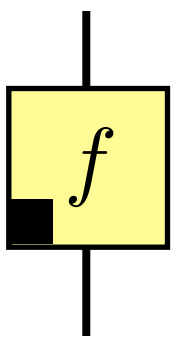


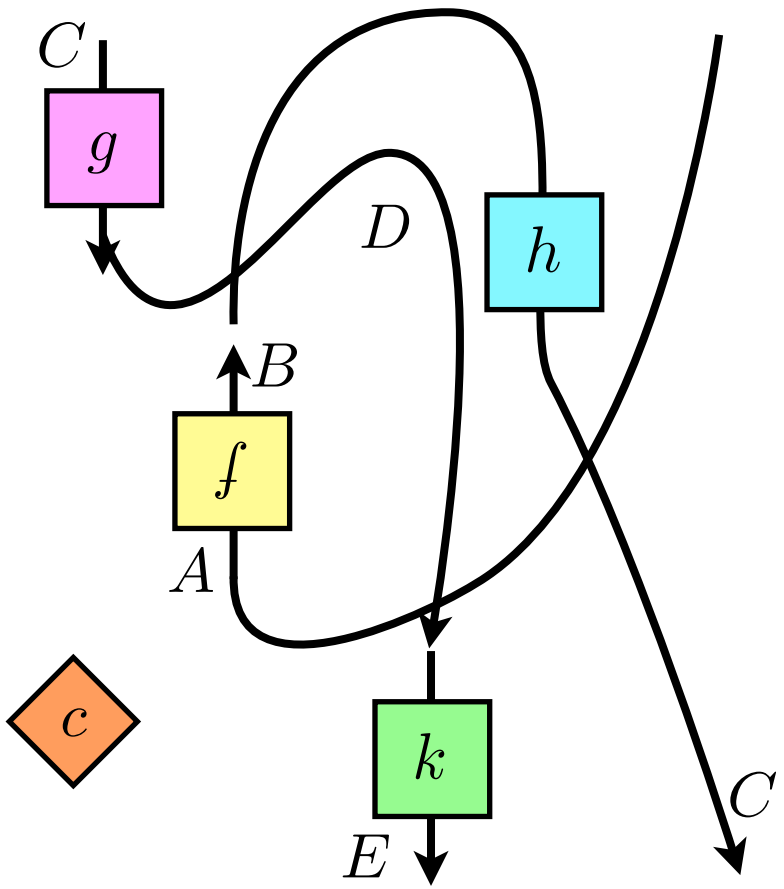
Compact Closure

$f =$   $= f^{*\dagger} =: f_*$

$f^\dagger =$   $= f^*$

Graphical Calculus Theorem

Thm: one diagram can be deformed to another if and only if their denotations are equal by the structural equations of the category.



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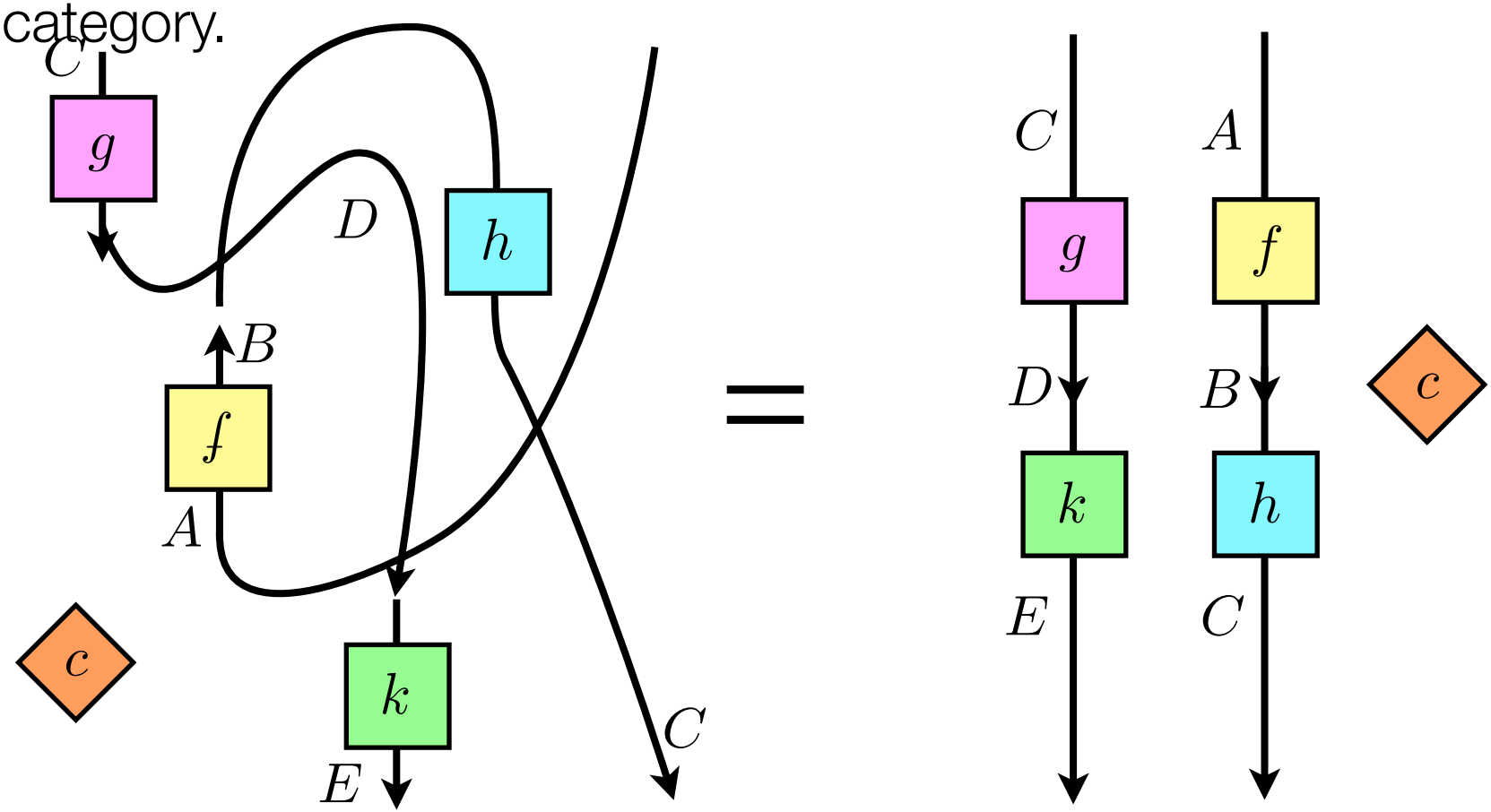


The Category **FRel**

FRel is the category of finite relations. It is †-compact with the following structure.

- Objects: finite sets, $X, Y, Z,$ etc
- Arrows: Relations $R \subseteq X \times Y$
- Tensor: Cartesian product; $X \otimes Y := X \times Y, \quad I = \{*\}$
- $f^\dagger$ is relational converse: $(x, y) \in f \Leftrightarrow (y, x) \in f^\dagger$

A relation $R \subseteq \{*\} \times X$ is simply a *subset* of X; its converse is also a subset.

Hence $S^\dagger \circ R : I \to I$ is non-empty iff $R \cap S \neq \emptyset$.

The Category **FDHilb**

FDHilb is the category of finite dimensional complex Hilbert spaces. It is †-monoidal with the following structure.

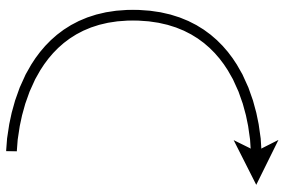
- Objects: finite dimensional Hilbert spaces, A, B, C, etc
- Arrows: all linear maps
- Tensor: usual (Kronecker) tensor product; $I = \mathbb{C}$
- $f^\dagger$ is the usual adjoint (conjugate transpose)

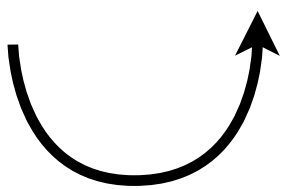
A linear map $\psi : I \to A$ picks out exactly one vector. It is a ket and $\psi^\dagger : A \to I$ is the corresponding bra.

Hence $\psi^\dagger \circ \phi : I \to I$ is the inner product $\langle \psi \mid \phi \rangle$

Compact Structure of **FDHilb**

In $\mathbf{FDHilb}$ the compact structure is given by the maps:





whenever $\{a_i\}_i$ is a basis for A and $\{\overline{a_i}\}_i$ is the corresponding basis for the dual space A^*

Compact Structure of **FDHilb**

In $\mathbf{FDHilb}$ the compact structure is given by the maps:

$$d : 1 \mapsto \sum_i a_i \otimes \overline{a_i} \qquad\qquad e : \overline{a_i} \otimes a_i \mapsto 1$$

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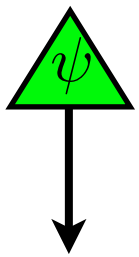
In the case of $\mathbb{C}^2$ the map d picks out the Bell state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

which is the simplest example of *quantum entanglement*.

Example: Quantum Teleportation

Aleks



Bob

Bennett et al 1993; Abramsky & Coecke 2004

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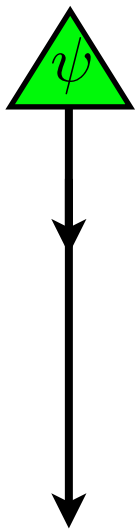

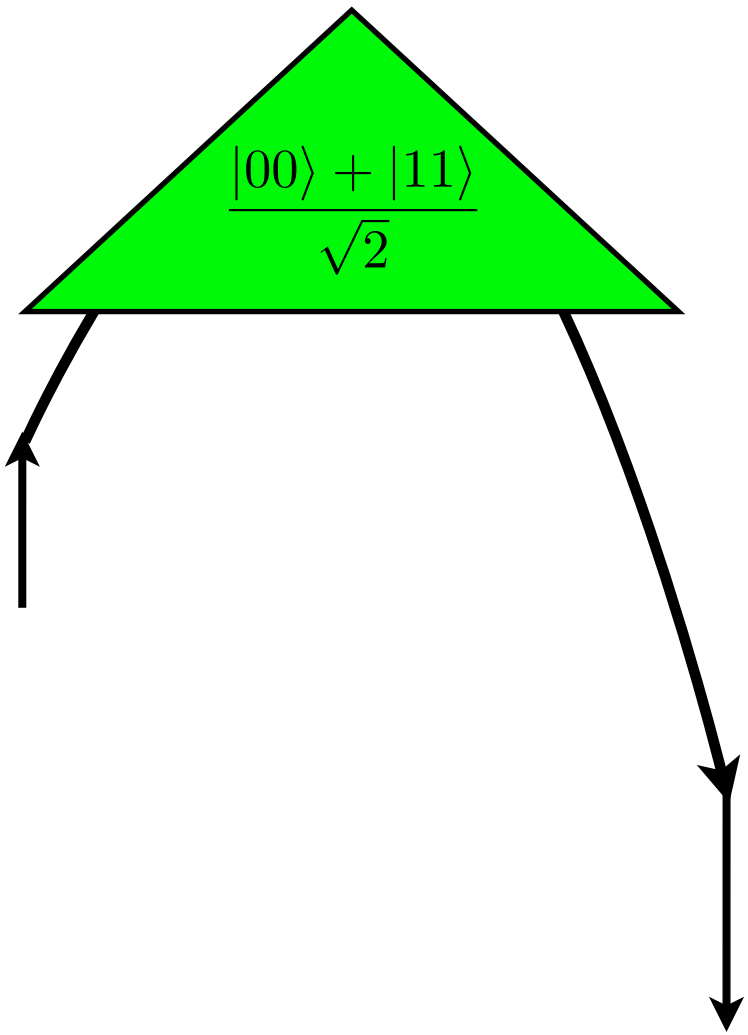

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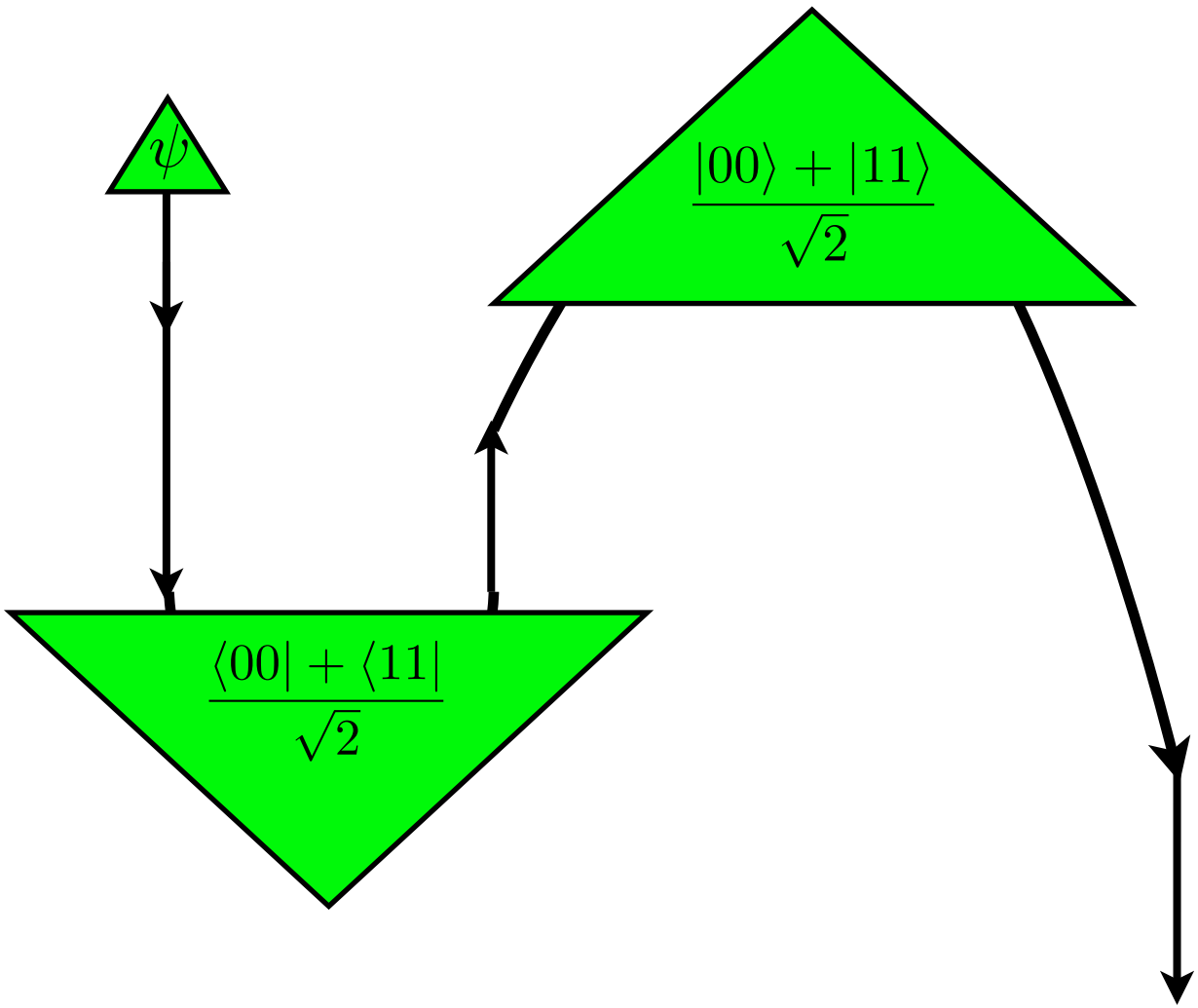

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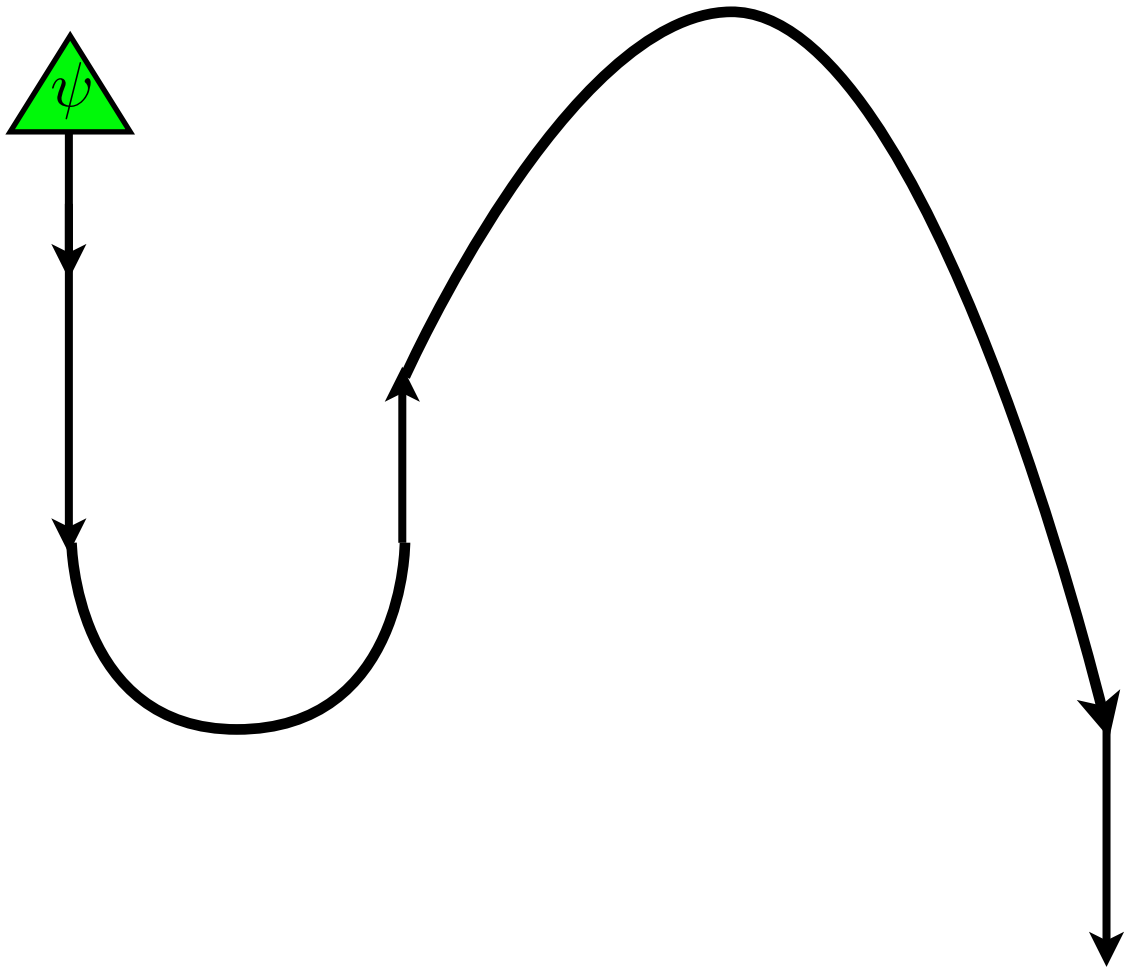

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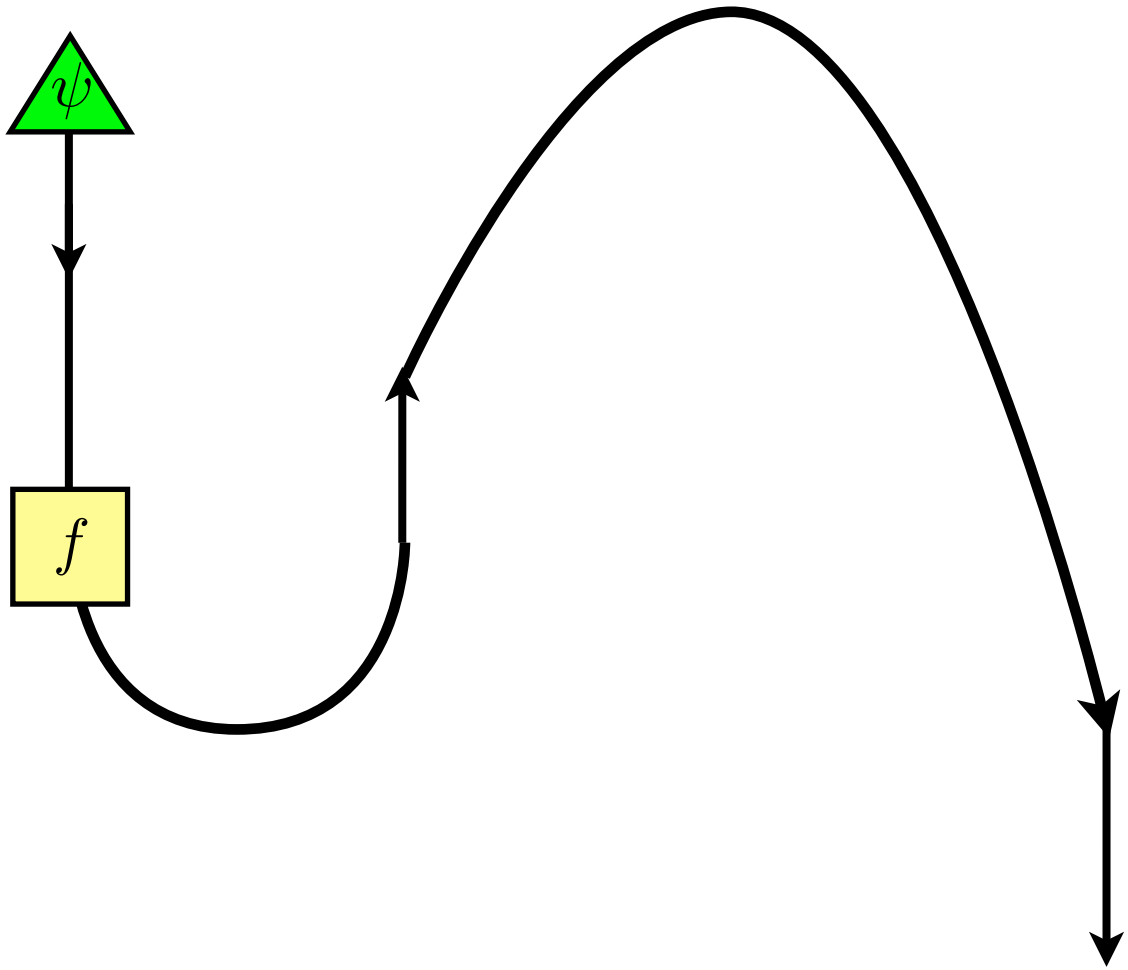

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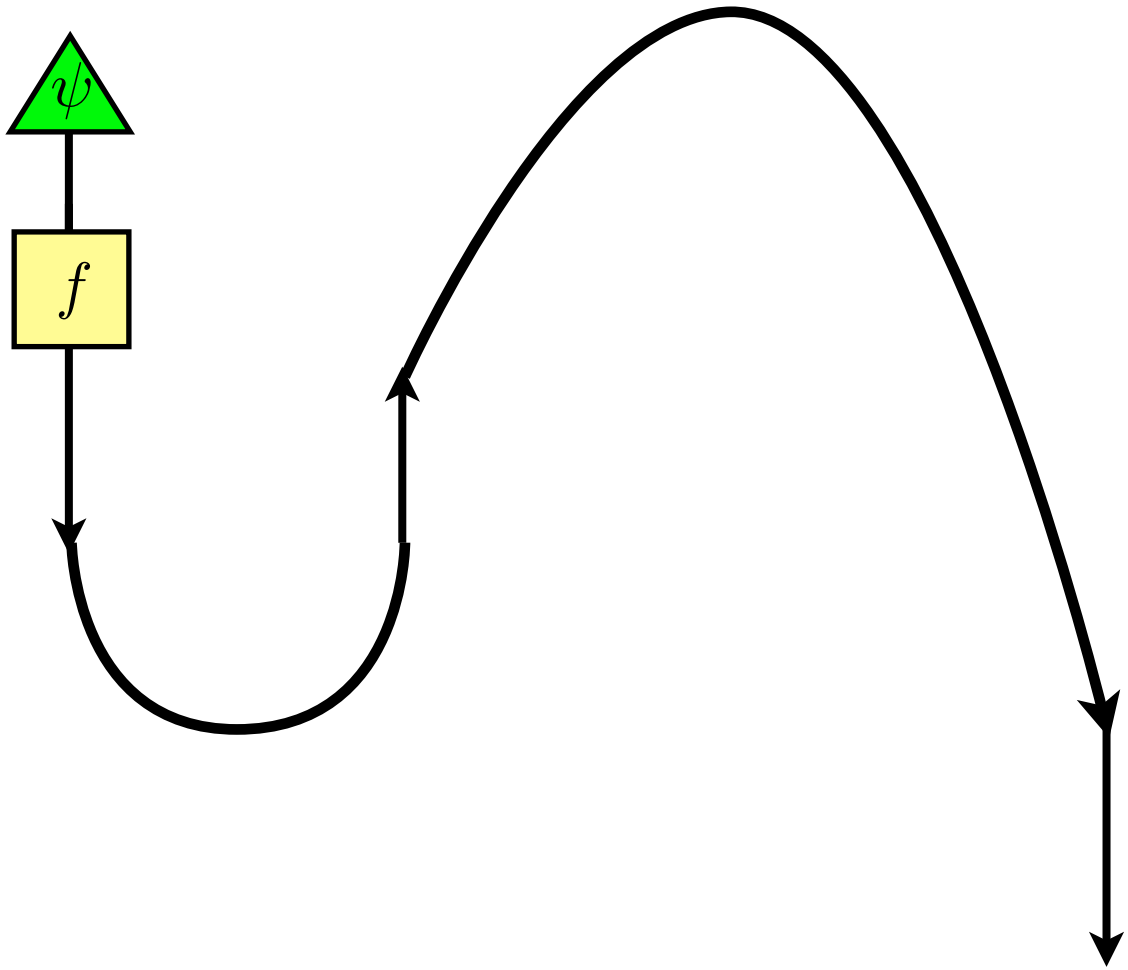

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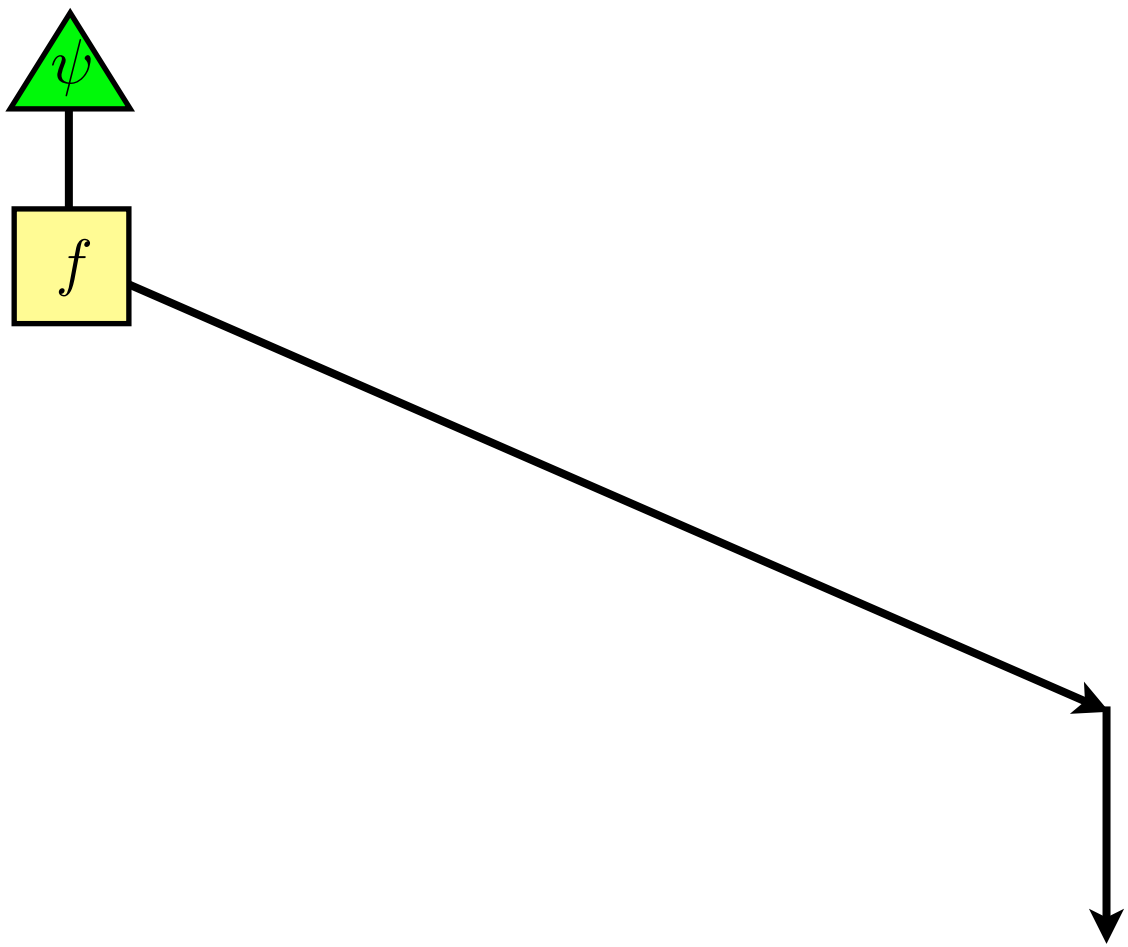

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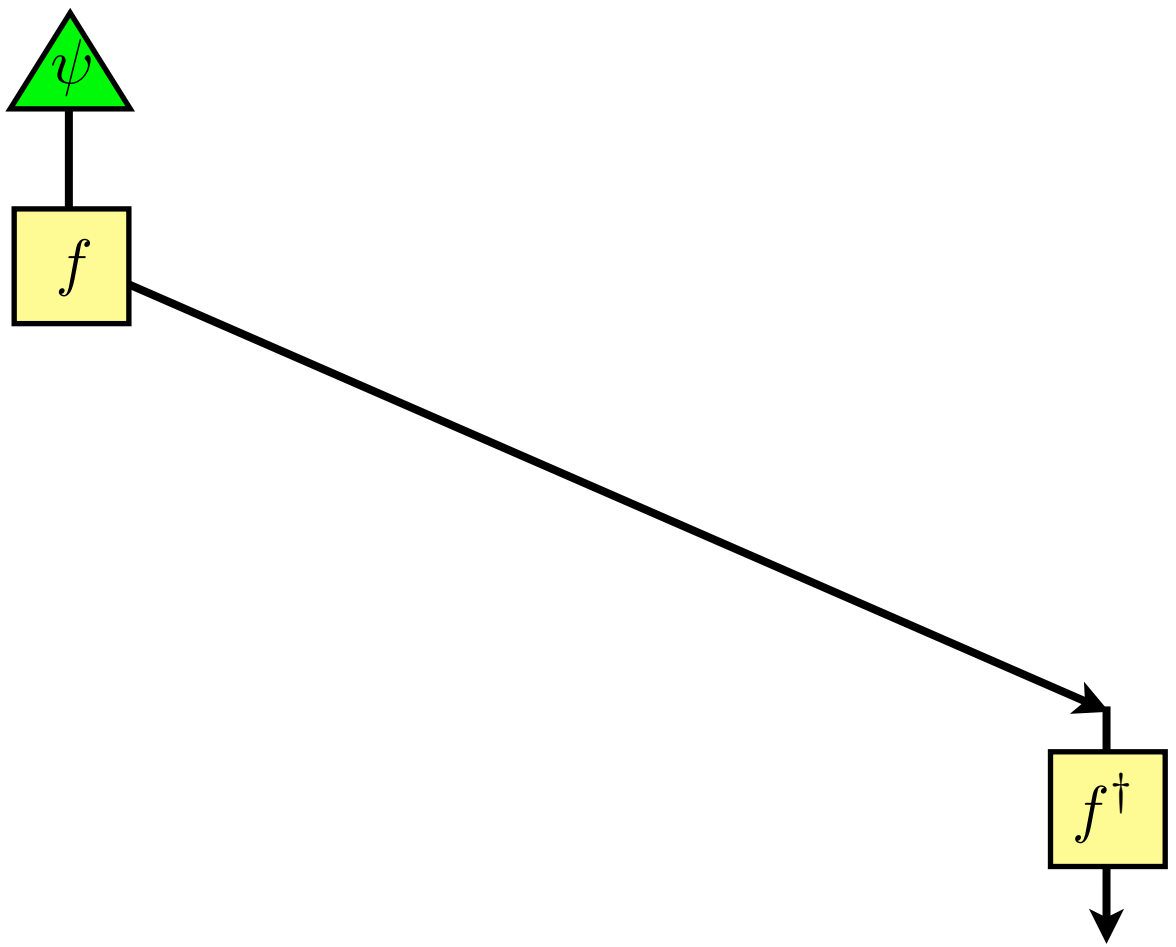

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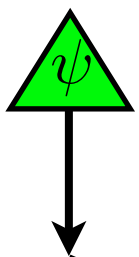
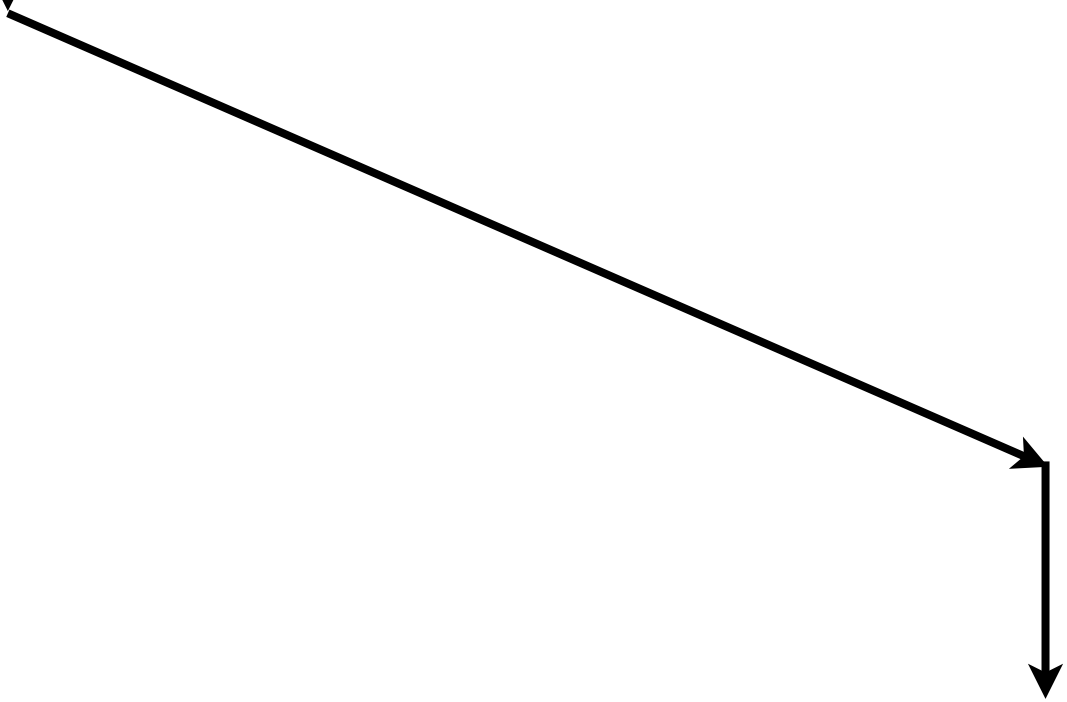




Bob

Bennett et al 1993; Abramsky & Coecke 2004

A compact caveat

If compact structure exists, *it need not be unique*.

For example, in $\mathbf{FDHilb}$, we have compact structures :

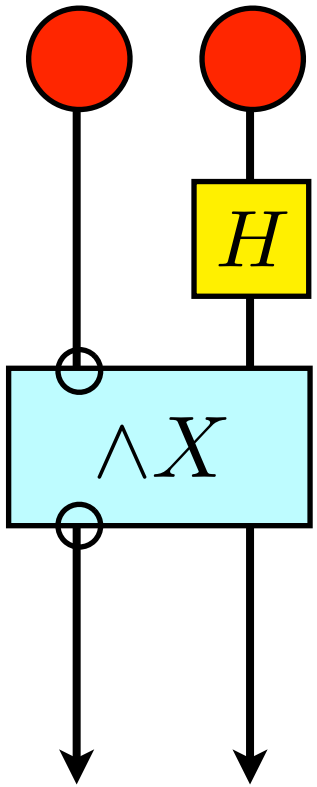
$$d_Z = 1 \mapsto |00\rangle + |11\rangle$$

and

$$d_Y = 1 \mapsto |00\rangle - |11\rangle$$

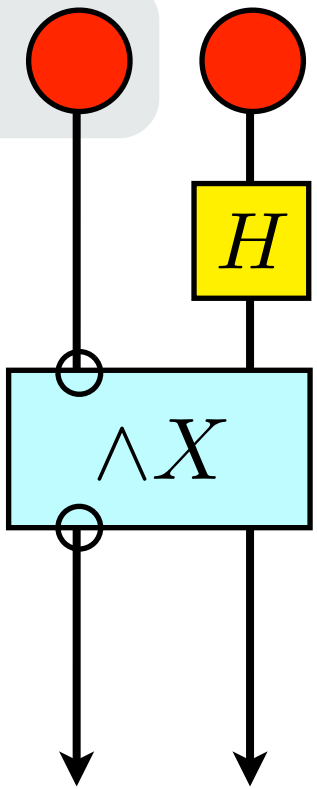
Therefore the $(\cdot)^*$ and $(\cdot)_*$ operations are not defined uniquely either.

Example: Preparing a Bell state



Example: Preparing a Bell state

$|0\rangle : I \to Q$



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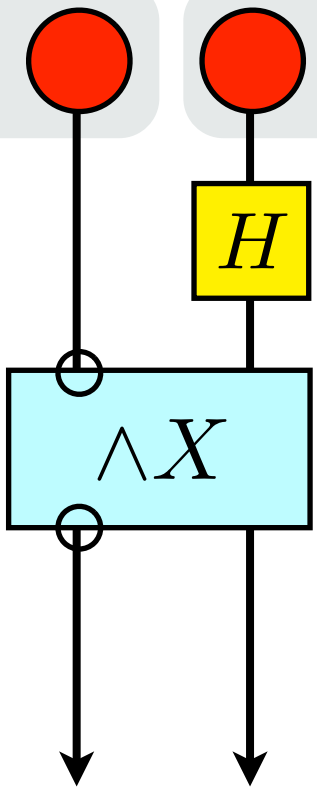

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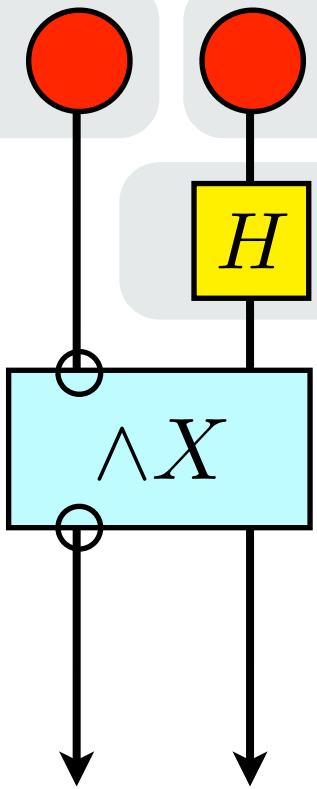
$H : Q \to Q$



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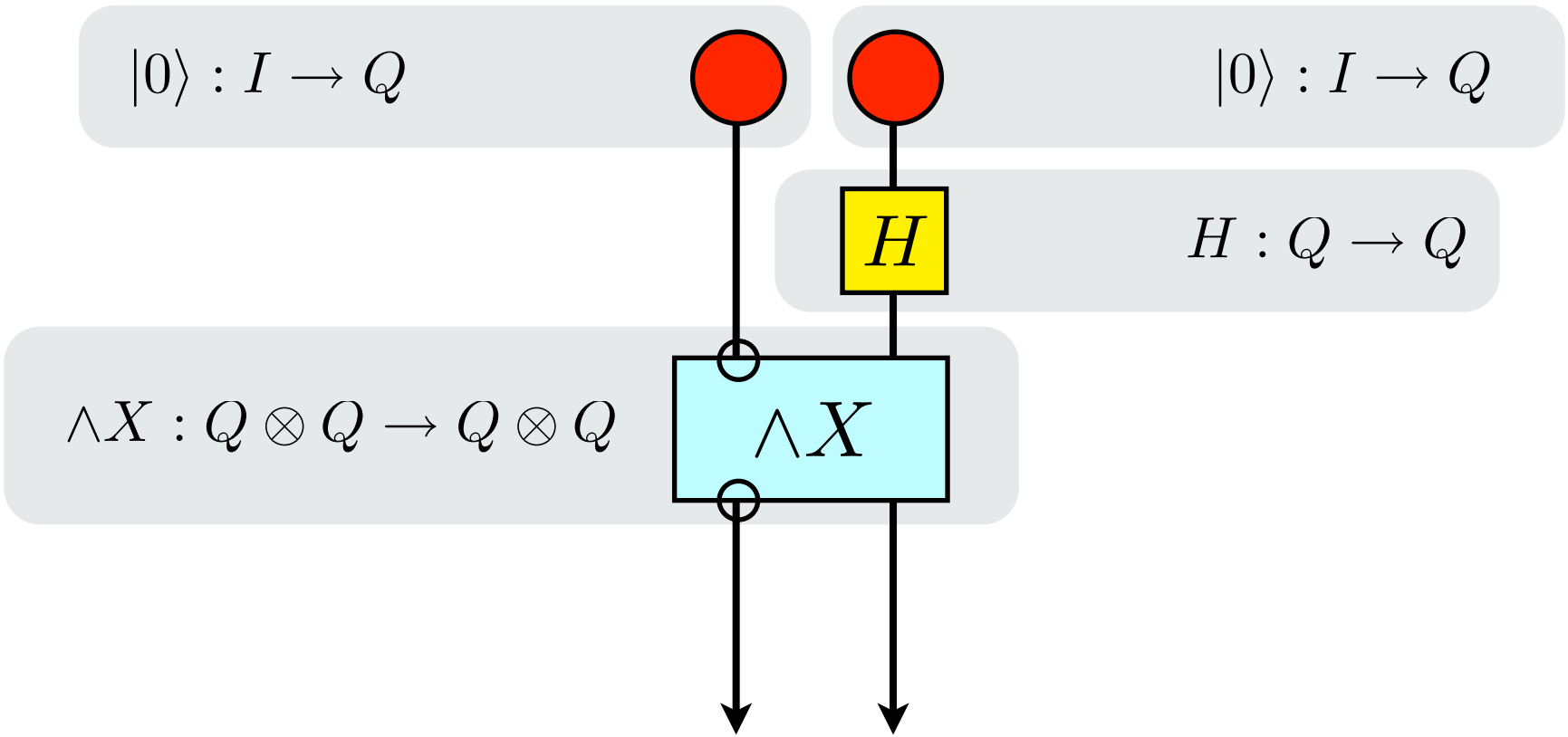
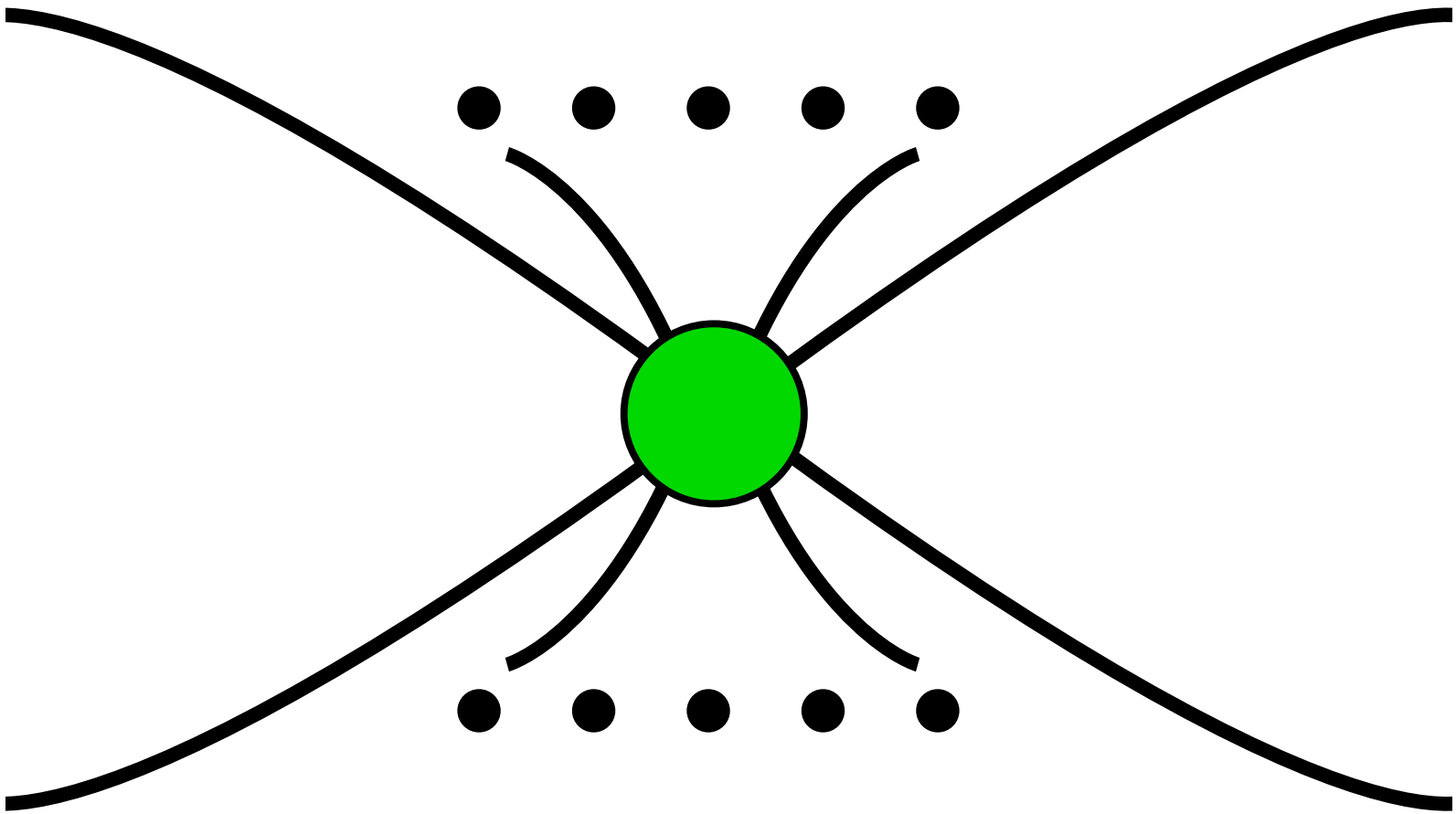


Classical Structure



Copying, deleting, and all that

No-Cloning and No-Deleting

Theorem: There are no unitary operations D such that

$$D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$$
$$D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no unitary operations E such that

$$E : |\psi\rangle \mapsto |0\rangle$$
$$E : |\phi\rangle \mapsto |0\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]

No-Cloning and No-Deleting, abstractly

Theorem: if a †-compact category has natural transformations

$$\delta : - \Rightarrow - \otimes -$$

$$\epsilon : - \Rightarrow I$$

then the category collapses [Abramsky 2007]

(Translation: in our abstract setting there are no universal cloning or deleting operations, just like in quantum mechanics)

“Classical” Quantum States

When can a quantum state be treated as if classical?

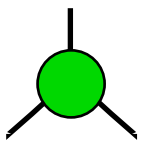
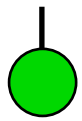
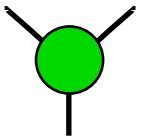
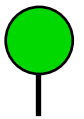
- no-go theorems allow copying and deleting of *orthogonal* states;

In other words:

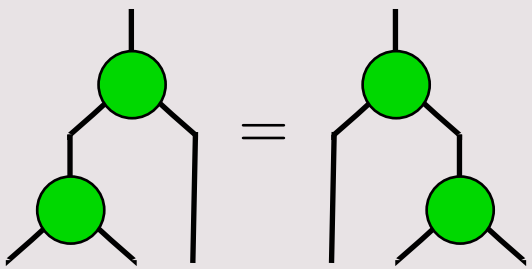
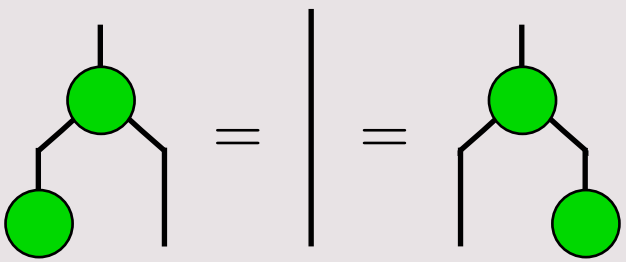
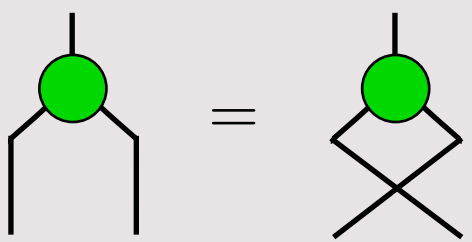
- A quantum state may be copied and deleted if it is an eigenstate of some *known observable.*

We’ll use this property to formalise *observables* in terms of *copying* and *deleting* operations.

Classical Structures

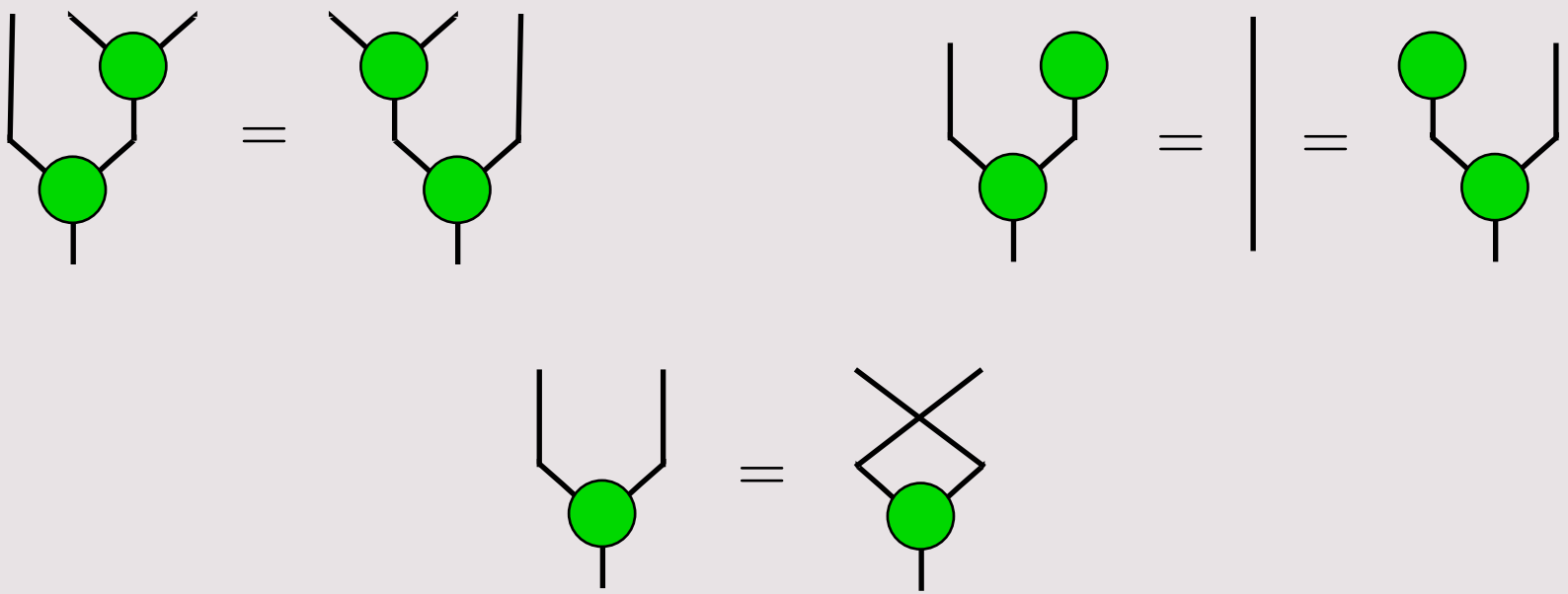
$\delta =$ 
$\epsilon =$ 
$\delta^\dagger =$ 
$\epsilon^\dagger =$ 

Comonoid Laws

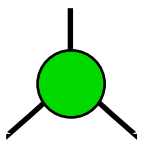
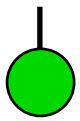
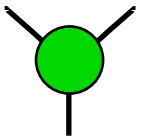
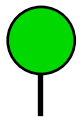




Classical Structures

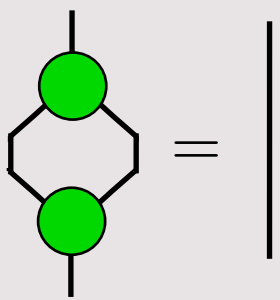
Monoid Laws



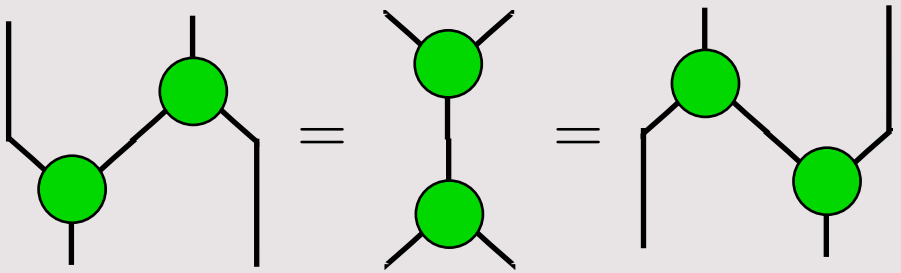
Classical Structures

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

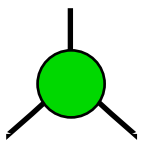
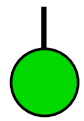
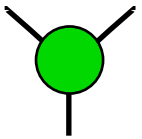
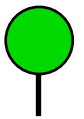
Isometry Law



Frobenius Law



Classical Structures

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

In other words: a classical structure is a *special commutative †-Frobenius algebra*

Classical Structures

Given any finite dimensional Hilbert space we can define a classical structure by

$$\delta : A \to A \otimes A :: a_i \mapsto a_i \otimes a_i$$
$$\epsilon : A \to I :: \sum_i a_i \mapsto 1$$

Example: $\delta : \begin{array}{l} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{array}$ $\qquad \epsilon : |0\rangle + |1\rangle \mapsto 1$

define a classical structure over qubits; the standard basis is copied and erased. Note however that:

$$\delta(|+\rangle) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

showing that not every state can be cloned.

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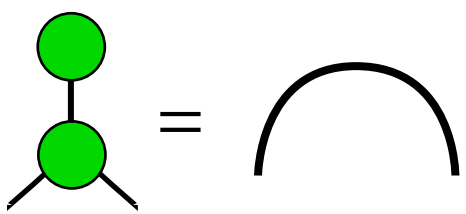

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Theorem: in **FDHilb**, classical structures are in bijective correspondence to bases. [Coecke, Pavlovic, Vicary]

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Each (well behaved) observable defines a basis, hence each observable defines a classical structure!

Example: $\mathbf{FRel}$

For any set X we can define a classical structure by:

$$D : x \sim (x, x) \qquad \forall x \in X$$
$$E : x \sim *$$

Perhaps surprisingly, there are others. Let $X = \{0,1\}$ then we have:

$$D_X : \quad \begin{array}{l} 0 \sim \{(0,0),(1,1)\} \\ 1 \sim \{(0,1),(1,0)\} \end{array}$$
$$E_X : \quad 0 \sim *$$

Spider Theorem

Theorem: any maps constructed from δ and ε , and their adjoints, whose graph is connected, is determined uniquely by the number of inputs and outputs.

Coecke & Paquette 2006

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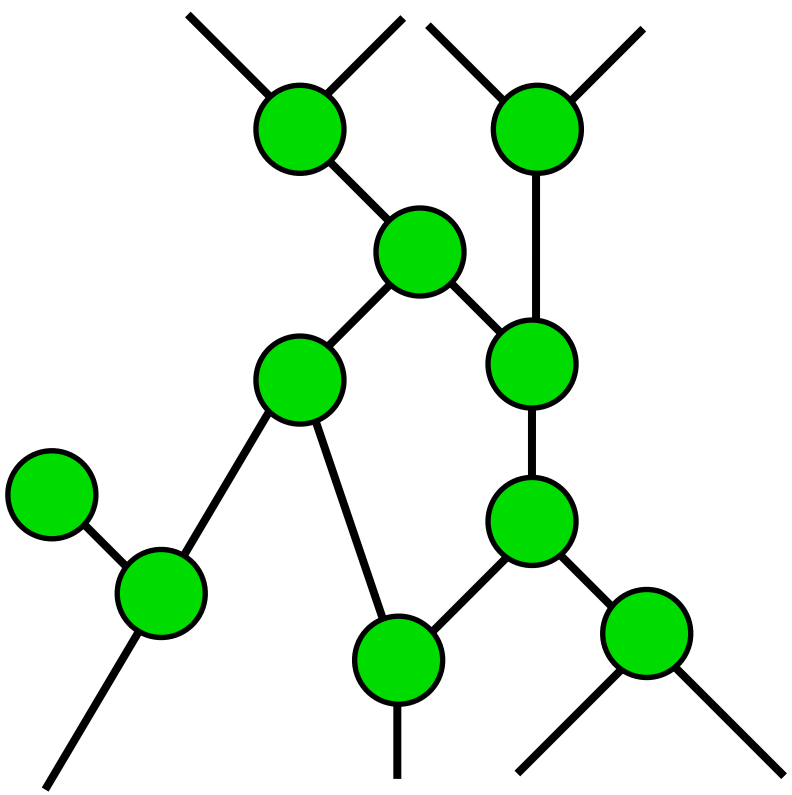

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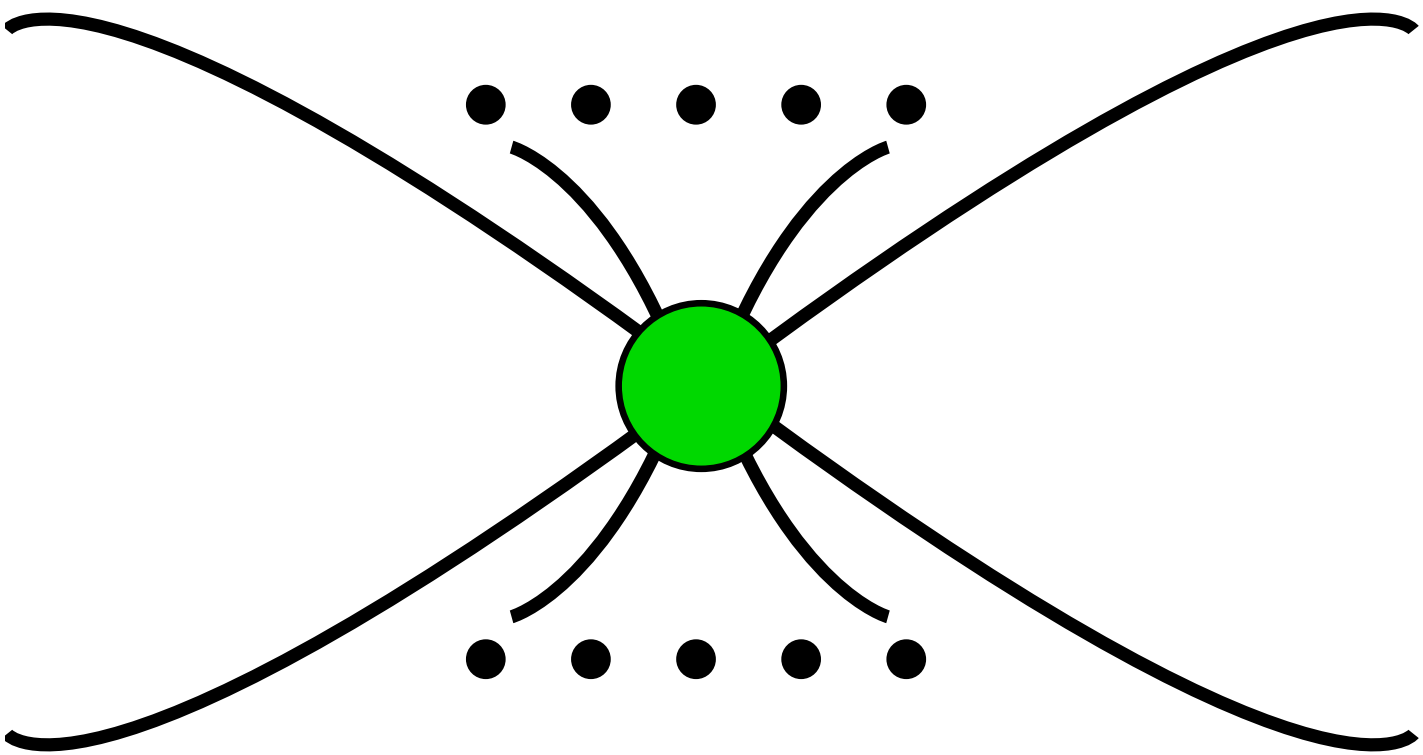
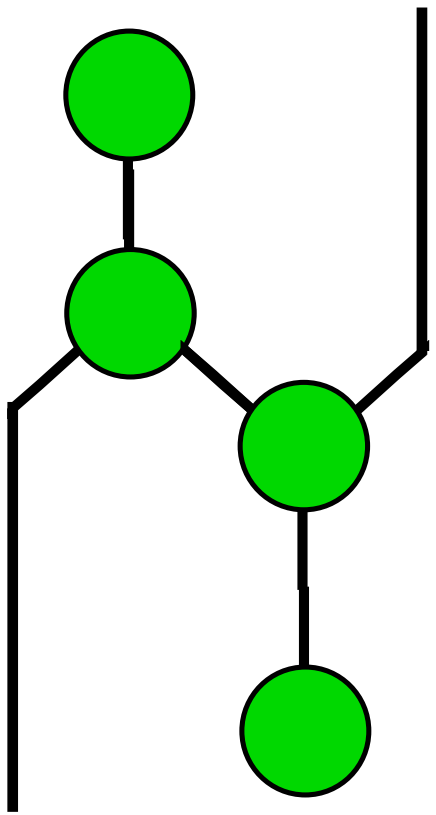


Coecke & Paquette 2006

Classical Structure begets Compact Structure



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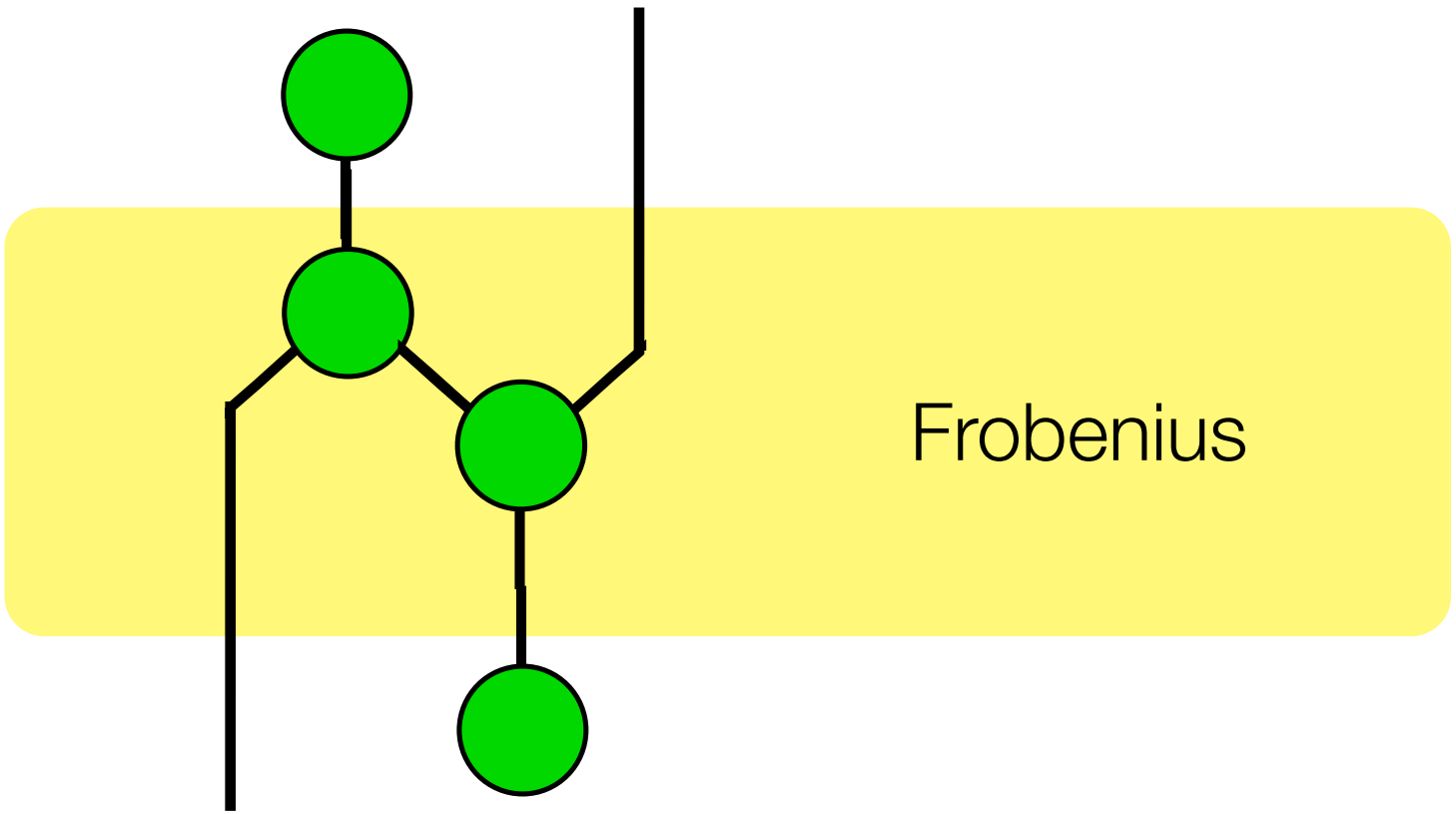

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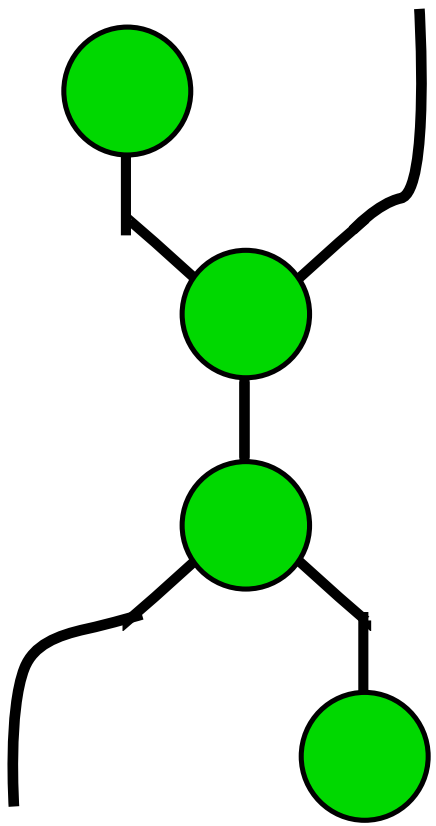

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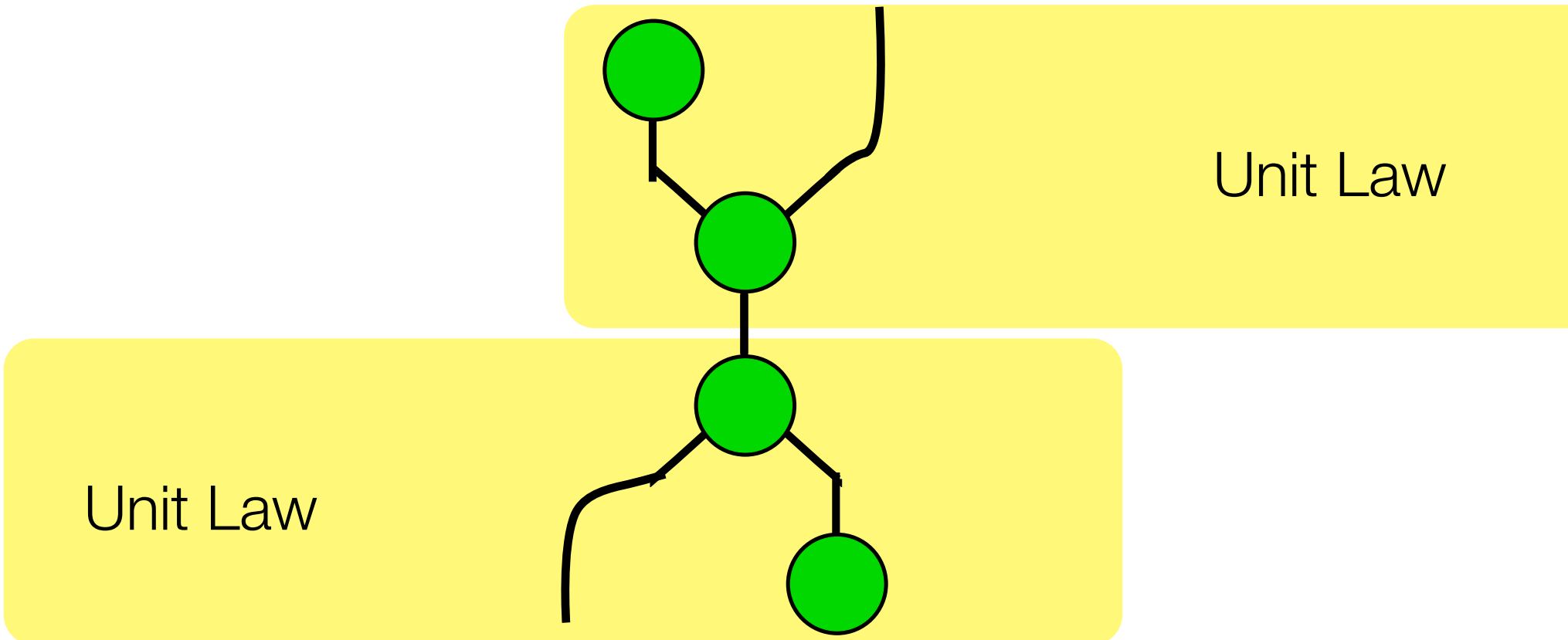

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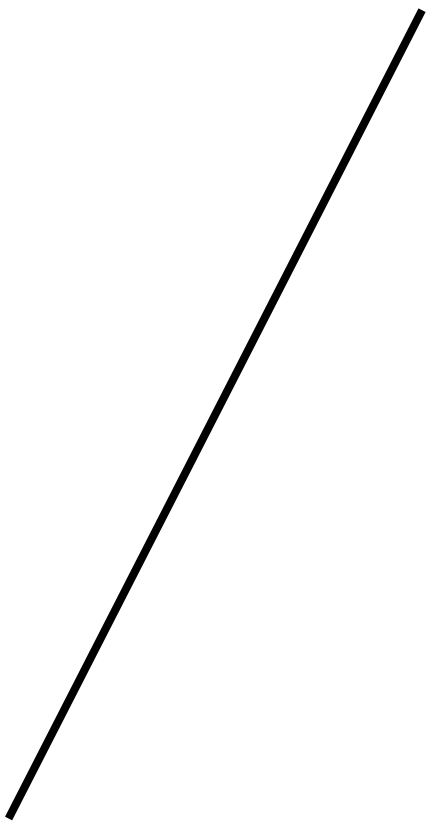

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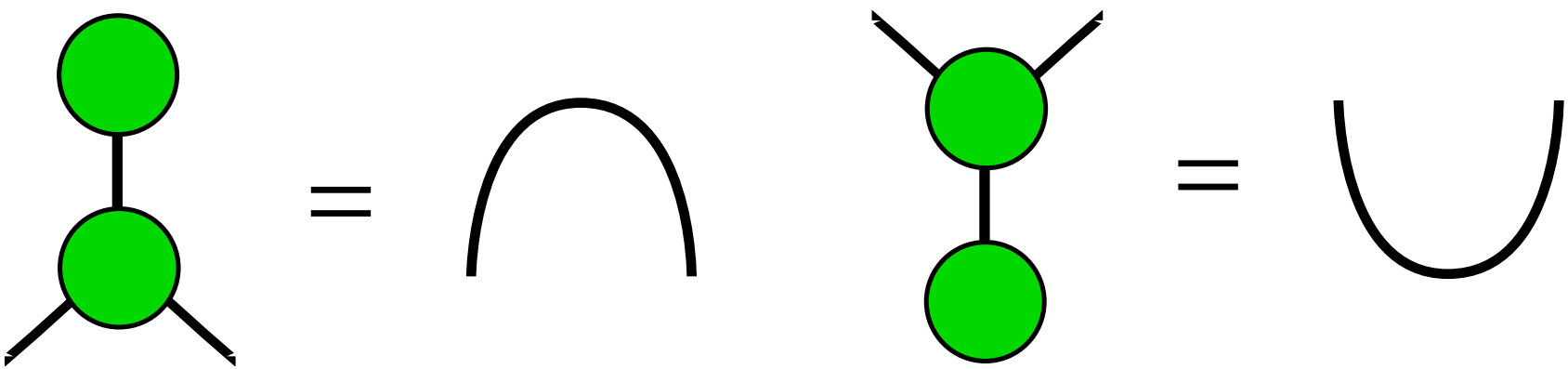
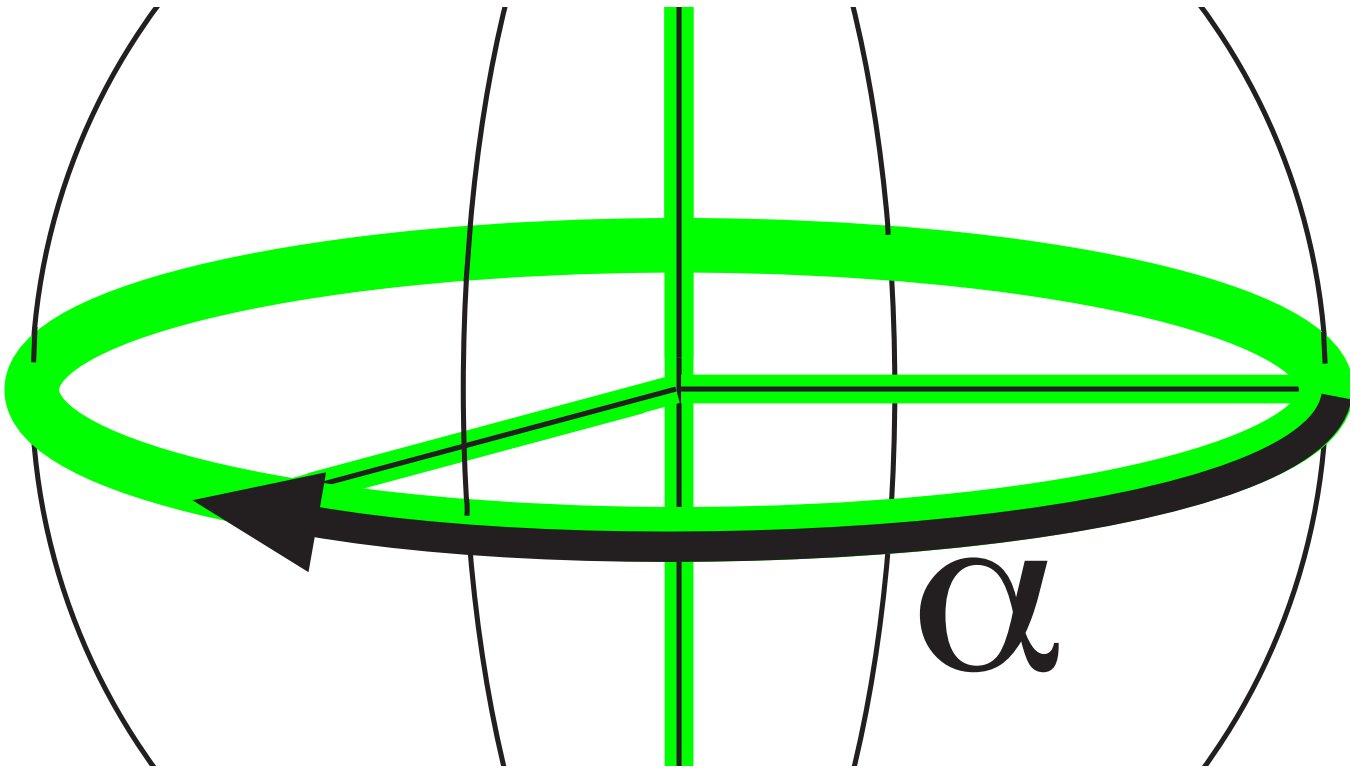


Each classical structure induces a self-dual compact structure.

Self duality is addressed in Coecke, Paquette, & Perdrix 2008

Phase Maps

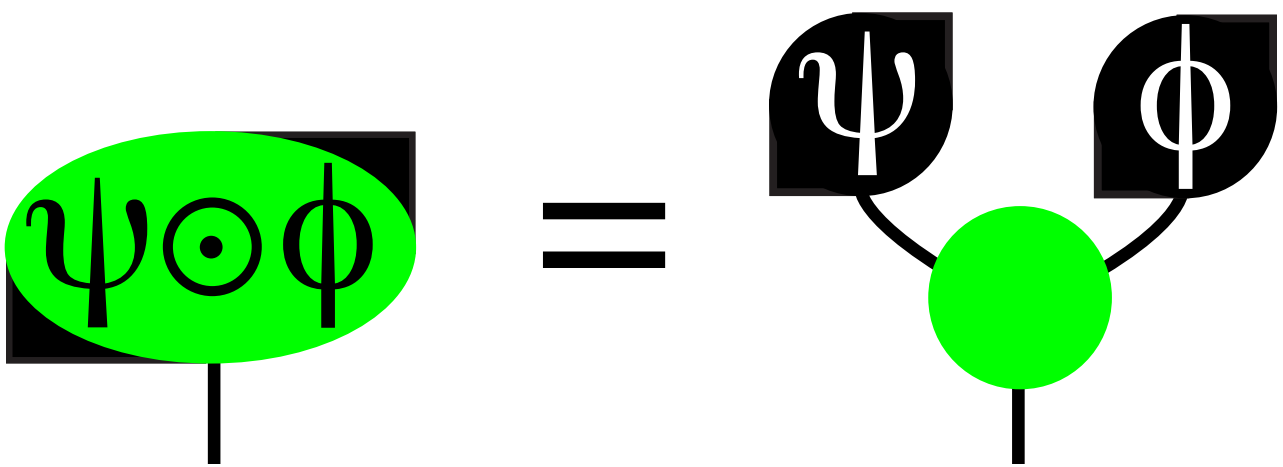


Spinning around an observable

Using the monoid operation

Let $\psi, \phi : I \to A$ be points of A ; we can combine them using the monoid operation $\delta^\dagger : A \otimes A \to A$

$$\psi \odot \phi := \delta^\dagger \circ (\psi \otimes \phi)$$



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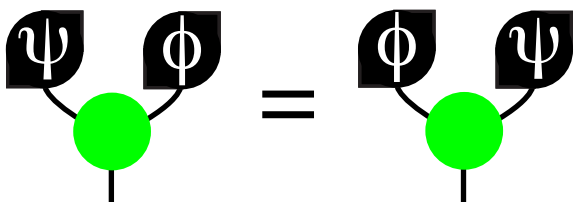

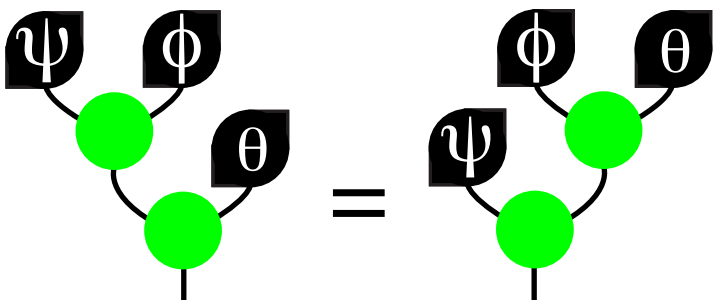

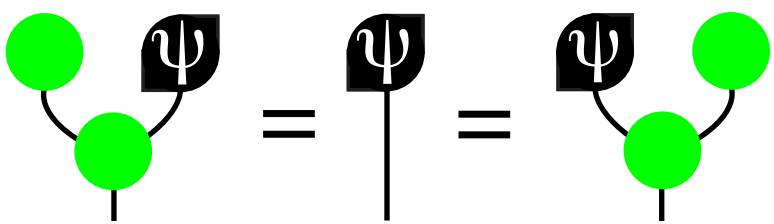

Example: qubits

$$\delta_Z^\dagger : Q \otimes Q \to Q$$

$$|\phi\rangle : I \to Q$$
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$$\delta_Z^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: qubits

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$$|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

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$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \psi_1\phi_1 \\ \psi_1\phi_2 \\ \psi_2\phi_1 \\ \psi_2\phi_2 \end{pmatrix}$$

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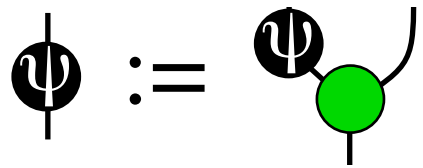
a.k.a. the *Schur product* or *convolution product*

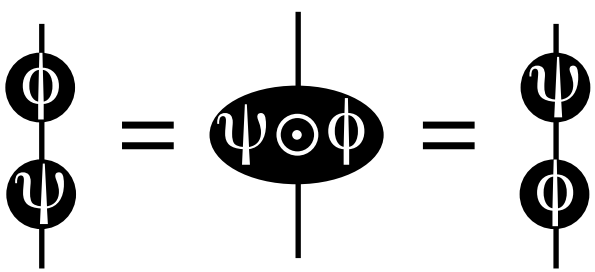
Using the monoid operation

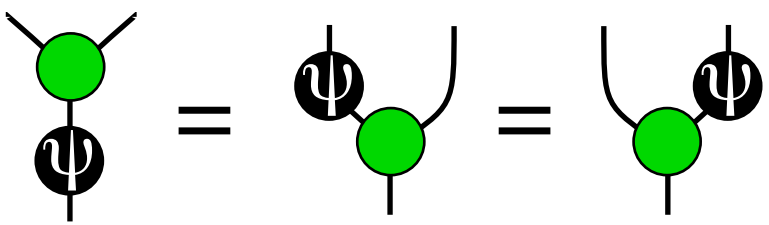
Moreover, each point $\psi : I \to A$ can be lifted to an endomorphism $\Lambda(\psi) : A \to A$

$$\Lambda(\psi) := \delta^\dagger \circ (\psi \otimes \mathrm{id}_A)$$



This yields a homomorphism of monoids so we have:





Example: qubits

$$\delta_Z^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Example: qubits

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$$\mathrm{id} \otimes |\psi\rangle = \begin{pmatrix} \psi_1 & 0 \\ \psi_1 & 0 \\ 0 & \psi_2 \\ 0 & \psi_2 \end{pmatrix}$$

$$\Lambda^Z(\psi) := \delta_Z^\dagger \circ (\mathrm{id} \otimes |\psi\rangle) = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$$

Generalised Spider Theorem

Theorem: any maps constructed from δ, ε, some points $\psi_i : I \to A$ and their adjoints, whose graph is connected, is determined by the number of inputs and outputs and the product $\bigodot_i \psi_i$.

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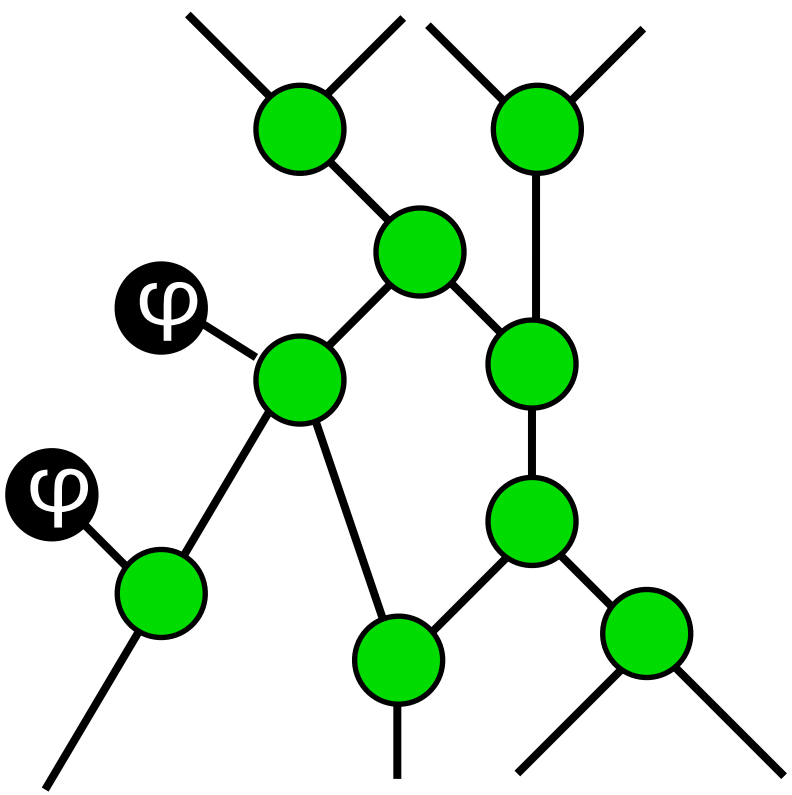

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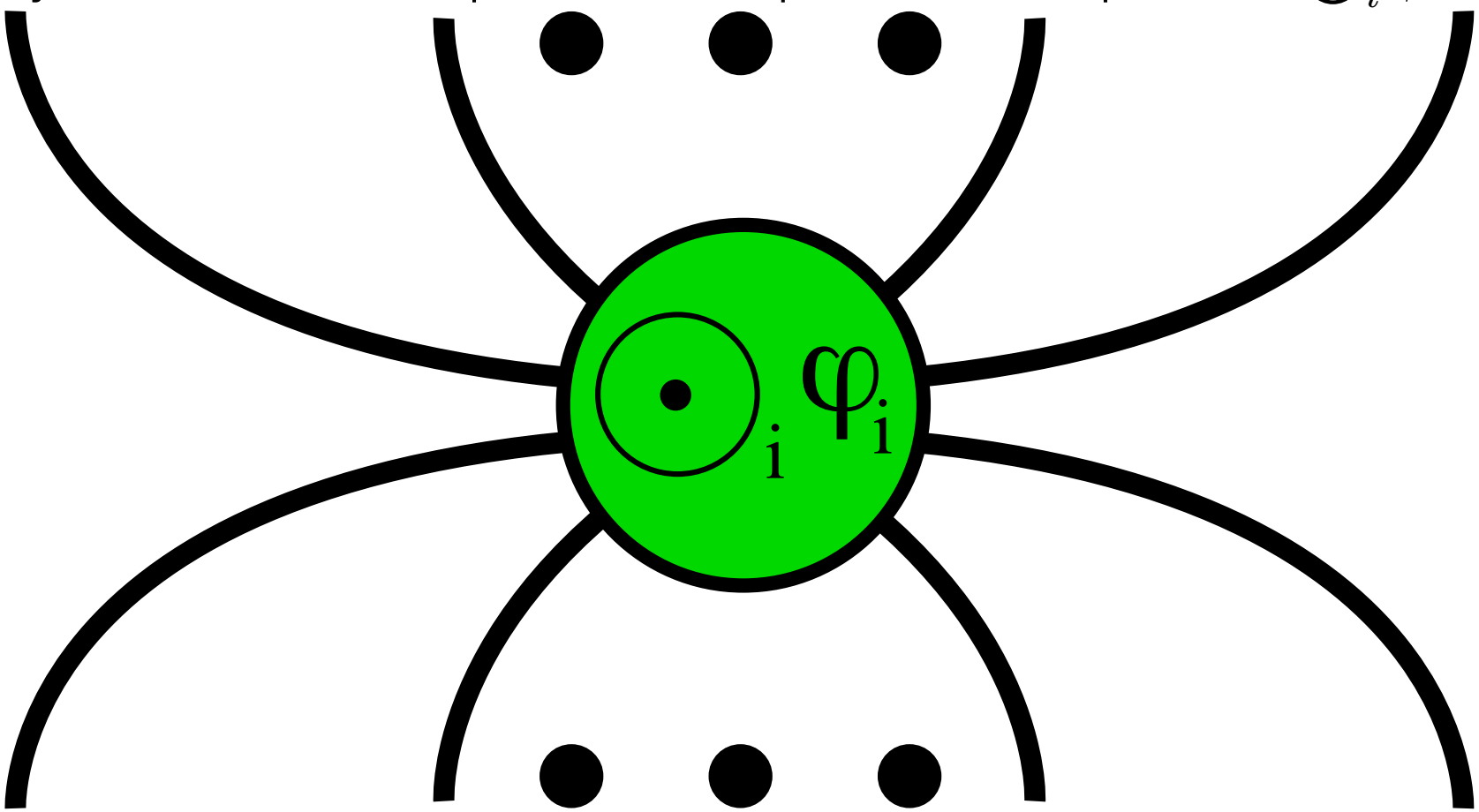
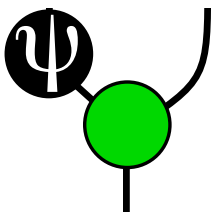


Unbiased Points

Q: When is  unitary?

A: In Hilbert spaces, $\Lambda(\psi)$ is unitary iff $|\psi\rangle$ is unbiased w.r.t. the basis copied by δ.

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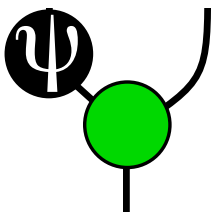
Q: When is 

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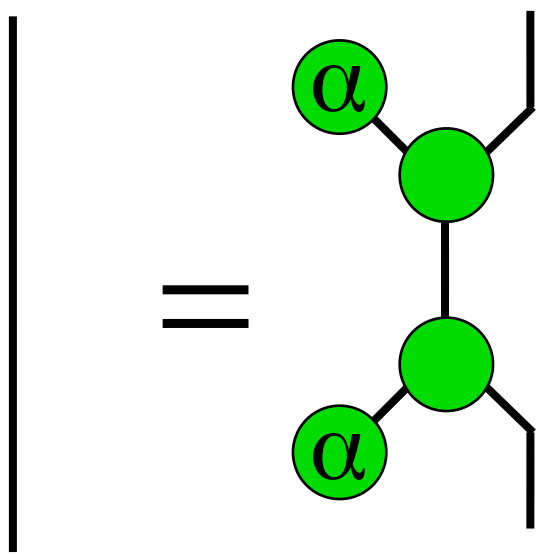

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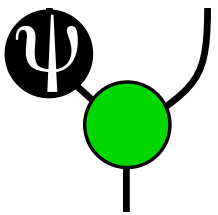
Q: When is 

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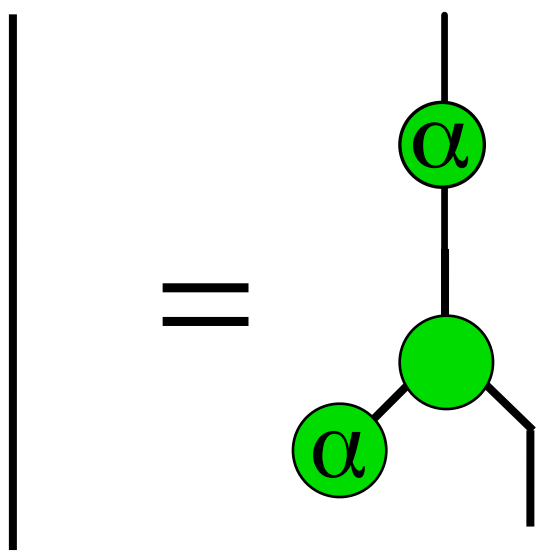

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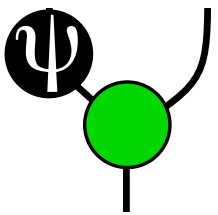
Q: When is 

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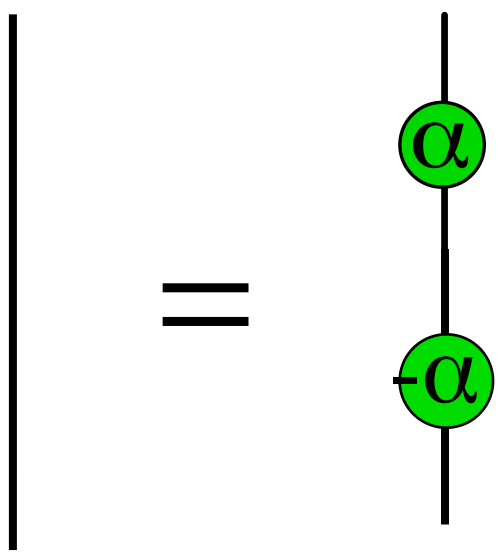

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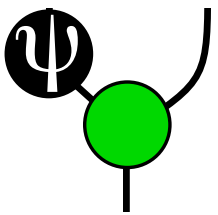
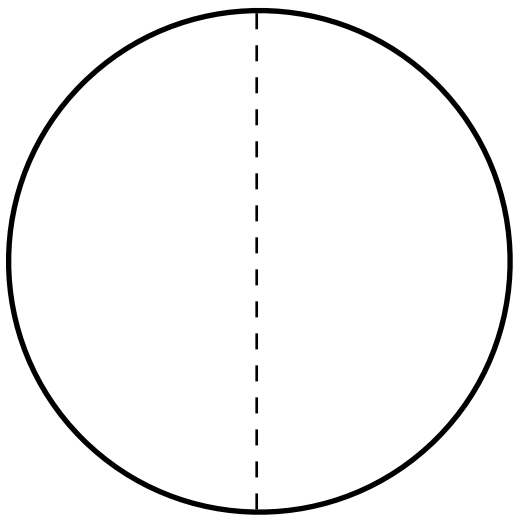
Q: When is  unitary?

A: In Hilbert spaces, $\Lambda(\psi)$ is unitary iff $|\psi\rangle$ is unbiased w.r.t. the basis copied by δ.

Prop:
1. the unbiased points for (δ, ϵ) form an abelian group w.r.t. $\odot$
2. the arrows generated by the unbiased points form an abelian group w.r.t. composition.

Example: qubits



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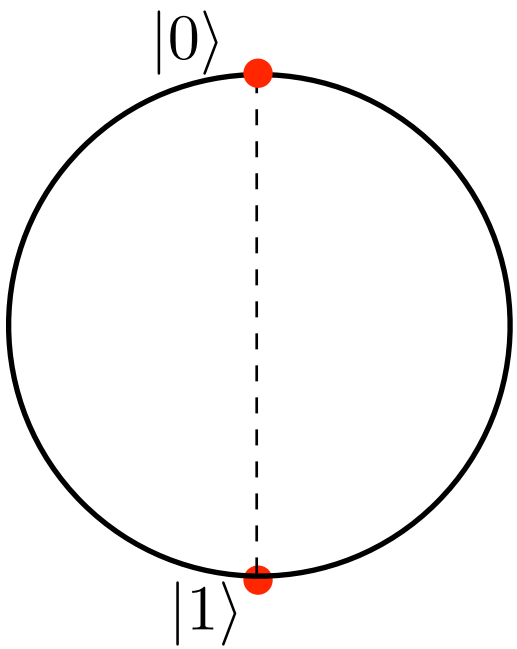

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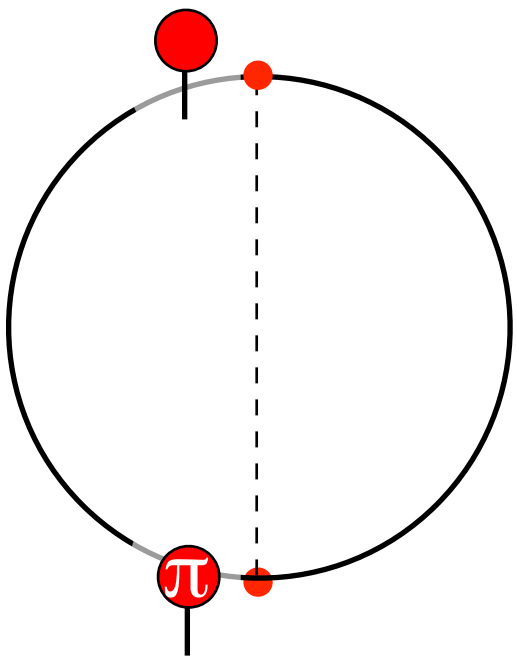

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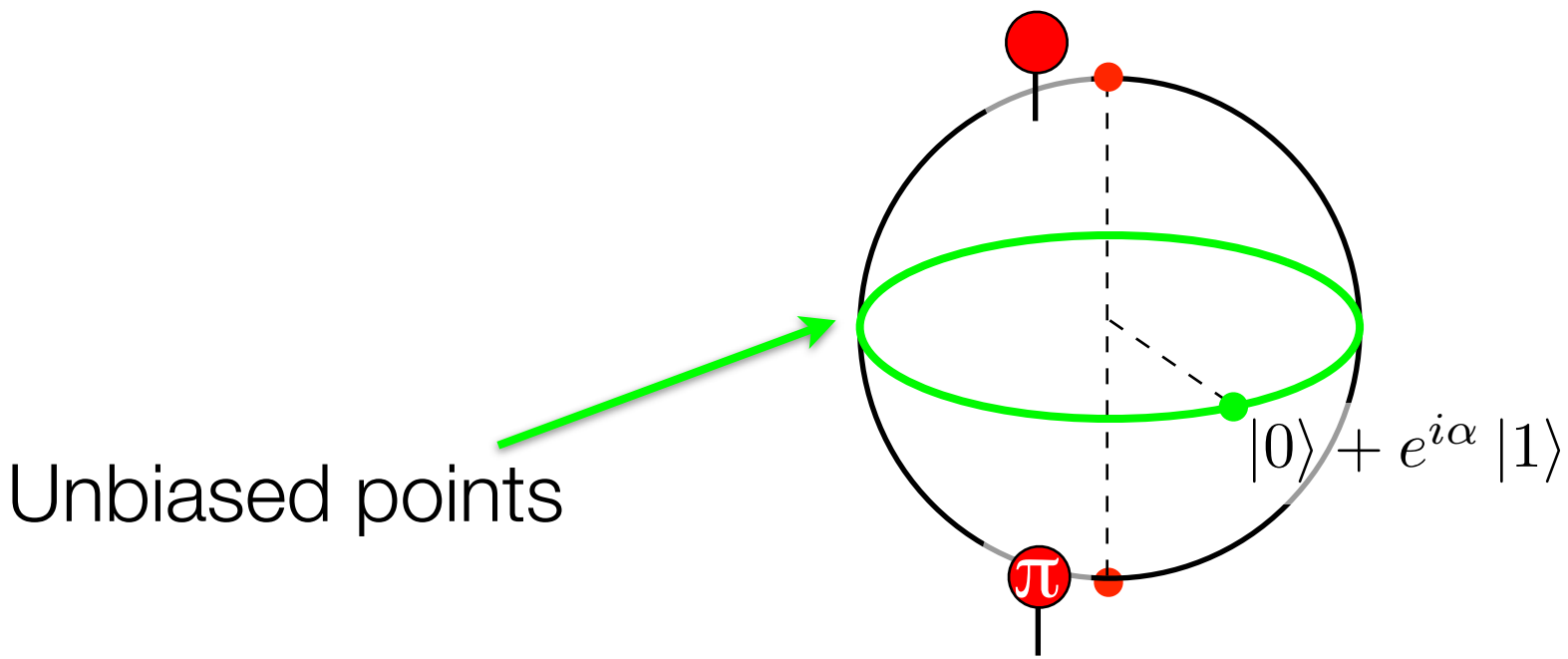

Example: qubits

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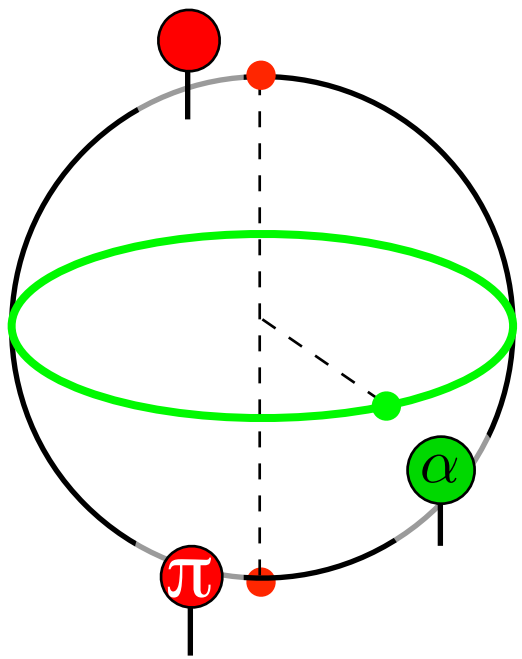

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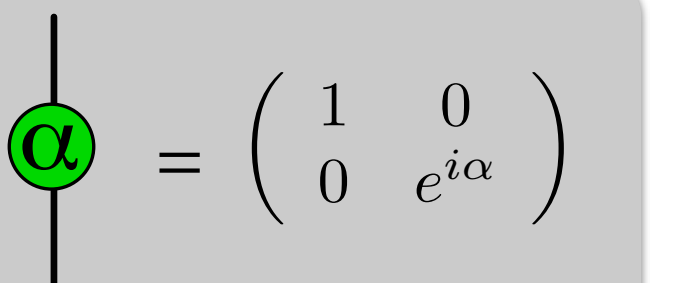

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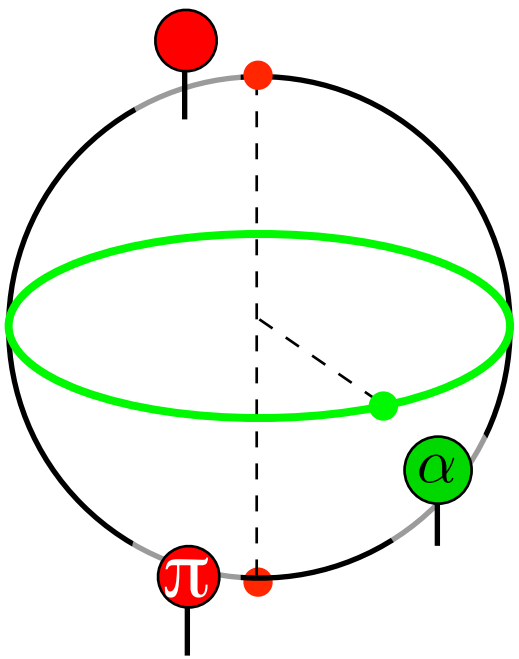


Example: $\mathbf{FRel}$

$$D^\dagger : (x, x) \sim x, \forall x \in X$$

$D^\dagger \circ (\mathrm{id} \otimes \psi)$ is unitary iff $\psi = X$

Hence the phase group is trivial.

Example: $\mathbf{FRel}$

Suppose that we are working with a two element set = {0,1}.
Then:

$$D : \begin{array}{l} 0 \sim \{(0,0),(1,1)\} \\ 1 \sim \{(0,1),(1,0)\} \end{array}$$

$$E : 0 \sim *$$

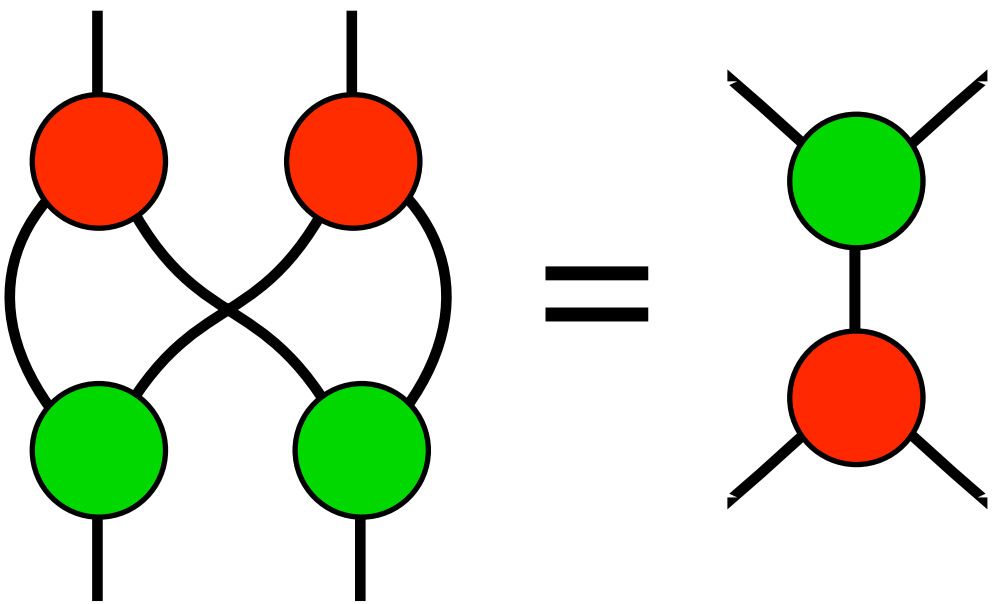
Defines a classical structure whose phase group is $\mathbb{Z}_2$

Example: $\mathbf{FRel}$

Theorem (Pavlovic) : The classical structures of $\mathbf{FRel}$ are exactly biproducts of Abelian groups.

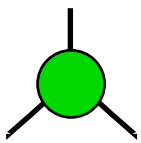
It's easy to check that in this gives a phase group which is just the product group

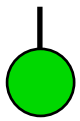
Complementary Observables

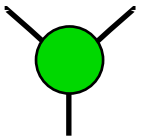


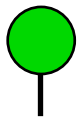
A very general theory of interference

Two kinds of points

$\delta =$ 

$\epsilon =$ 

$\delta^\dagger =$ 

$\epsilon^\dagger =$ 

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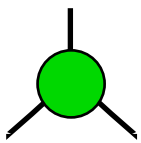
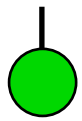
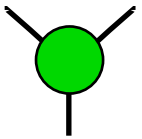
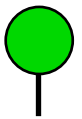
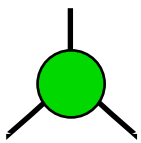
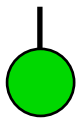
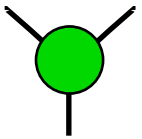
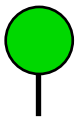
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Classical Points

Those points which can be copied by δ

Two kinds of points

$\delta =$ 
$\epsilon =$ 
$\delta^\dagger =$ 
$\epsilon^\dagger =$ 

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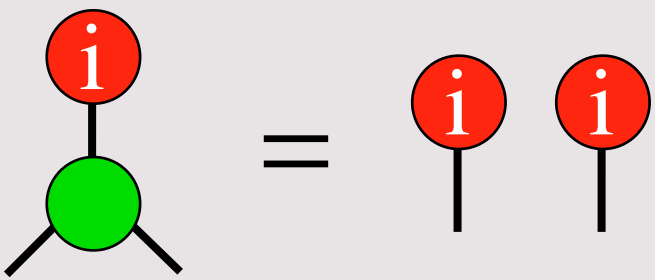
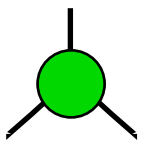
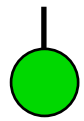
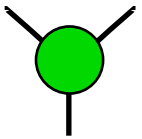
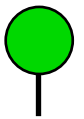


Those points which can be copied by δ

Two kinds of points

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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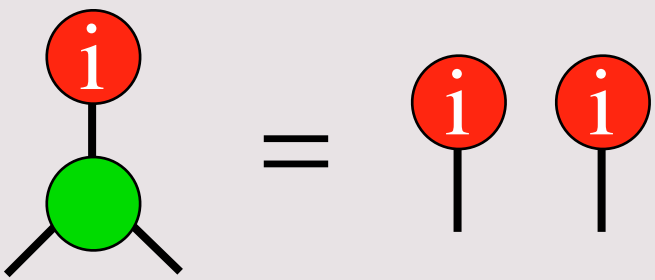
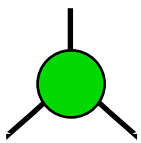
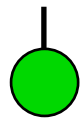
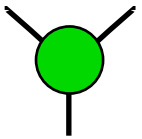
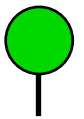


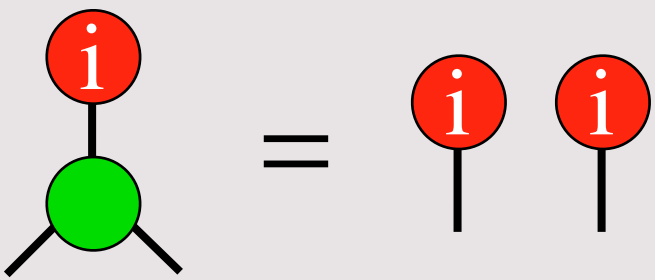
Those points which can be copied by δ

* to keep the pictures tidy, a scalar factor has been omitted (and everywhere from here on)

Two kinds of points

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

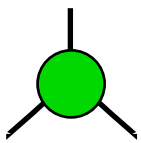
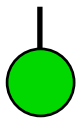
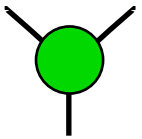
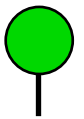
Classical Points



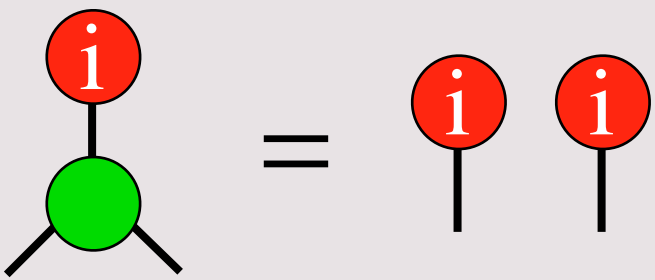
Those points which can be copied by δ

Unbiased Points

Two kinds of points

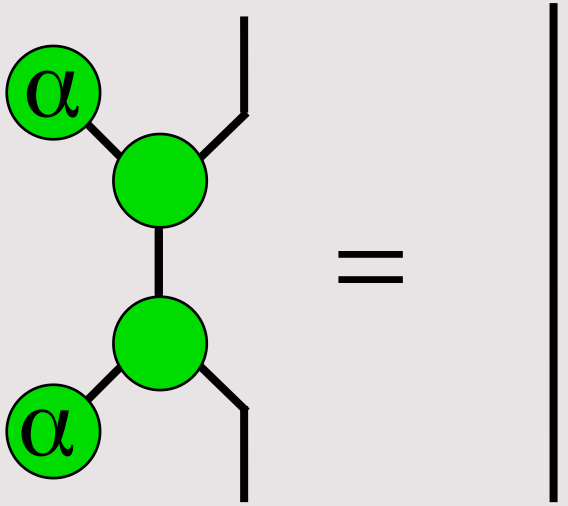
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Classical Points

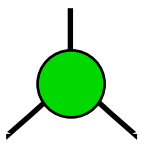
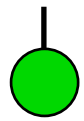
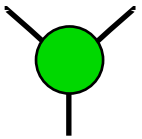
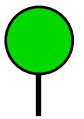


Those points which can be copied by δ

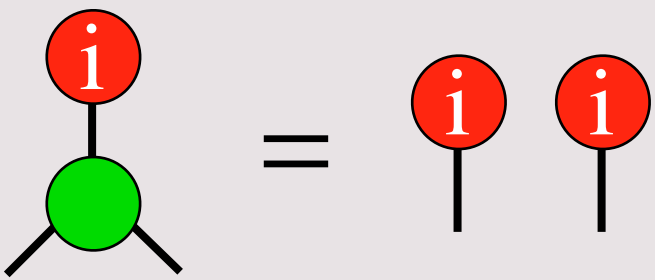
Unbiased Points



Two kinds of points

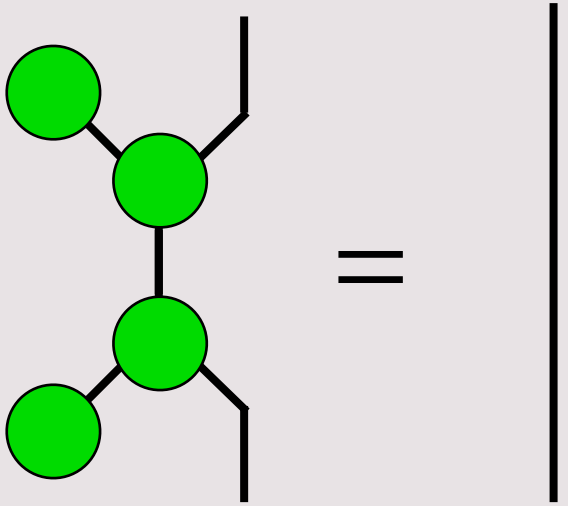
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Classical Points

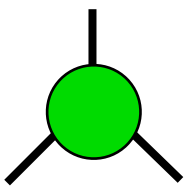
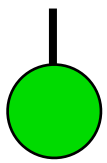


Those points which can be copied by δ

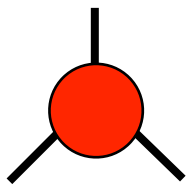
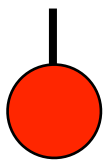
Unbiased Points



Complementary Classical Structures

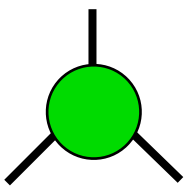
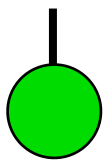
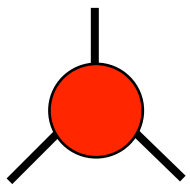
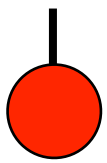
$\delta_Z =$  $\epsilon_Z =$ 

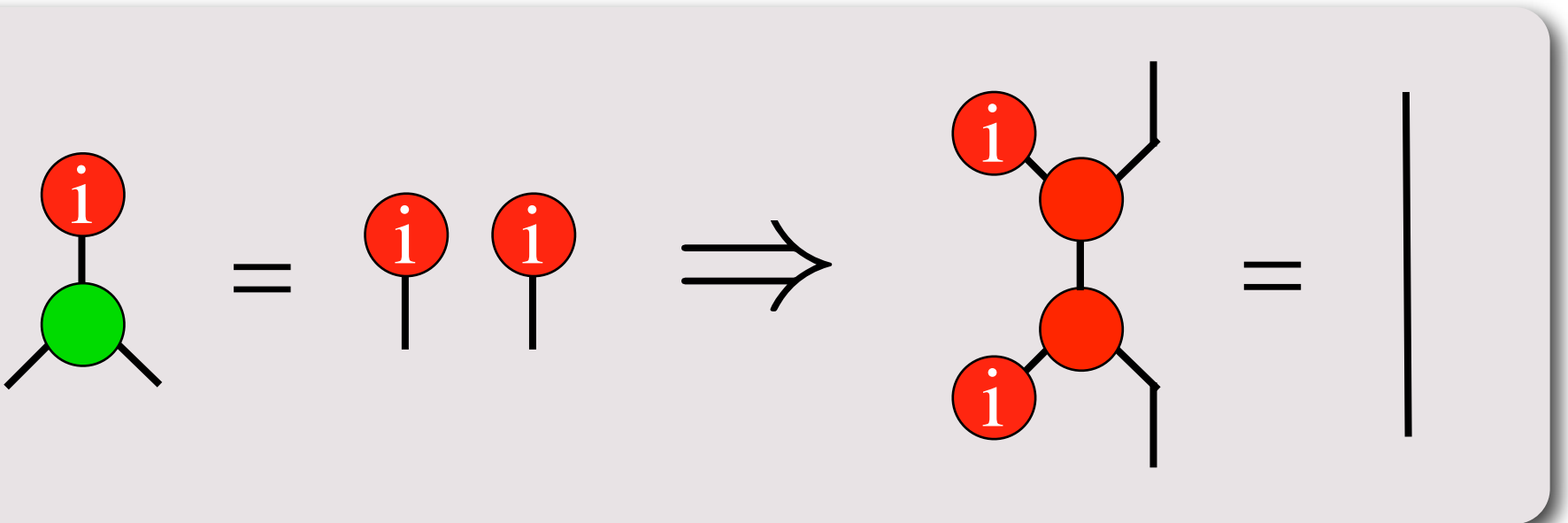
$|i\rangle \mapsto |ii\rangle$ $|+\rangle \mapsto 1$

$\delta_X =$  $\epsilon_X =$ 

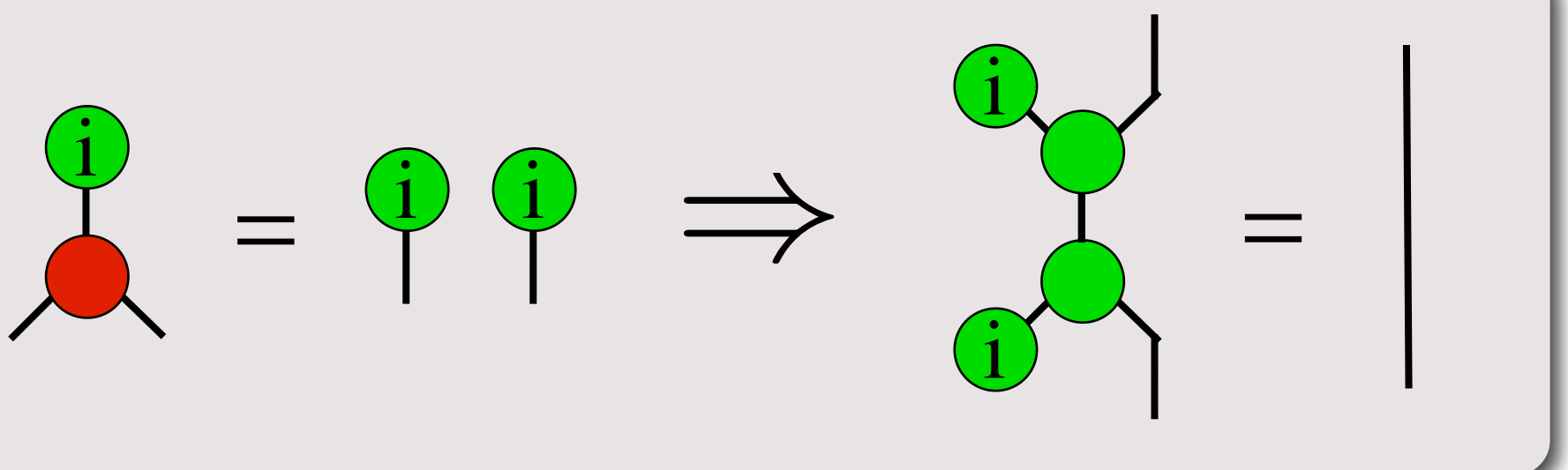
$|\pm\rangle \mapsto |\pm\pm\rangle$ $|0\rangle \mapsto 1$

Complementary Classical Structures

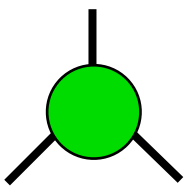
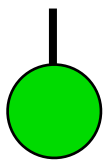
$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

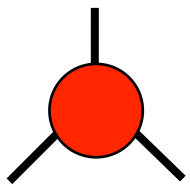
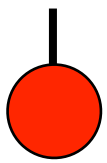


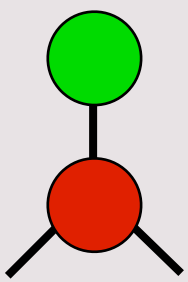
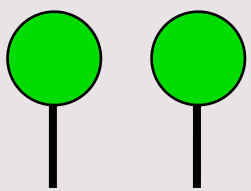
Complementary Classical Structures

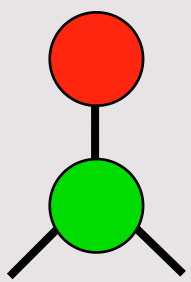
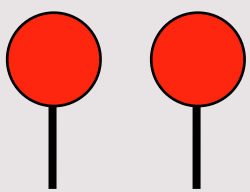


Complementary Classical Structures

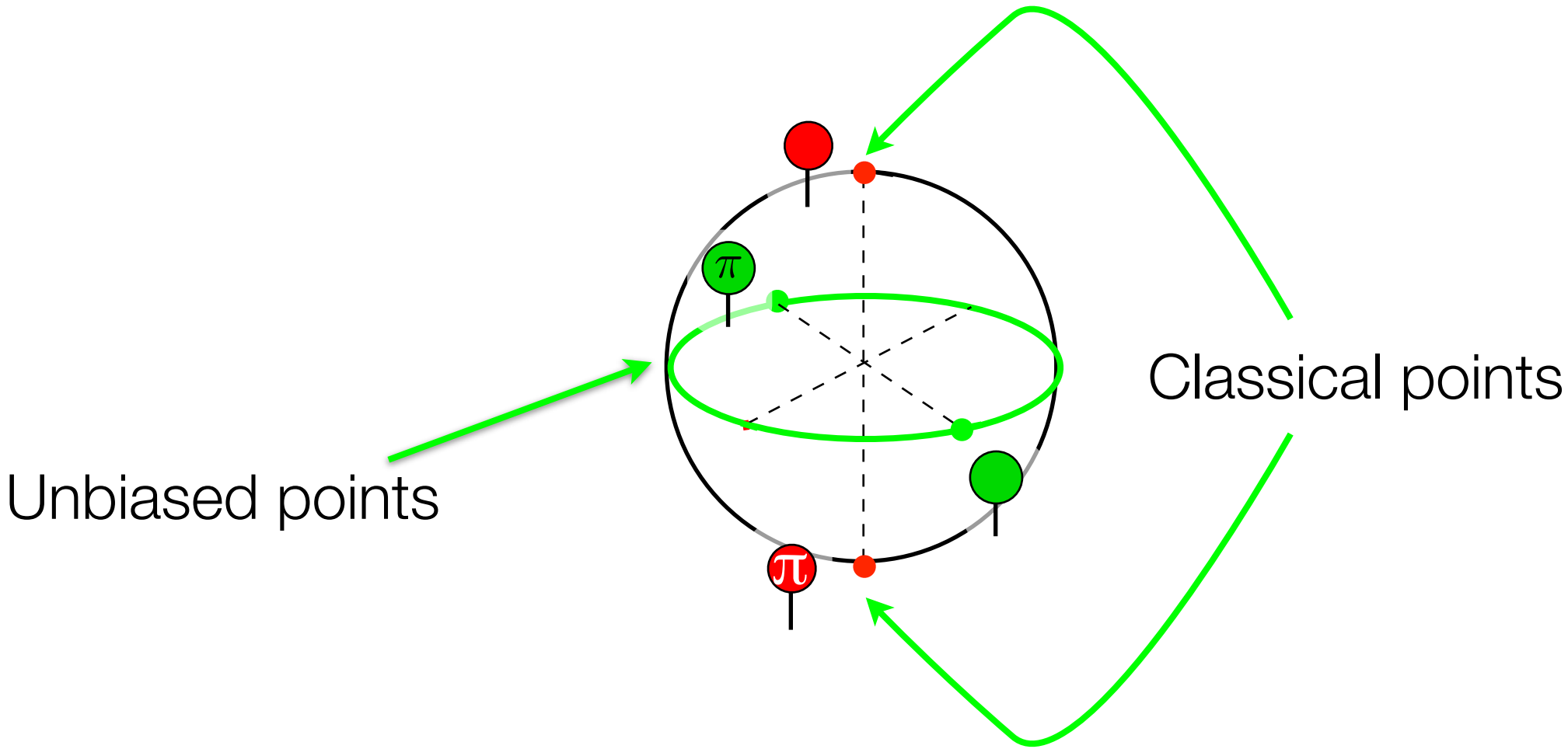
$\delta_Z =$  $\epsilon_Z =$ 

$\delta_X =$  $\epsilon_X =$ 

 = 

 = 

Example: qubits

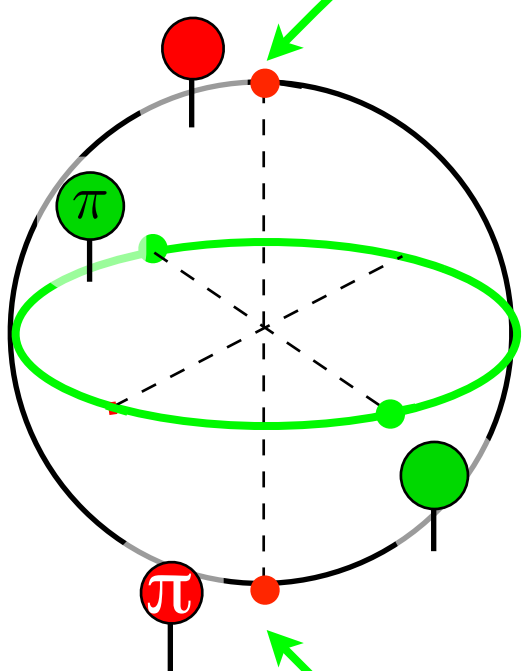


Example: qubits

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$|0\rangle =$$

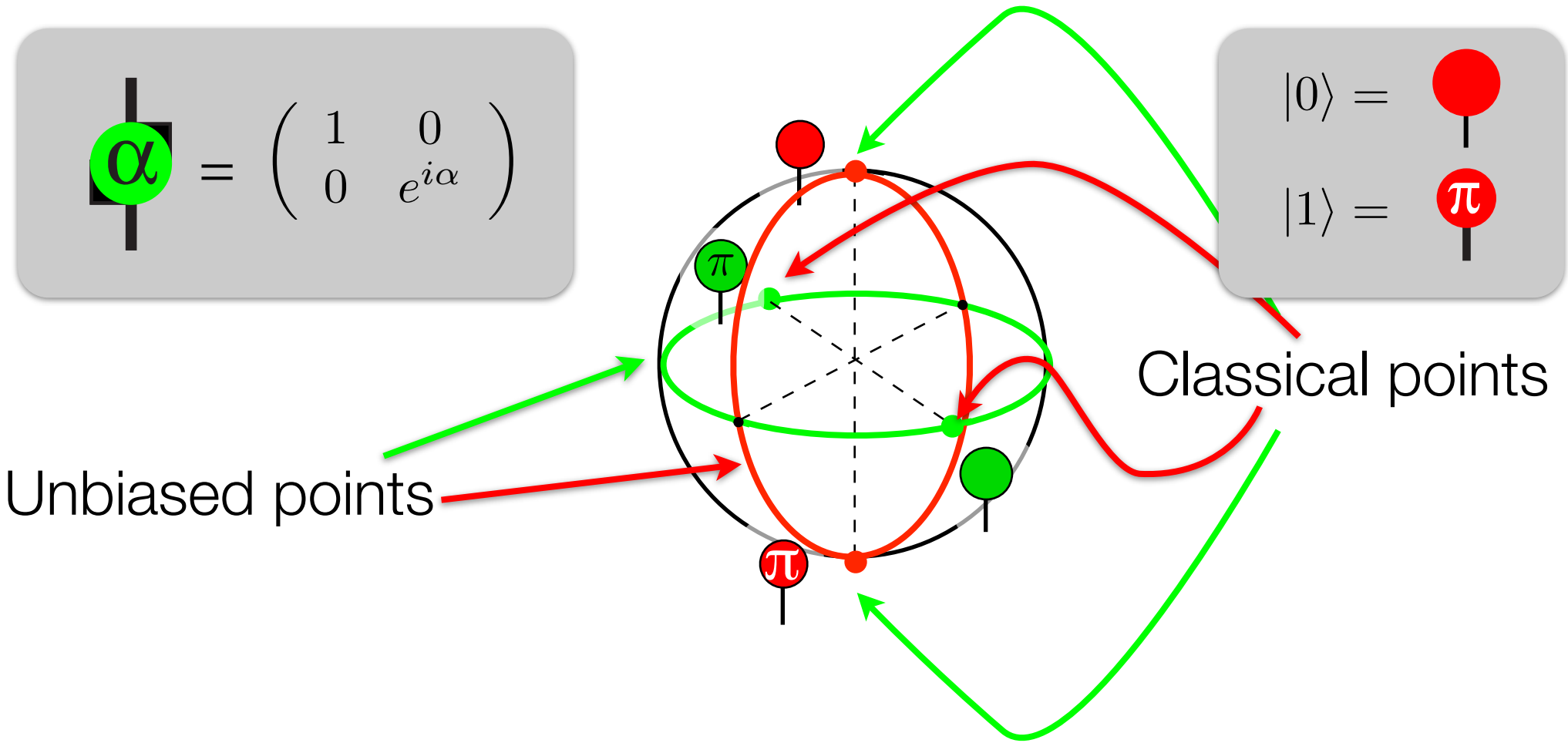
$$|1\rangle =$$



Unbiased points

Classical points

Example: qubits



Example: qubits

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

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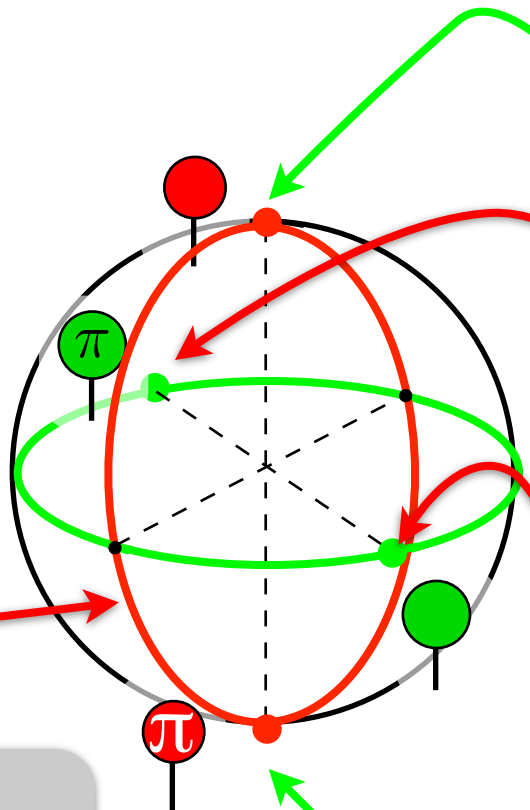


Classical points

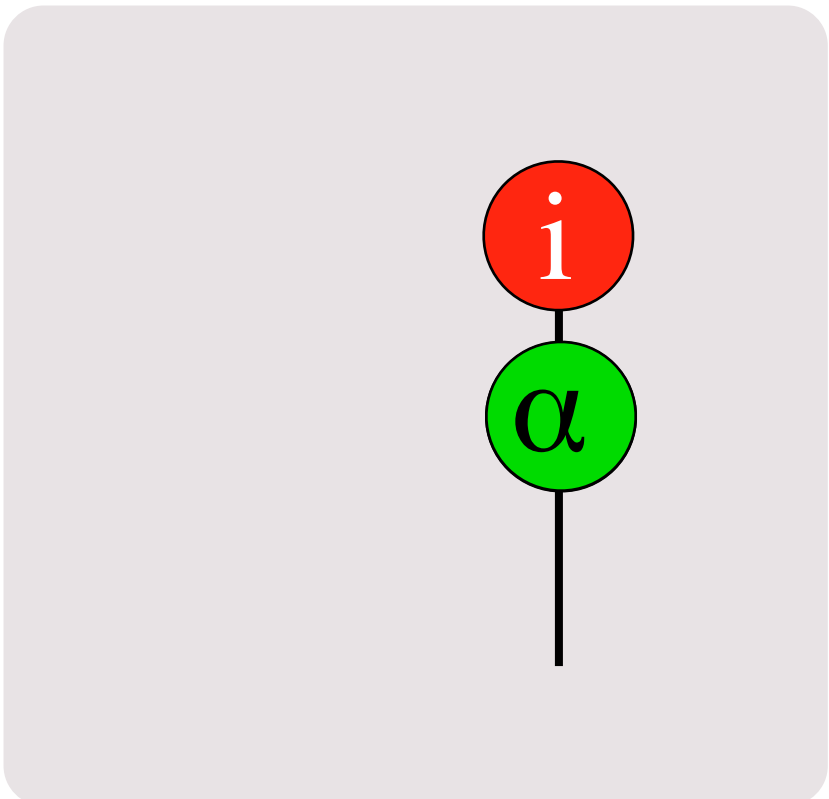
Unbiased points

$$\alpha = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

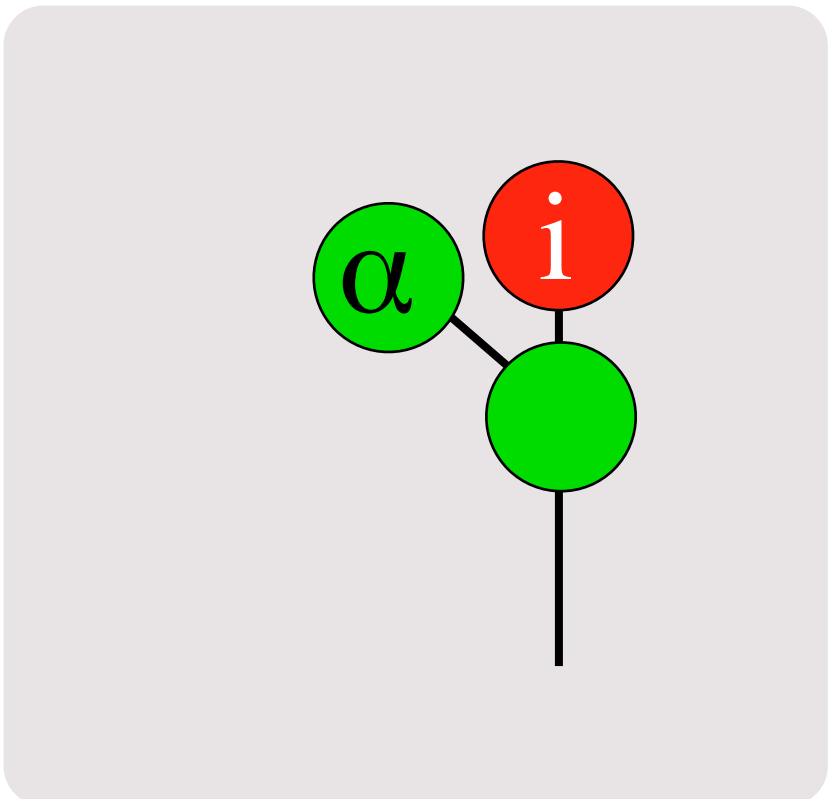
$$|+\rangle =$$

$$|-\rangle =$$

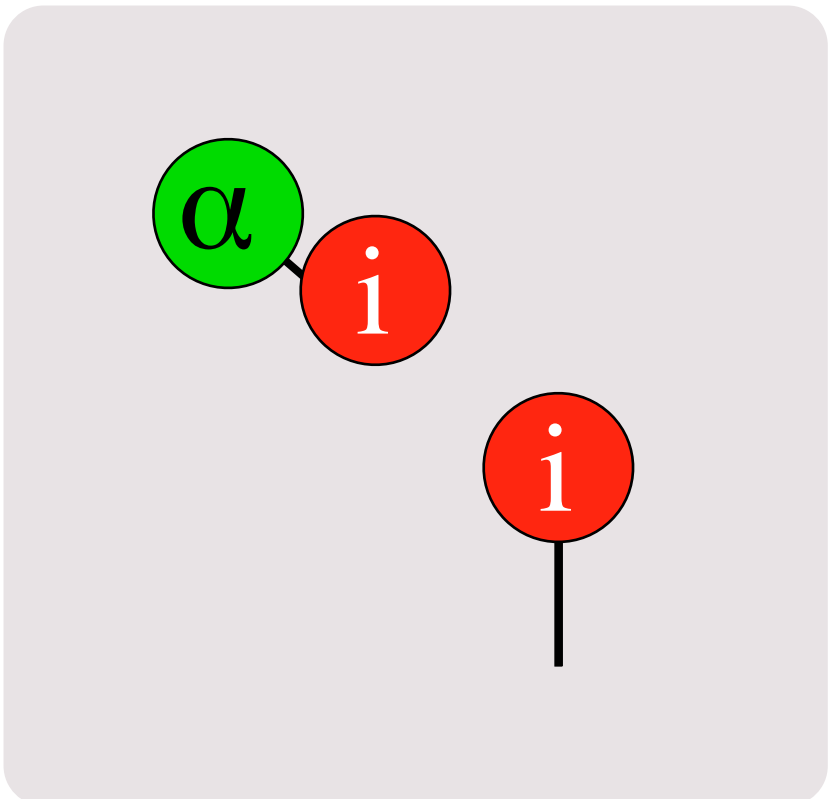
Classical points are eigenvectors



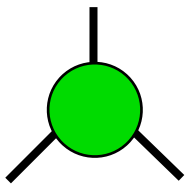
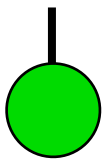
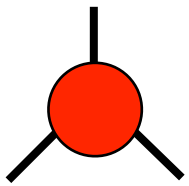
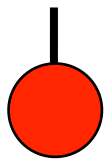
Classical points are eigenvectors



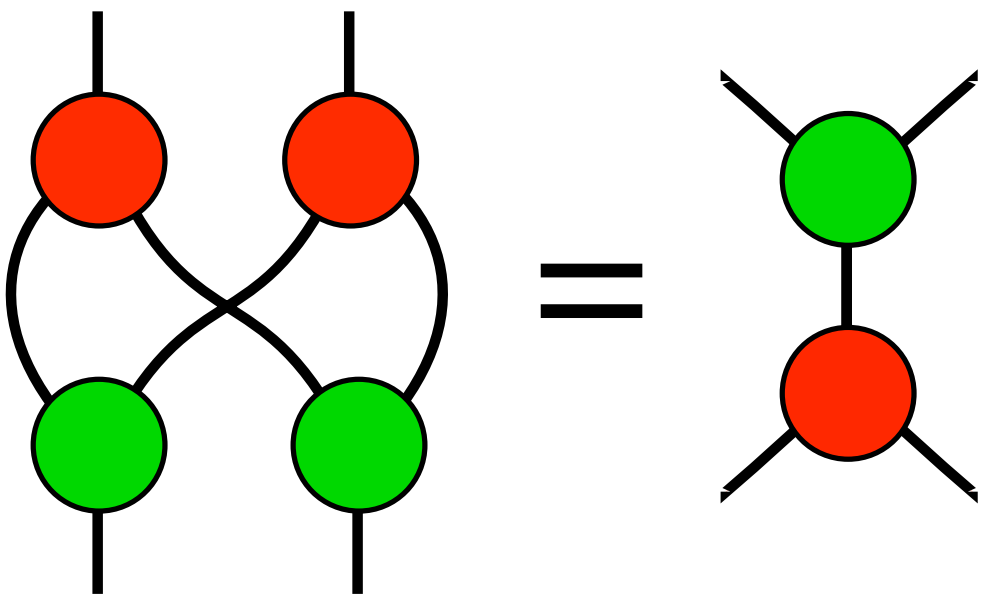
Classical points are eigenvectors



Strong Complementarity

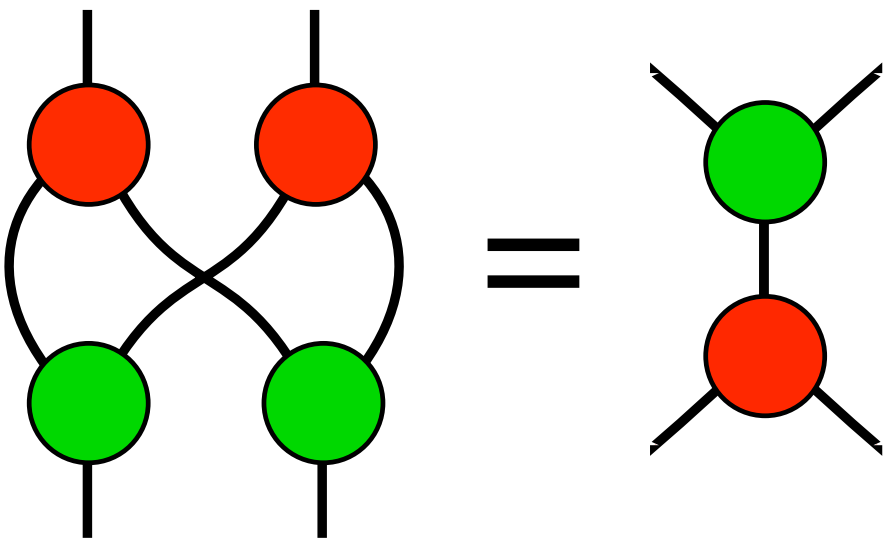
$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

A pair of complementary observables are called *strongly complementary* when they satisfy this equation:



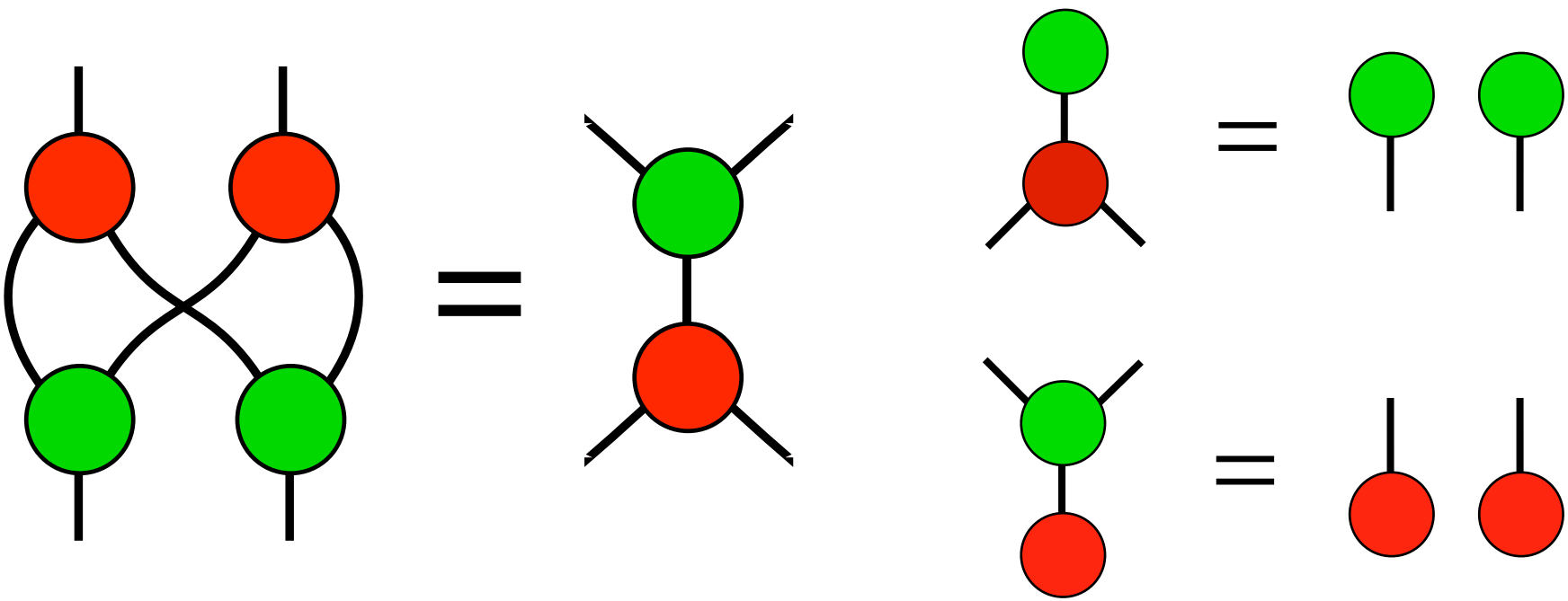
Strong Complementarity

Corollary: strongly complementary observables form a *bialgebra*

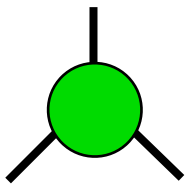
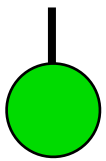


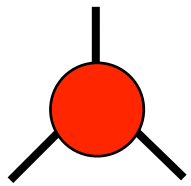
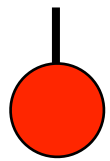
Strong Complementarity

Corollary: strongly complementary observables form a *bialgebra*

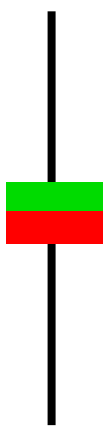
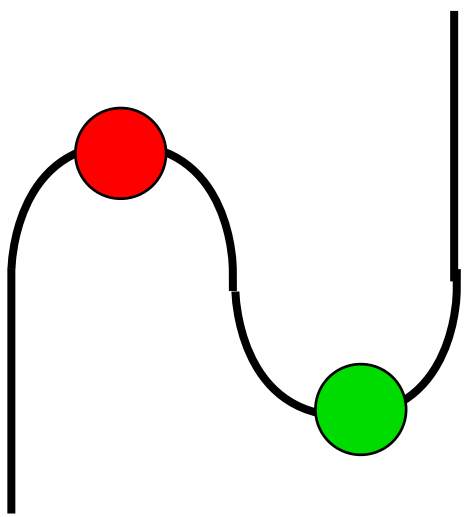


The antipode

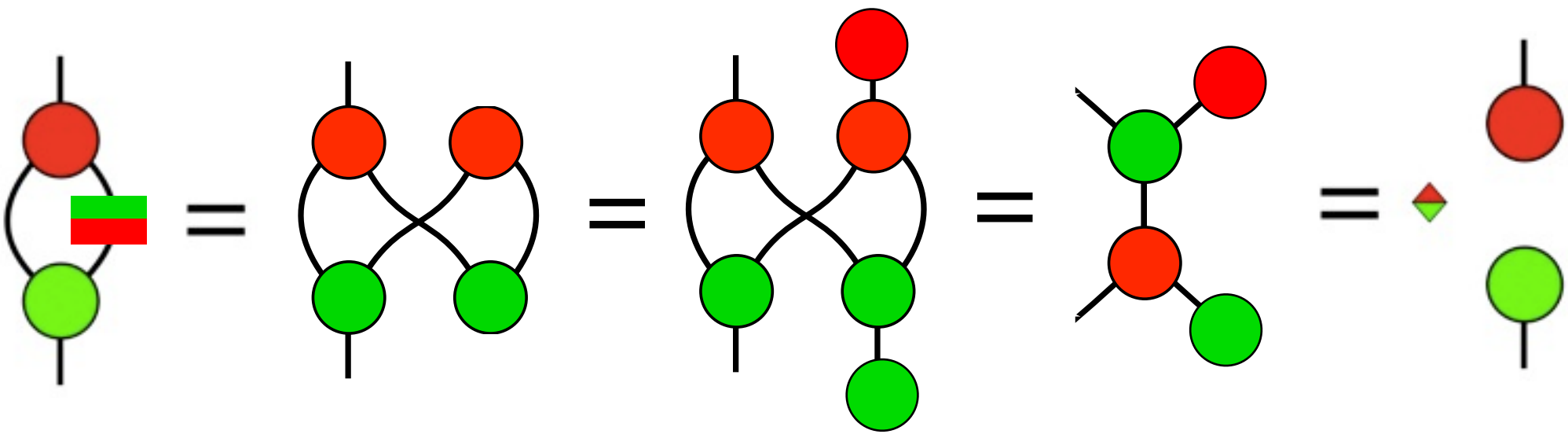
$\delta_Z =$  $\epsilon_Z =$ 

$\delta_X =$  $\epsilon_X =$ 

Define the map S by:

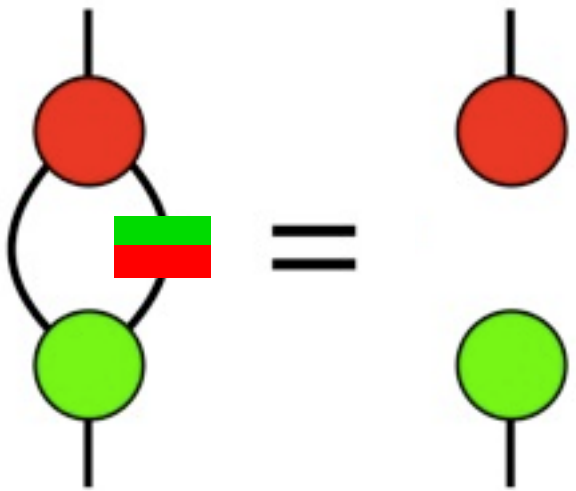
$S =$  $:=$ 

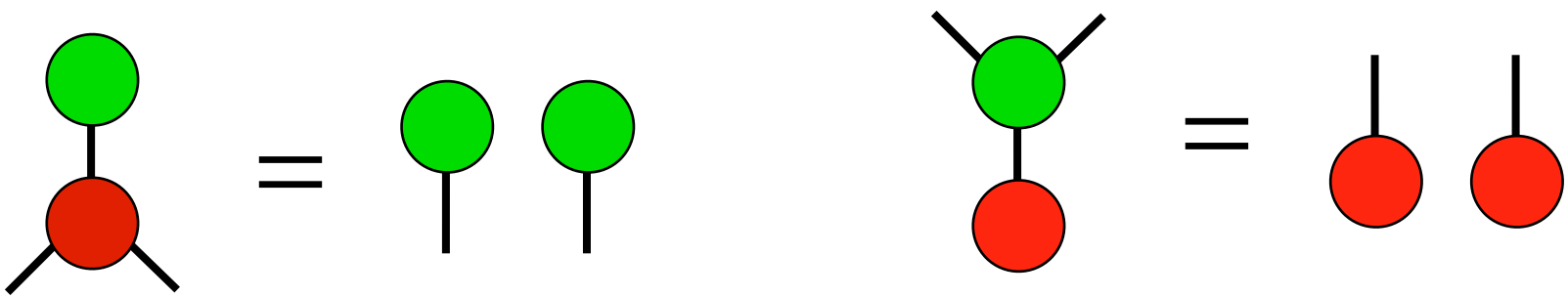
The antipode



Strongly complementary observables form Hopf algebras

Theorem:



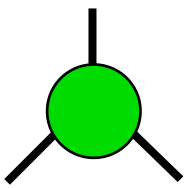
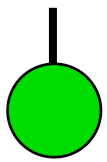


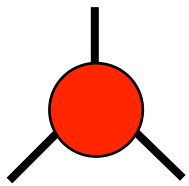
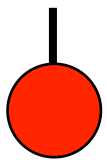
Remark: under the assumption of *enough classical points* the “strong” assumption is not needed; simple complementarity suffices

Strong Complementarity

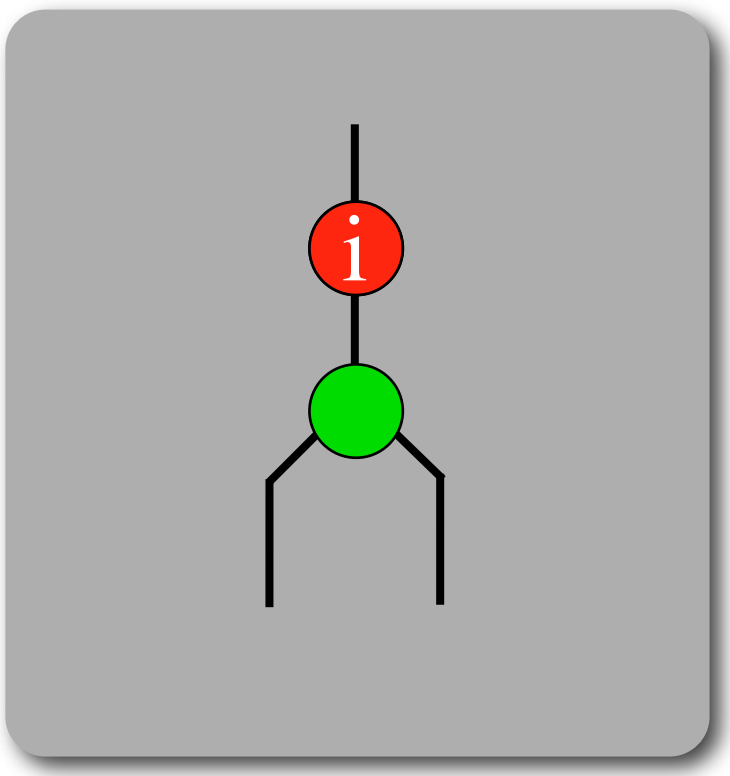
The following are equivalent characterisations of strong complementarity:

Comonoid Homomorphism

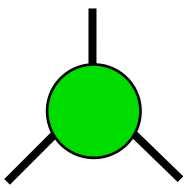
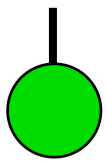
$\delta_Z =$  $\epsilon_Z =$ 

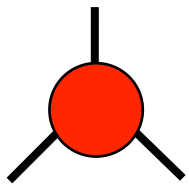
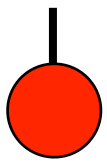
$\delta_X =$  $\epsilon_X =$ 

The classical maps are comonoid homomorphisms

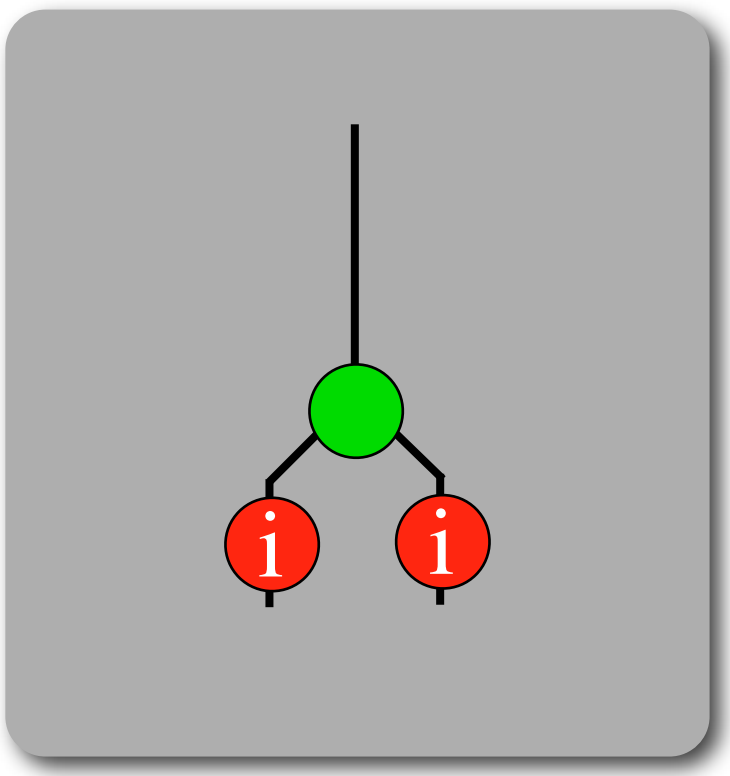


Comonoid Homomorphism

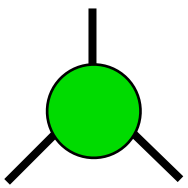
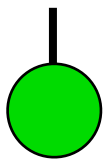
$\delta_Z =$  $\epsilon_Z =$ 

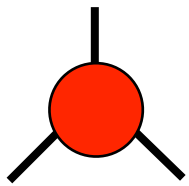
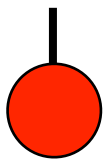
$\delta_X =$  $\epsilon_X =$ 

The classical maps are comonoid homomorphisms

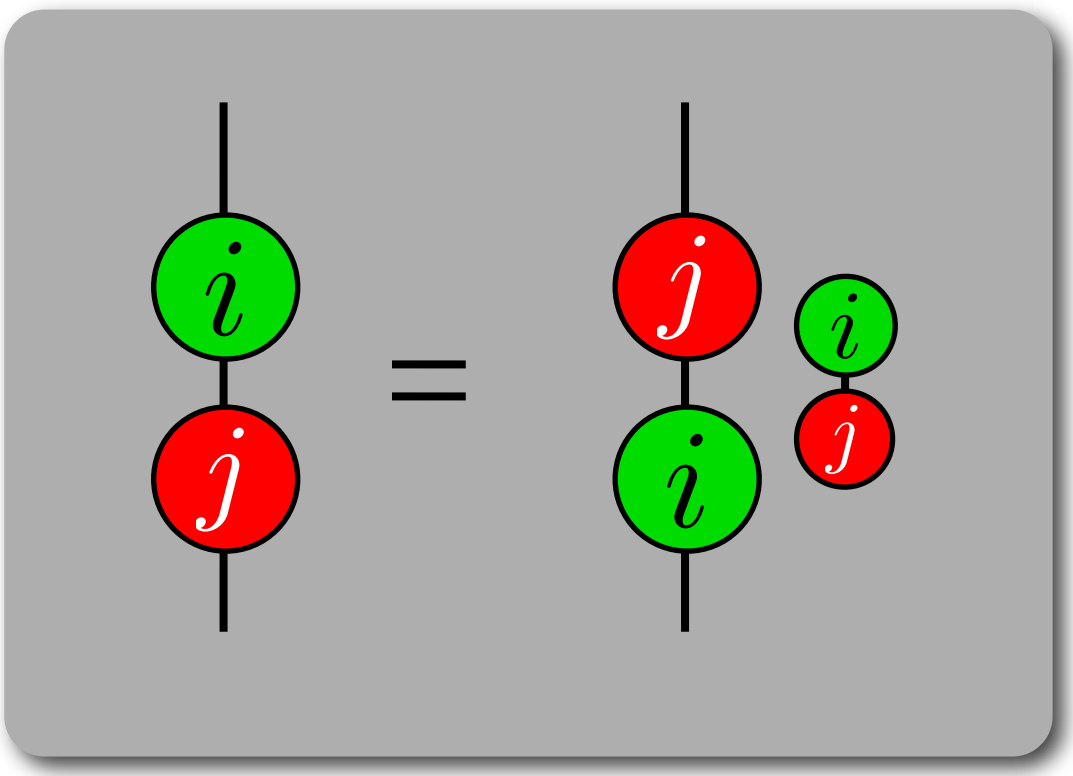


Classical Phases Commute

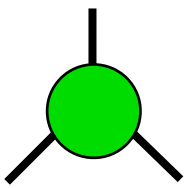
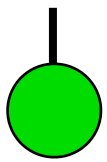
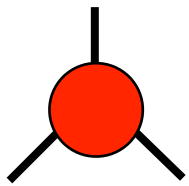
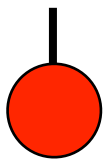
$\delta_Z =$  $\epsilon_Z =$ 

$\delta_X =$  $\epsilon_X =$ 

The classical maps satisfy canonical commutation relations

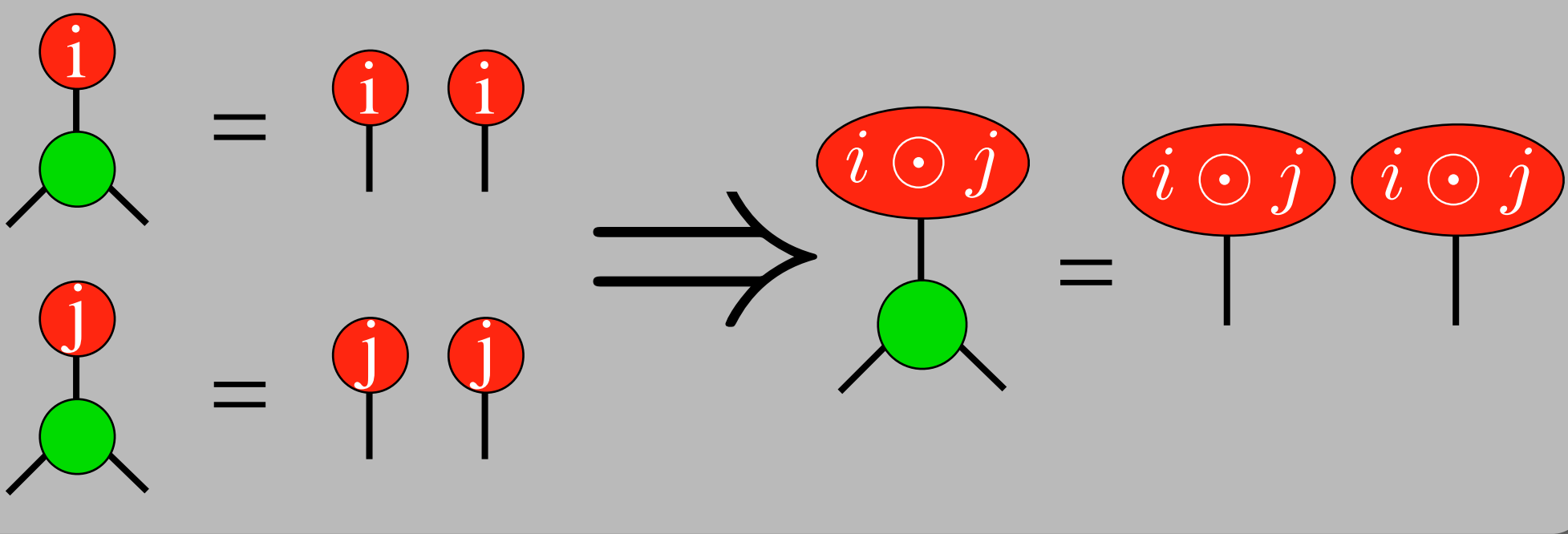


Closedness Property

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

Defn: complentary classical structures are called *closed when...*

Closedness Property



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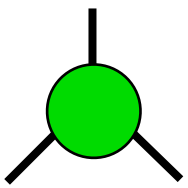
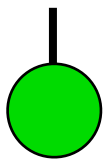
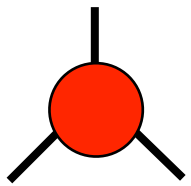
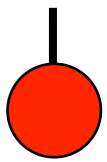
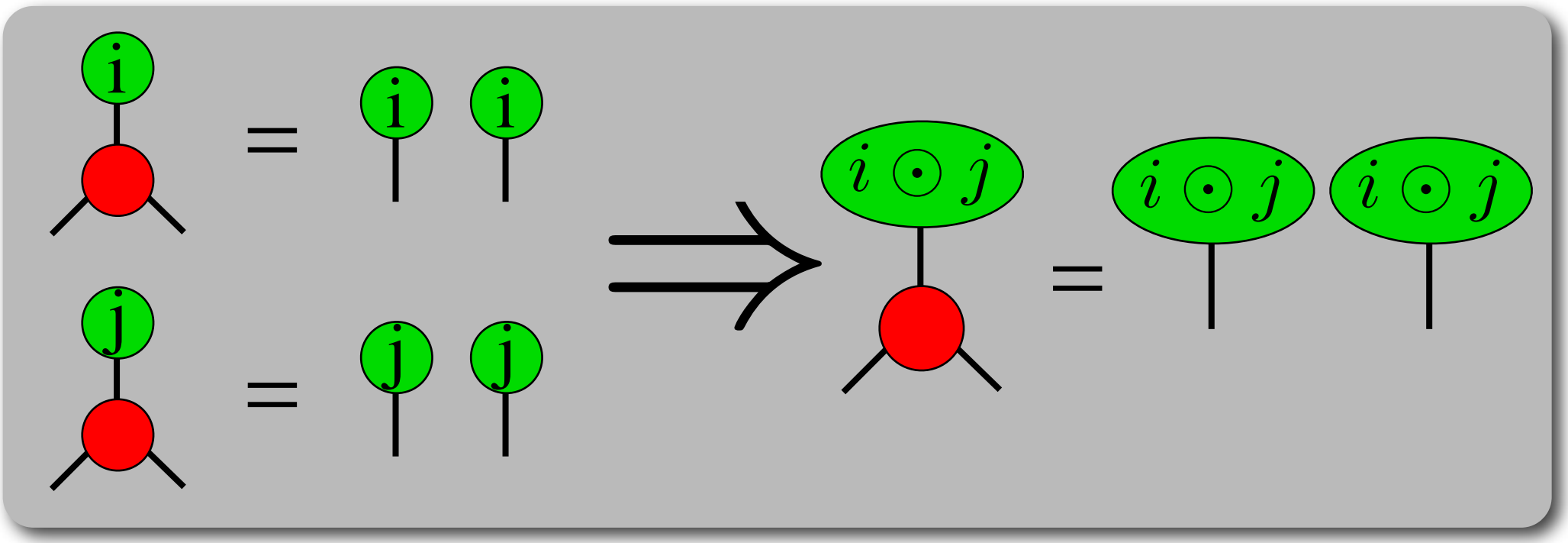
$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 



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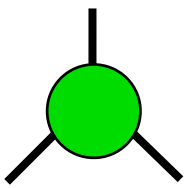
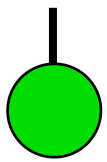
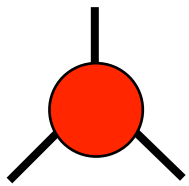
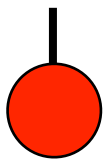
$\delta_Z =$  $\epsilon_Z =$ 

$\delta_X =$  $\epsilon_X =$ 

In **fdHilb** it is always possible to construct a pair of strong complementary observables...

... but in big enough dimension, it is possible to construct MUBs which are not closed.

Classical Points form a Subgroup

By the definition of complementarity, we have that:

$$C_X \subseteq U_Z$$
$$C_Z \subseteq U_X$$

Prop: If the observable structure is closed, and there are finitely many classical points, then they form a *subgroup* of the unbiased points, i.e.

$$(C_Z, \odot_X) \leq (U_X, \odot_X)$$
$$(C_X, \odot_Z) \leq (U_Z, \odot_Z)$$

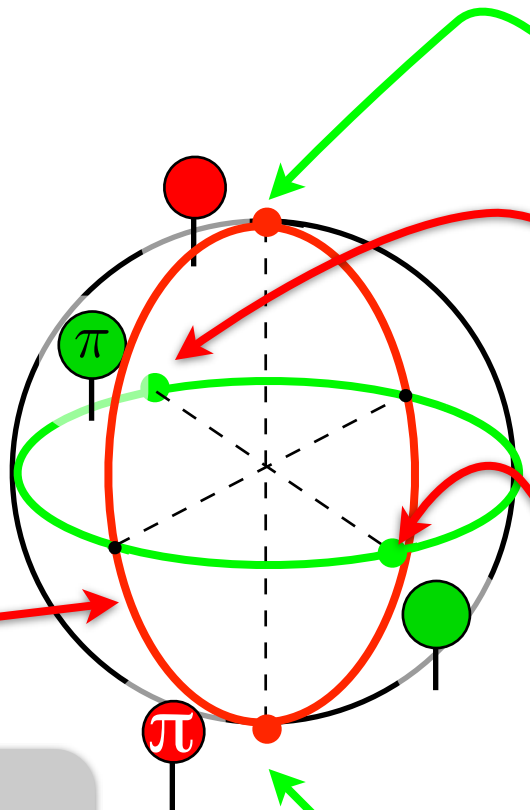
In particular, each $\Lambda^X(z_i)$ is a permutation on C_Z.

Example: qubits

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$|0\rangle =$$

$$|1\rangle =$$



Classical points

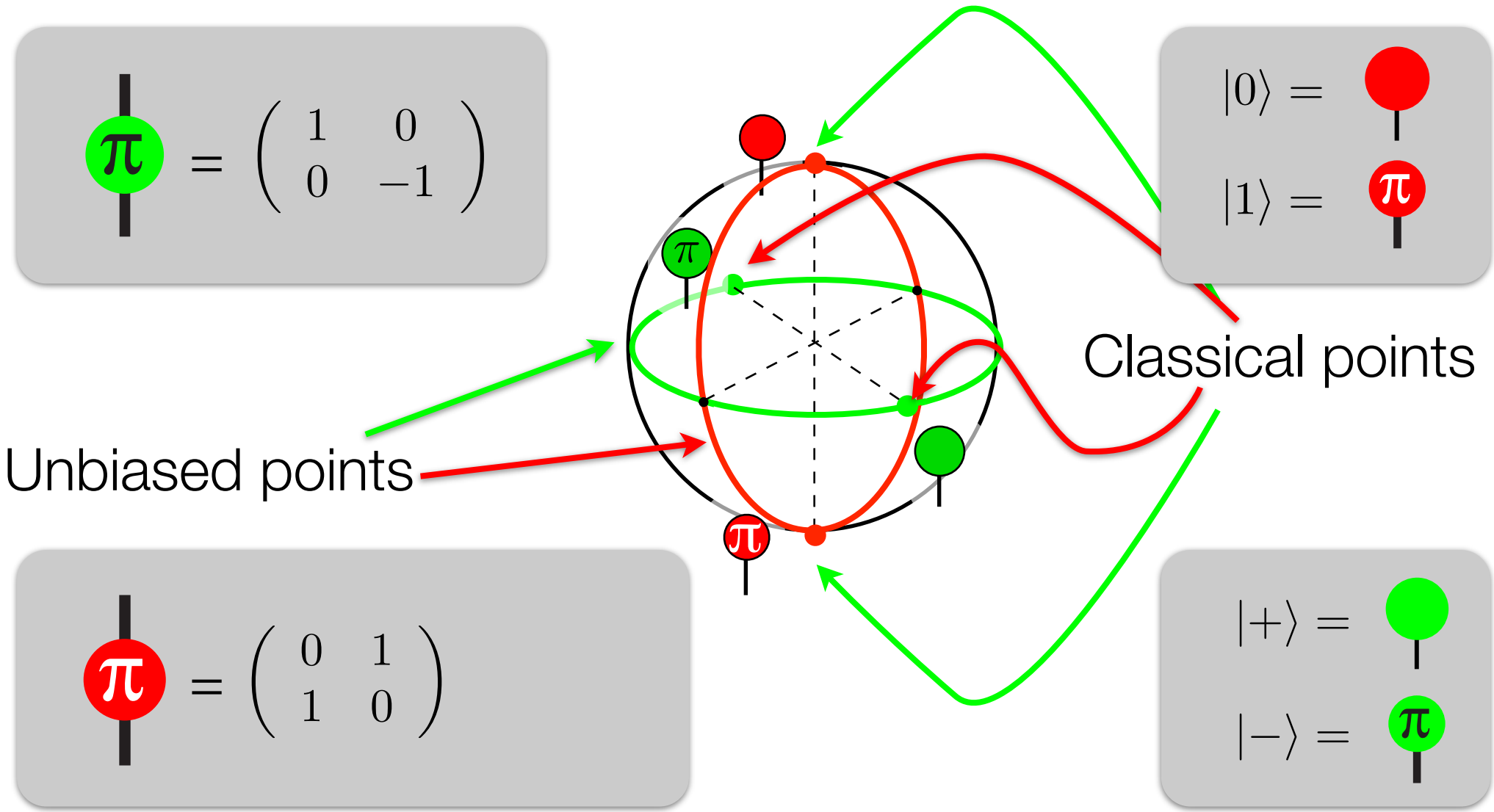
Unbiased points

$$\alpha = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$|+\rangle =$$

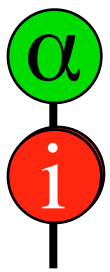
$$|-\rangle =$$

Example: qubits



Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:



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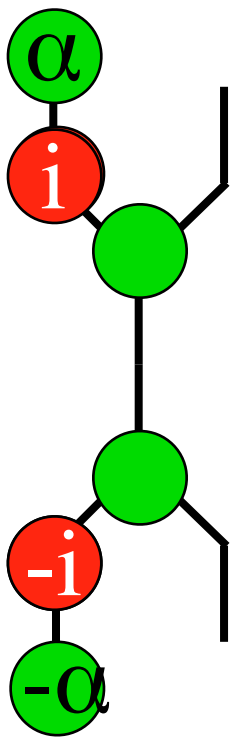

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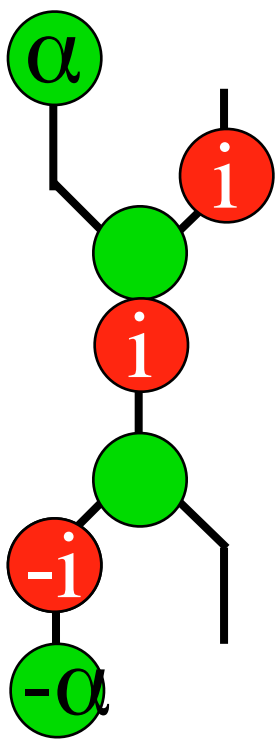

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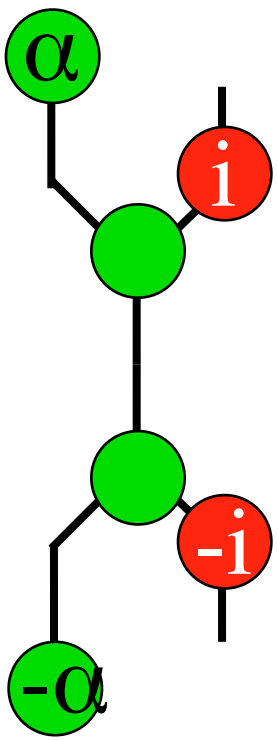

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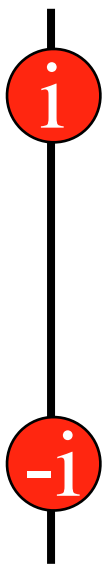

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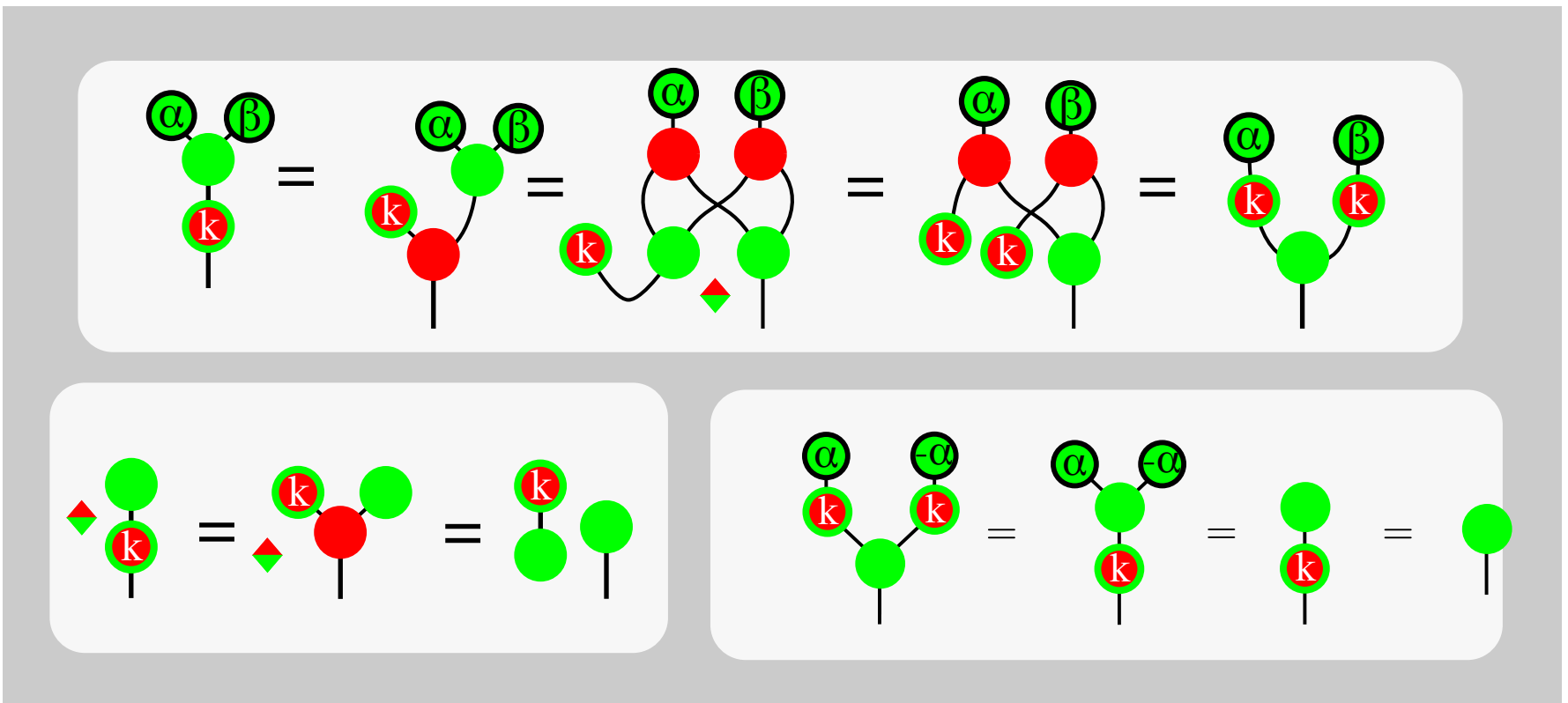


Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:

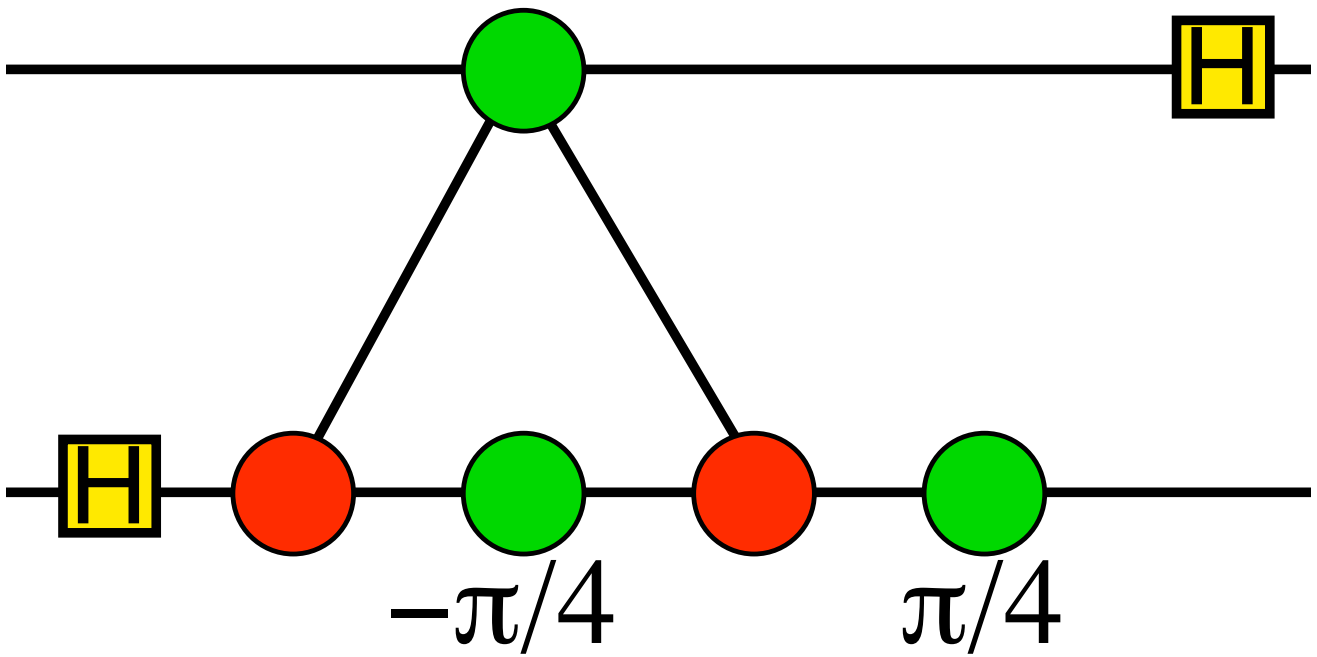
Classical points are symmetries of the phase group.

Why care about phase groups?

The phase group seems like a rather random structure. Why do we care? The phase group:

- ... determines the underlying *arithmetic*
- ... fixes the possibility for *interference*
- ... determines the possible results of CHSH-type experiments, and hence the possibility for non-locality.
  Aleks Kissinger will explain all about this in a later talk.)

The ZX-calculus

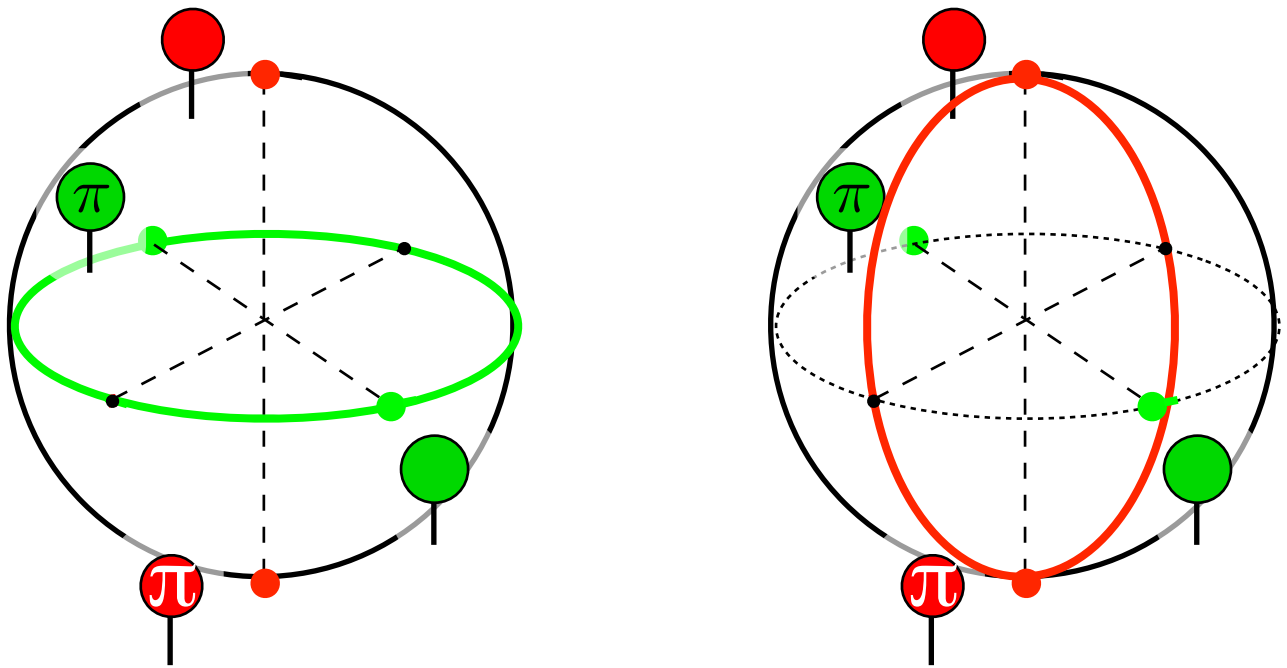


Quantum processes, diagrammatically

Z and X Observables

The Pauli **Z** and **X** observables are strongly complementary, with some additional features:

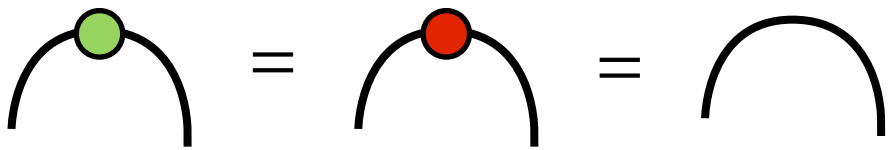
- The phase group is $[0,2\pi)$



- Dimension 2 implies two classical points
- The action of the classical group is $\alpha \mapsto -\alpha$

Z and X Observables

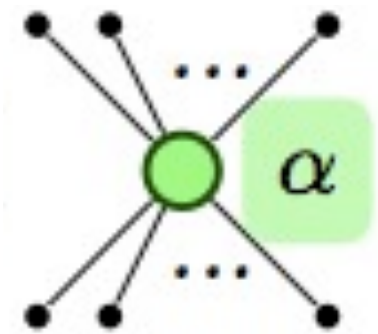
- Both observables generate the same compact structure

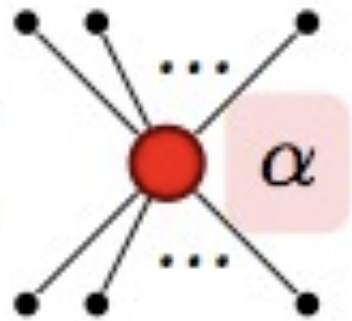


- can just treat the diagram as an undirected graph

- the Z and X are related by a definable unitary
  - gives rise to colour change rule

ZX-calculus syntax



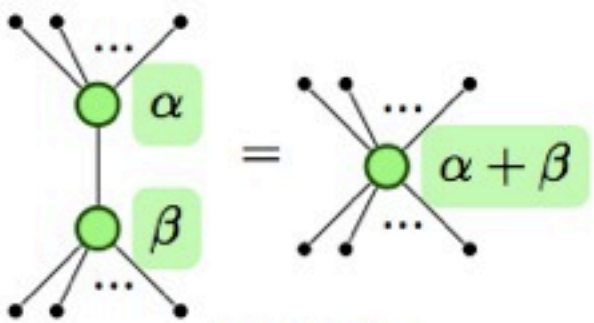


Defn: A *diagram* is an undirected open graph generated by the above vertices.

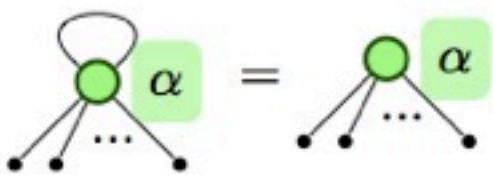
Defn: let $\mathbb{D}$ be the dagger compact category of diagrams s.t.

$$(\cdot)\dagger : \alpha \mapsto -\alpha$$

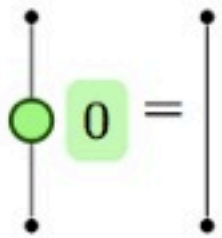
Equations



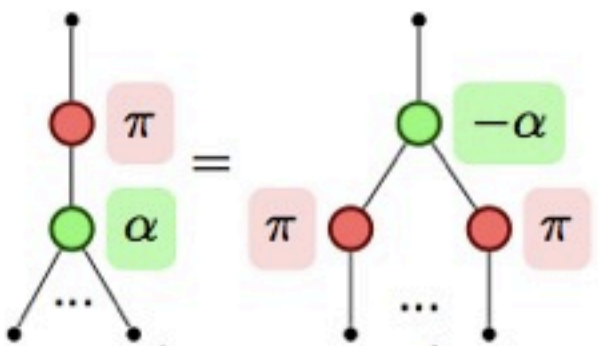
(spider)



(anti-loop)

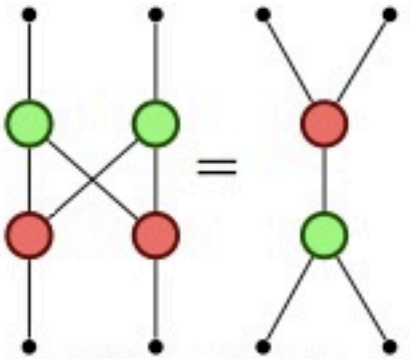


(identity)

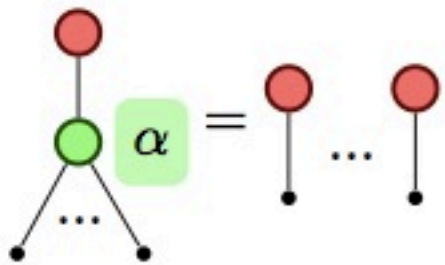


(π-commute)

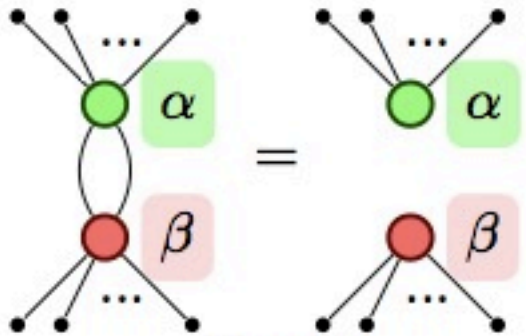
Equations



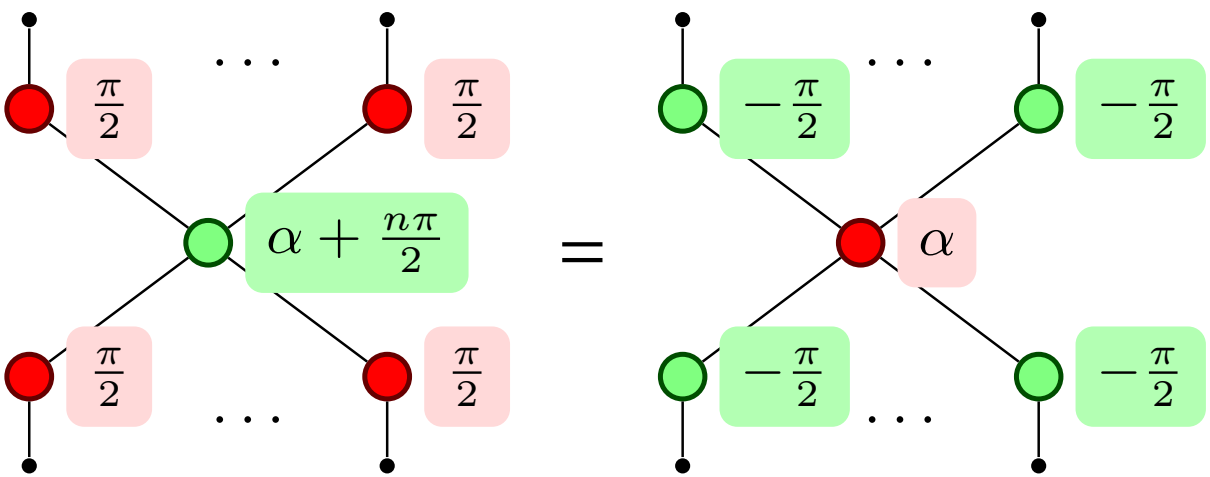
(bialgebra)



(copying)



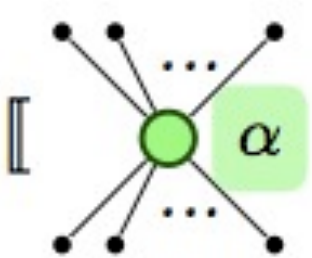
(hopf)



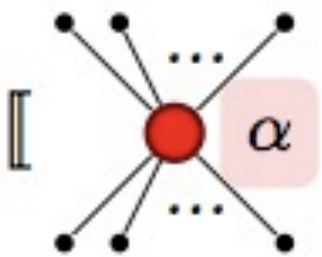
(colour change)

ZX-calculus semantics

Defn: Let $[\![\cdot]\!] : \mathbb{D} \to \boldsymbol{FDHilb}$ be the traced monoidal functor such and that $[\![A]\!] = \mathbb{C}^{2^n}$ and define its action on generators by:



$$[\![\quad]\!] = \begin{cases} |0\rangle^{\otimes m} \mapsto |0\rangle^{\otimes n} \\ |1\rangle^{\otimes m} \mapsto e^{i\alpha} |1\rangle^{\otimes n} \end{cases}$$



$$[\![\quad]\!] = \begin{cases} |+\rangle^{\otimes m} \mapsto |+\rangle^{\otimes n} \\ |-\rangle^{\otimes m} \mapsto e^{i\alpha} |-\rangle^{\otimes n} \end{cases}$$

Representing Qubits

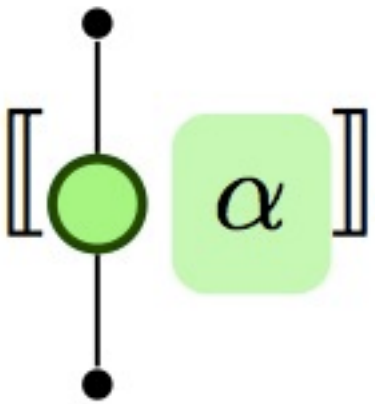
$$[\![\; \text{ } \;]\!] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$[\![\; \text{ } \;]\!] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

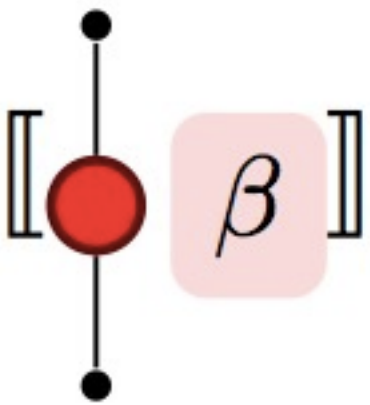
$$[\![\; \text{ } \;]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

$$[\![\; \text{ } \;]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$$

Representing Phase shifts



$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



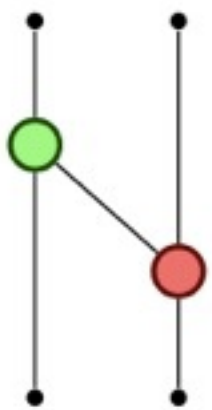
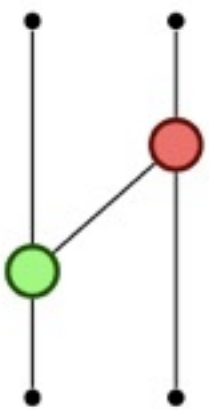
$$= \begin{pmatrix} \cos\frac{\beta}{2} & -i\sin\frac{\beta}{2} \\ -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

Representing Paulis

$$[\![\; \pi \;]\!] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

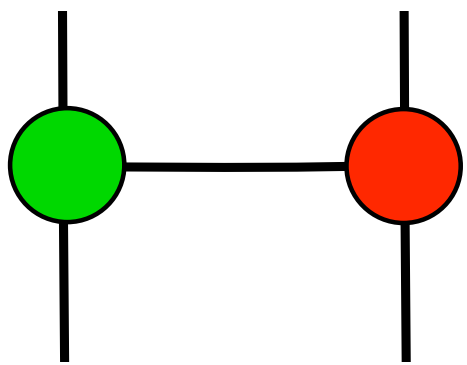
$$[\![\; \pi \;]\!] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Representing CNot

$$\wedge X = [\![$$  $$]\!] = [\![$$  $$]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Representing Logic Gates

$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



Representing Logic Gates

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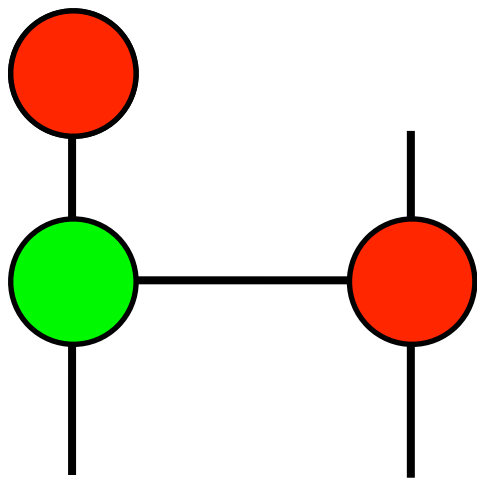
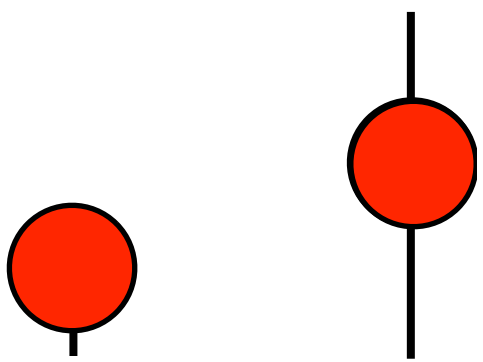


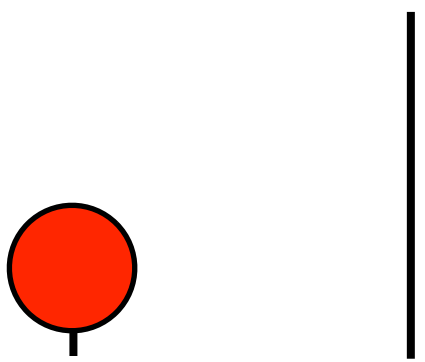
Representing Logic Gates

$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



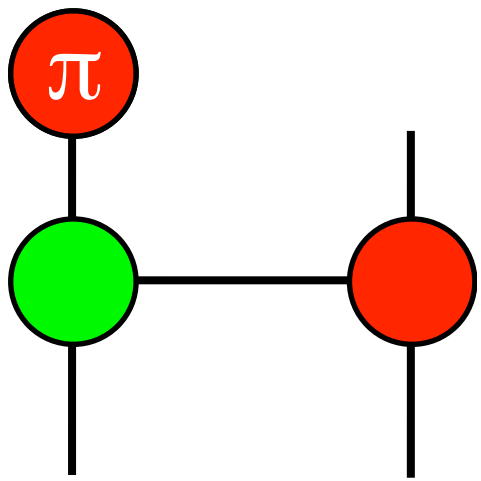
Representing Logic Gates

$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



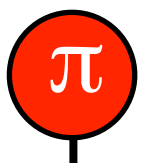
Representing Logic Gates

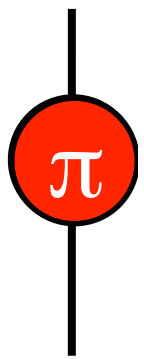
$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



Representing Logic Gates

$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



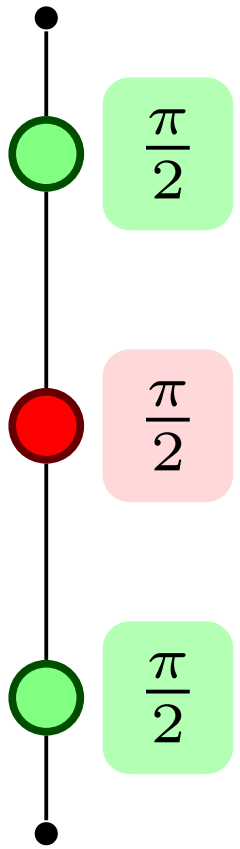


The ZX-calculus is universal

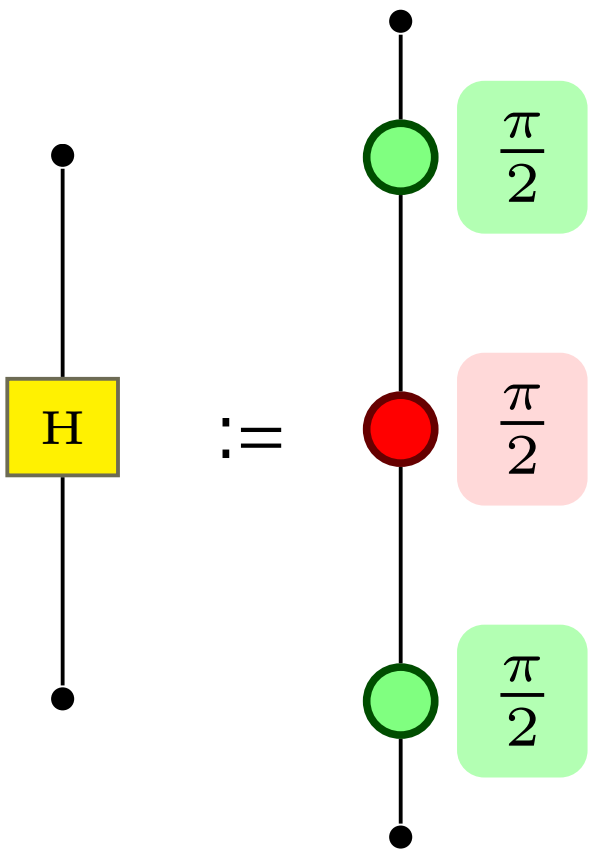
Theorem: Let U be a unitary map on n qubits; then there exists a ZX-calculus term D such that:

$$[\![D]\!] = U$$

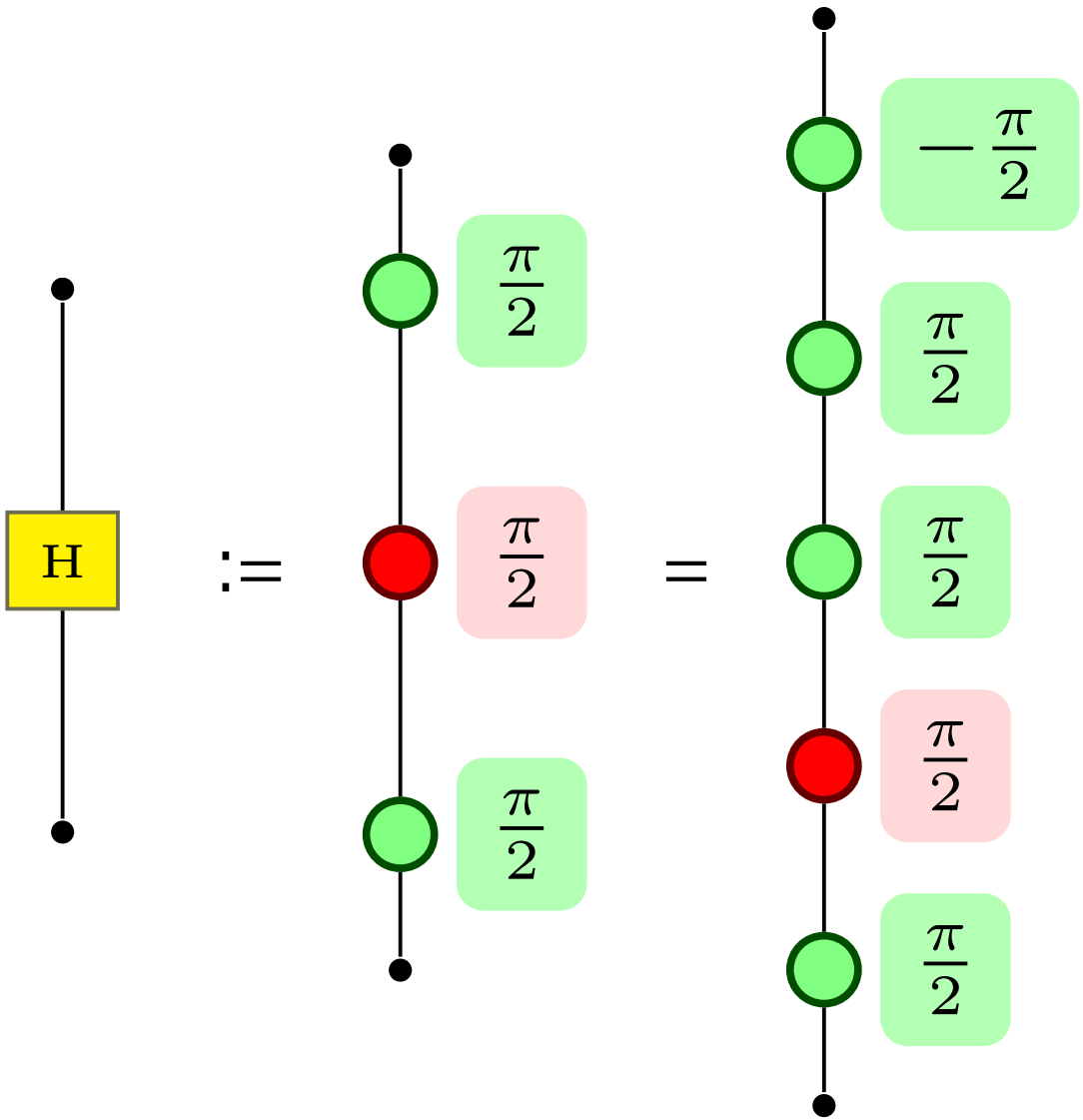
Representing Hadamard

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = [\![$$  $$]\!]$$

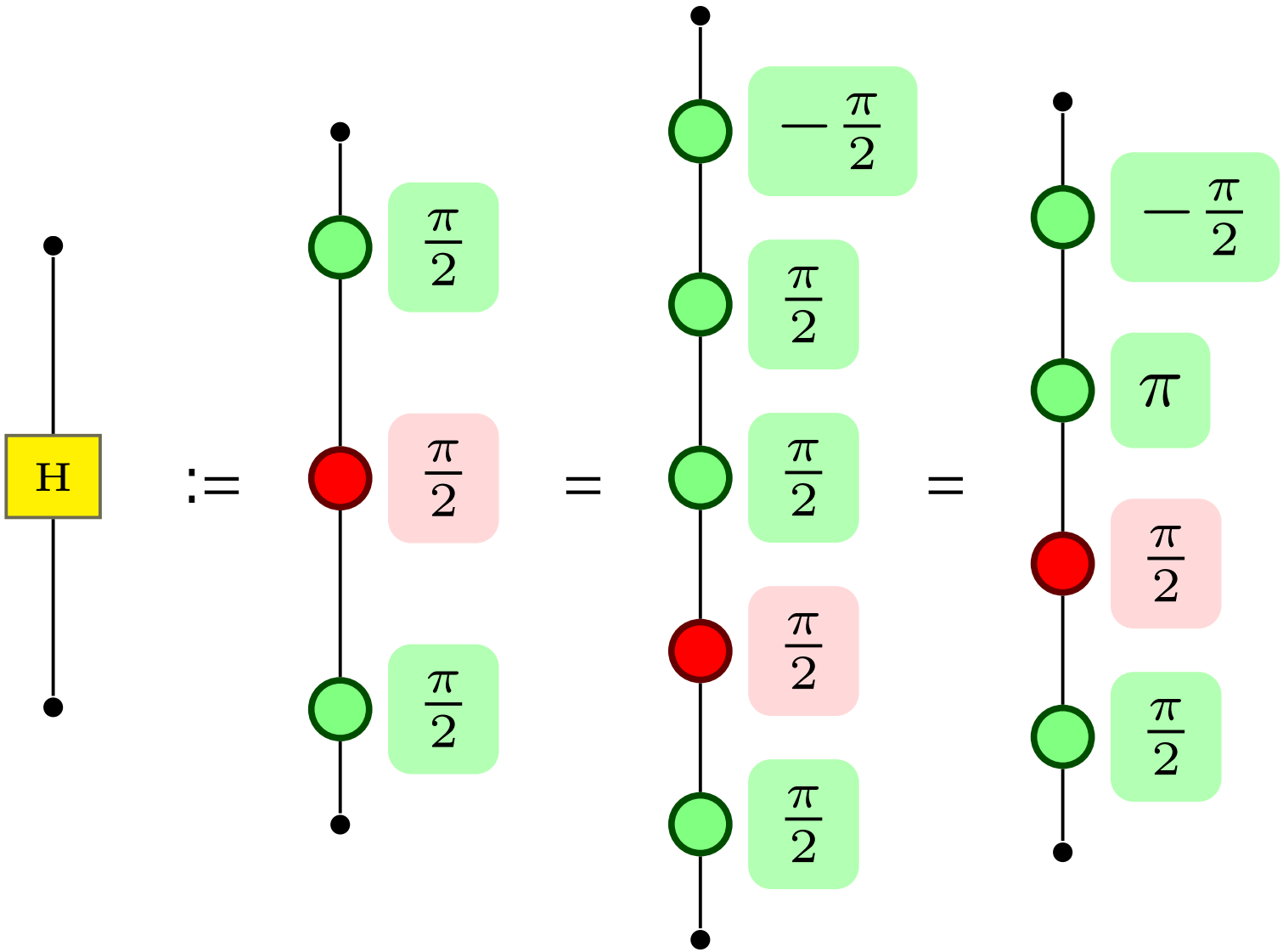
More on the Hadamard



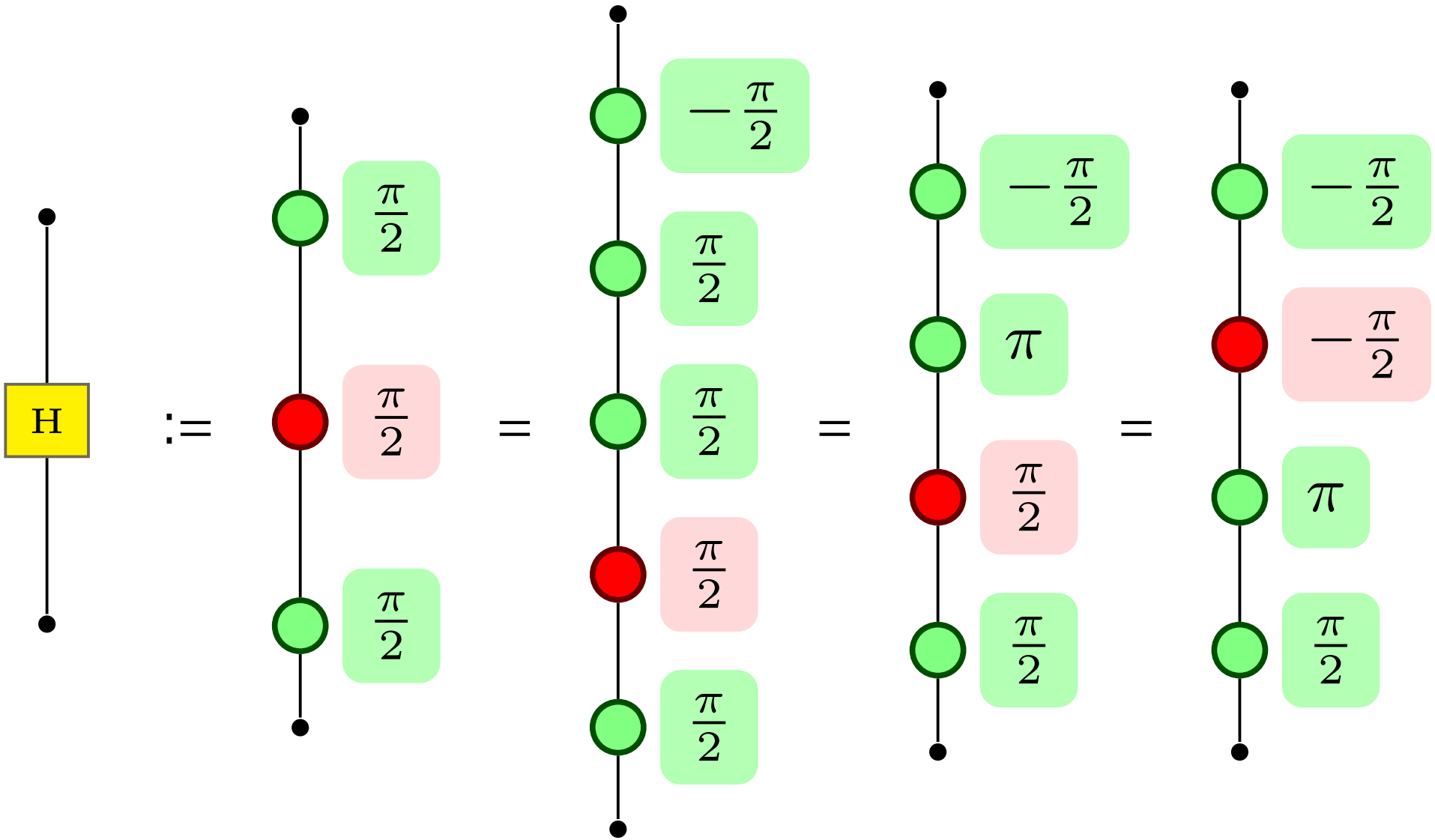
More on the Hadamard



More on the Hadamard

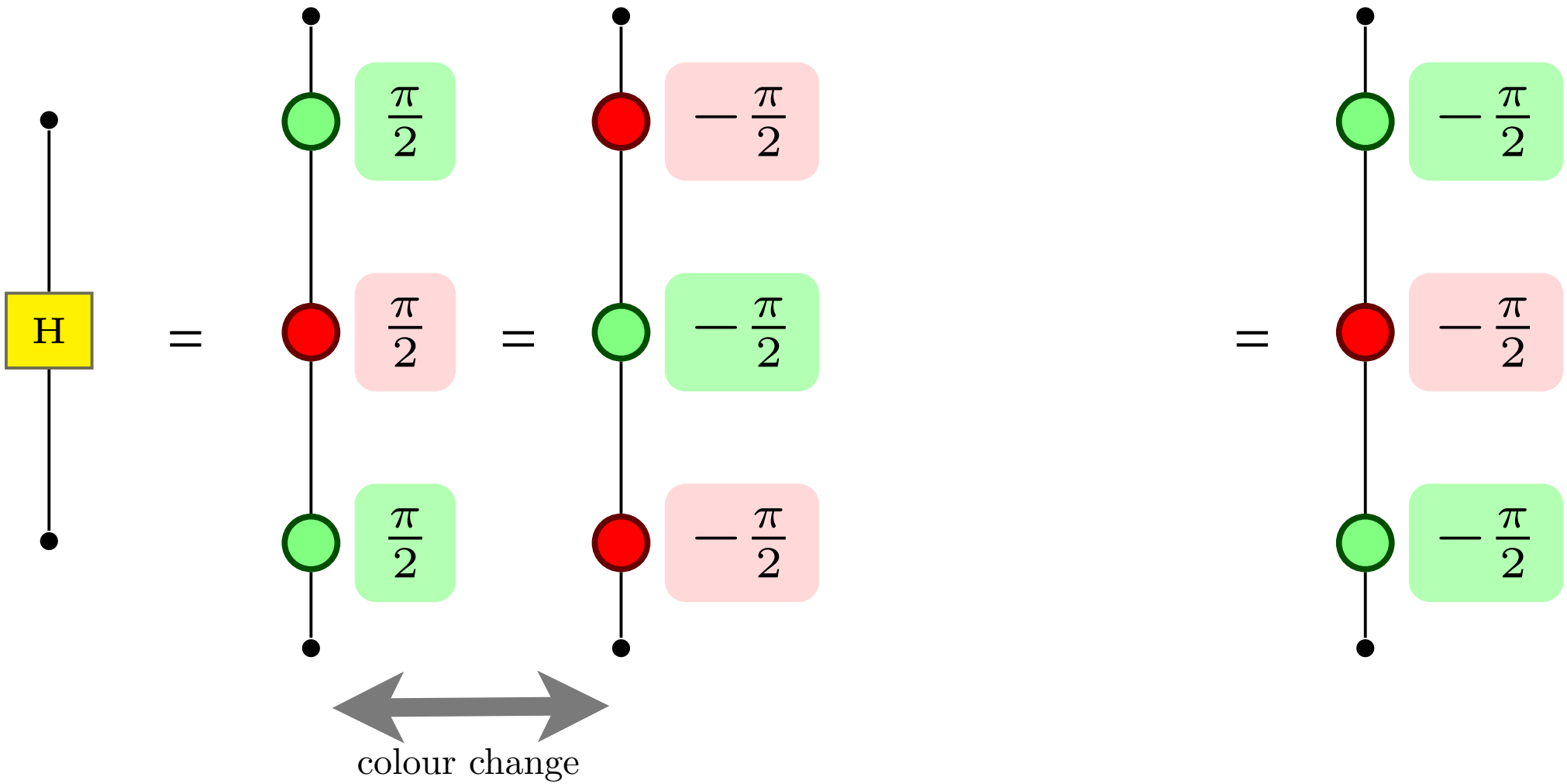


More on the Hadamard

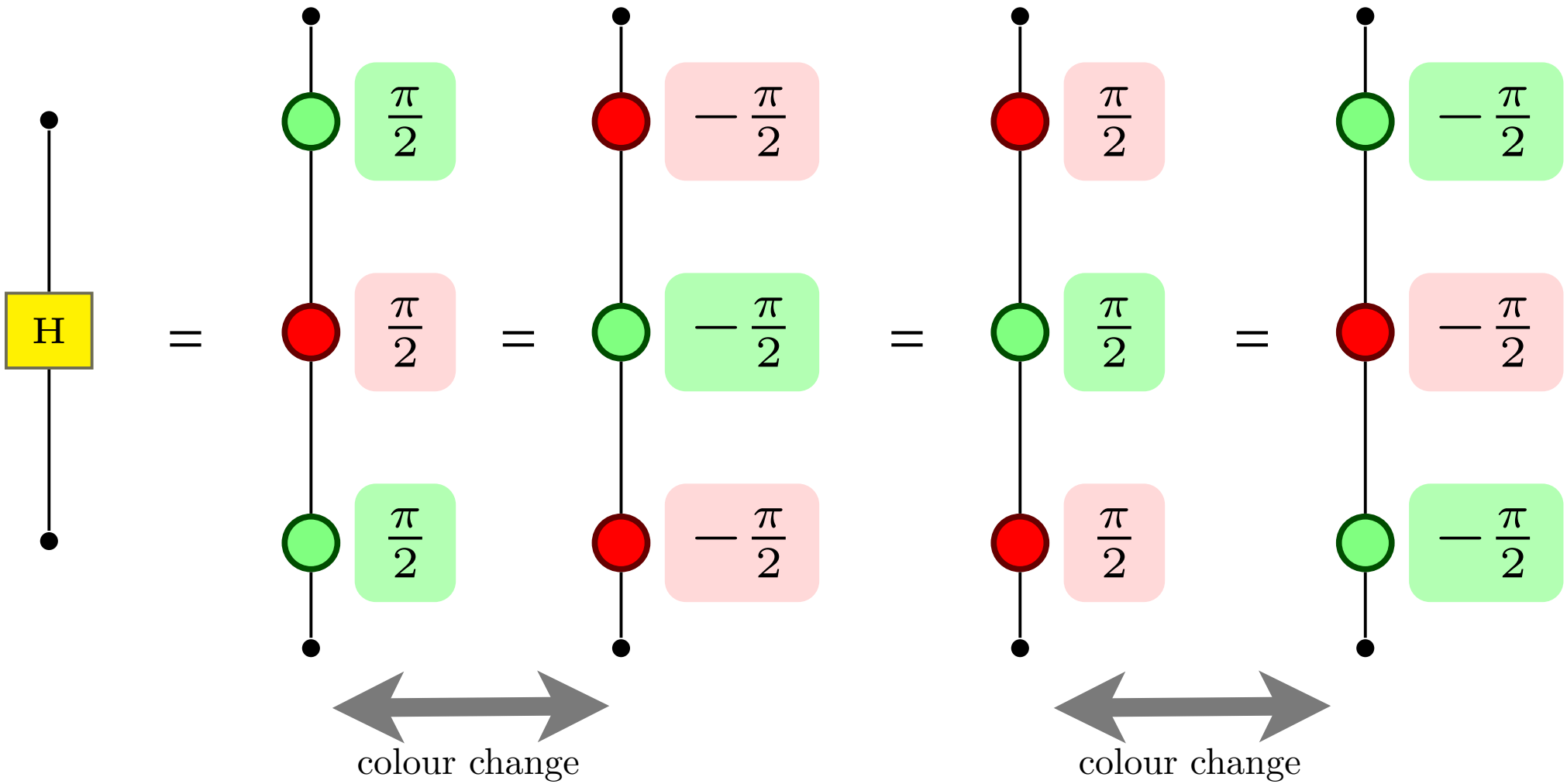


More on the Hadamard

More on the Hadamard

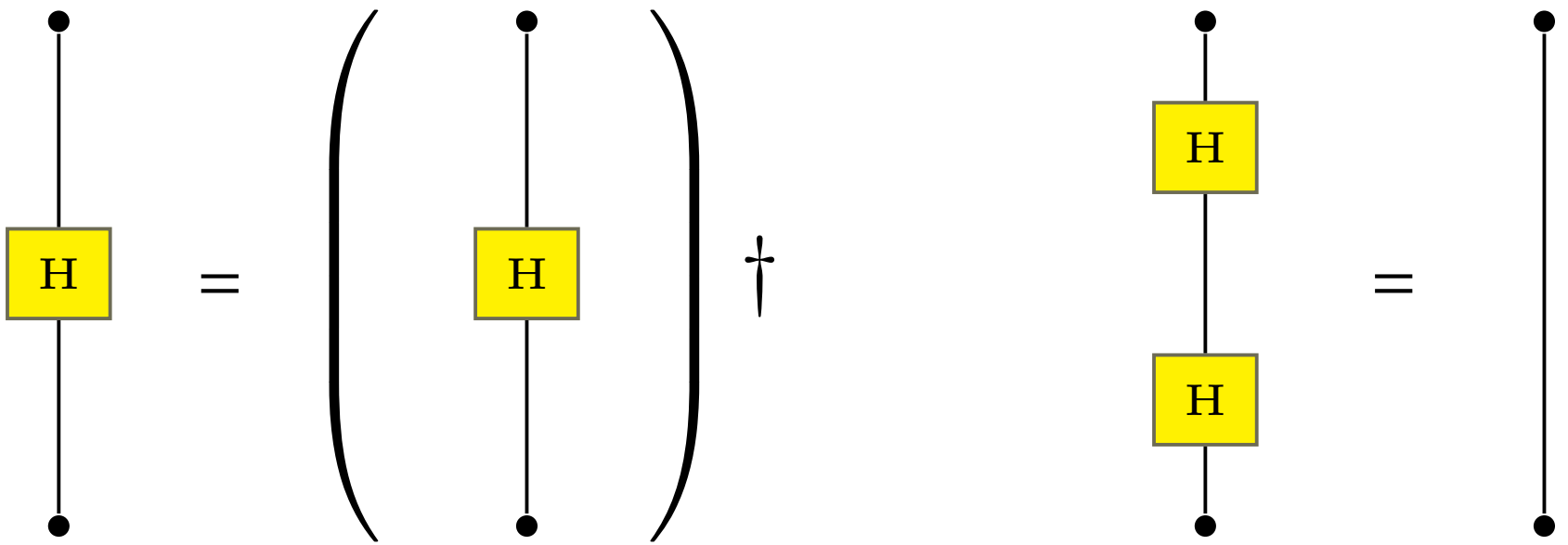


More on the Hadamard



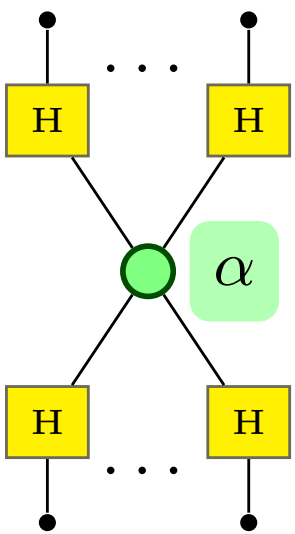
More on the Hadamard

Corollary 1:



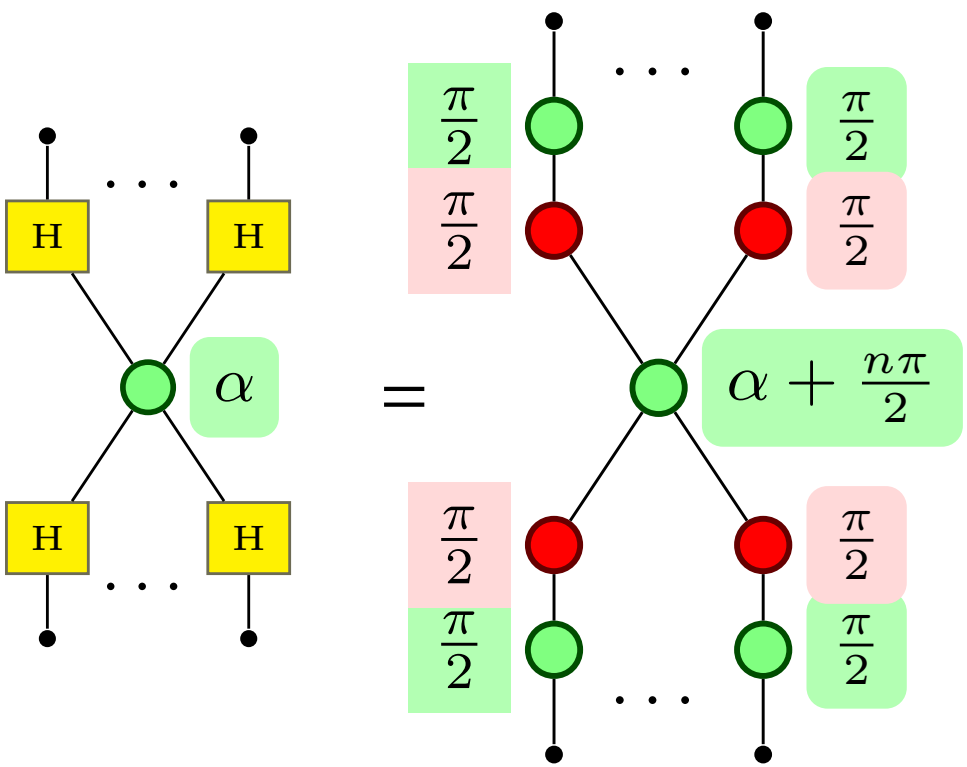
More on the Hadamard

Corollary 2:



More on the Hadamard

Corollary 2:



More on the Hadamard

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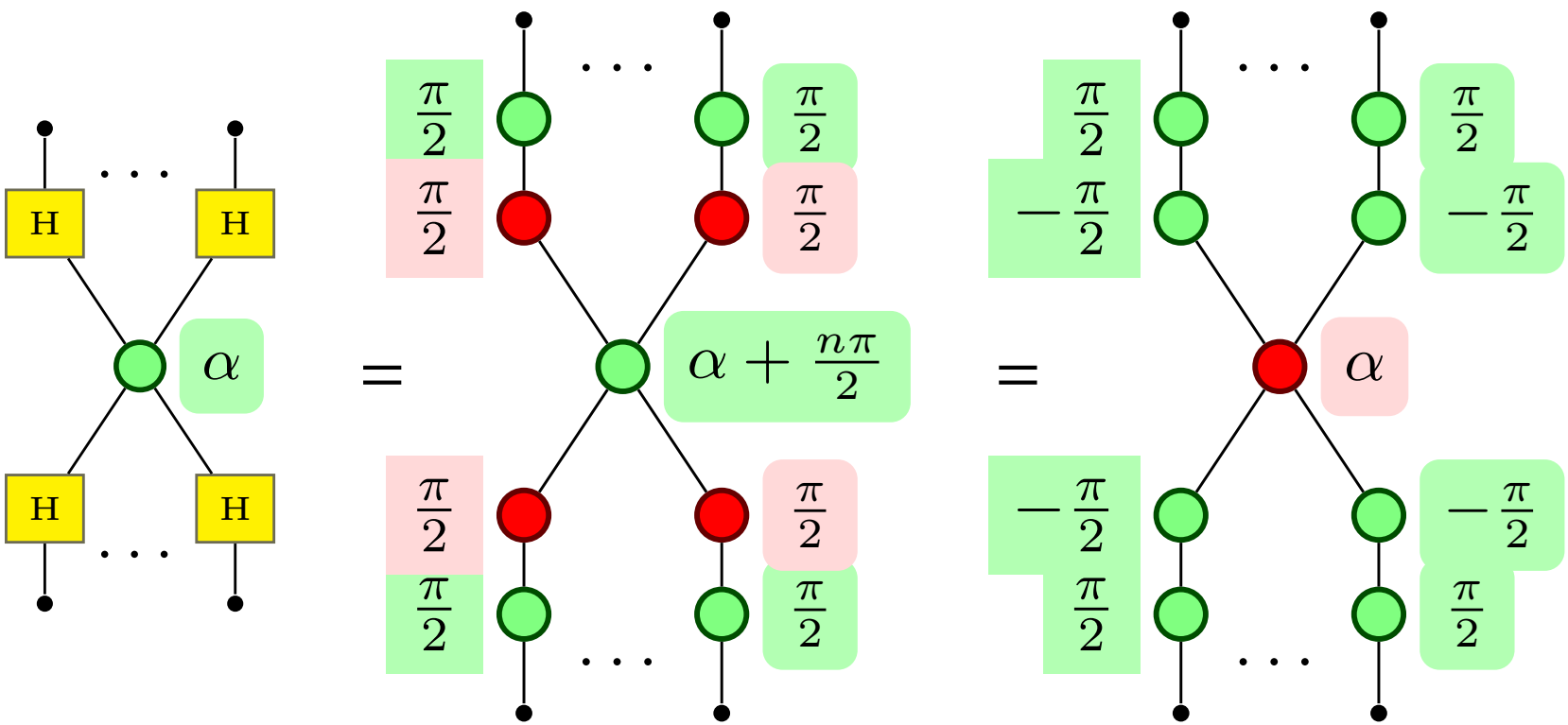

More on the Hadamard

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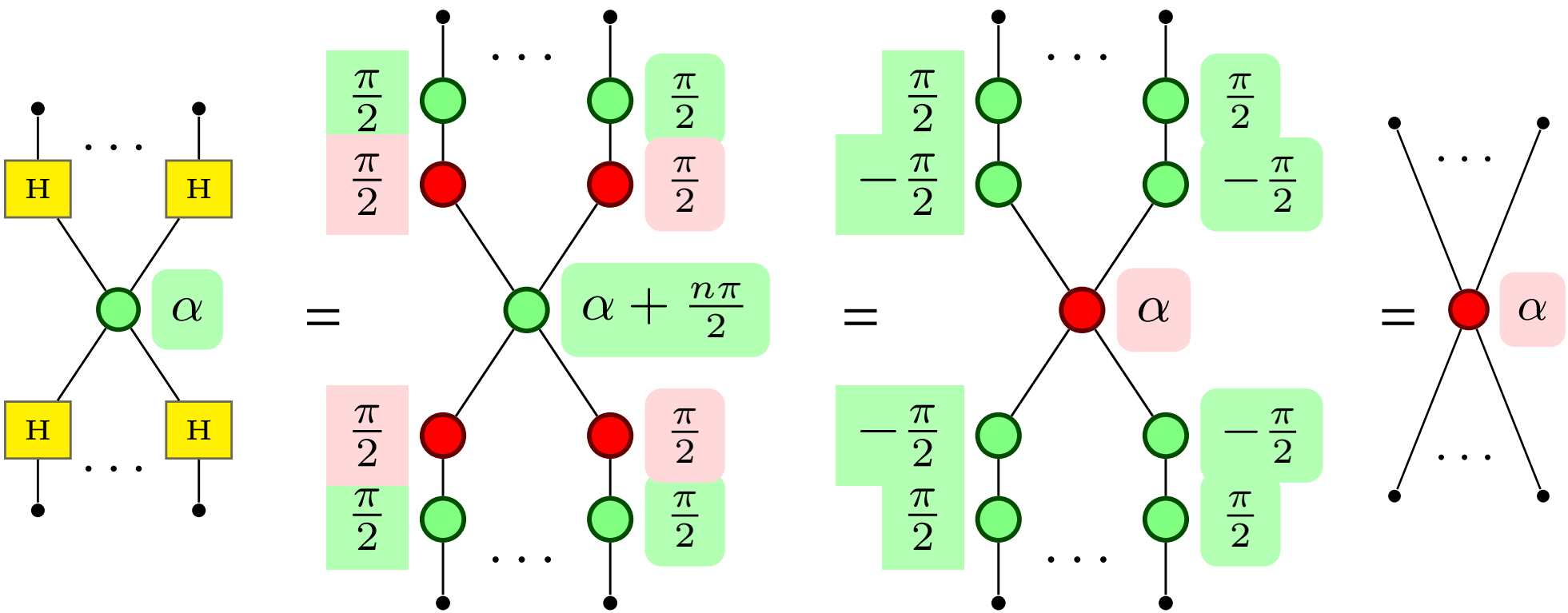
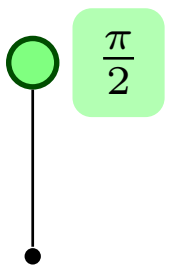


Corollary 2.1 :total symmetry between red and green

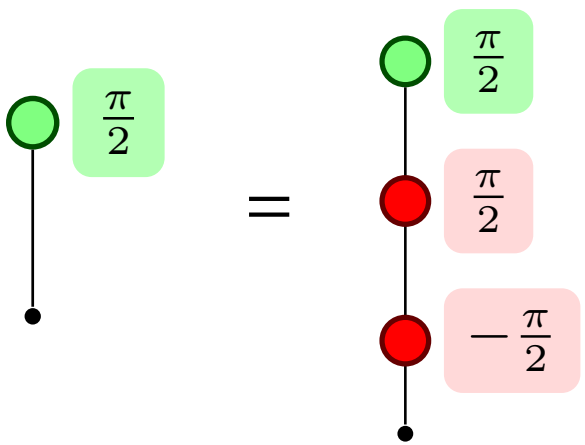
More on the Hadamard

Corollary 3:



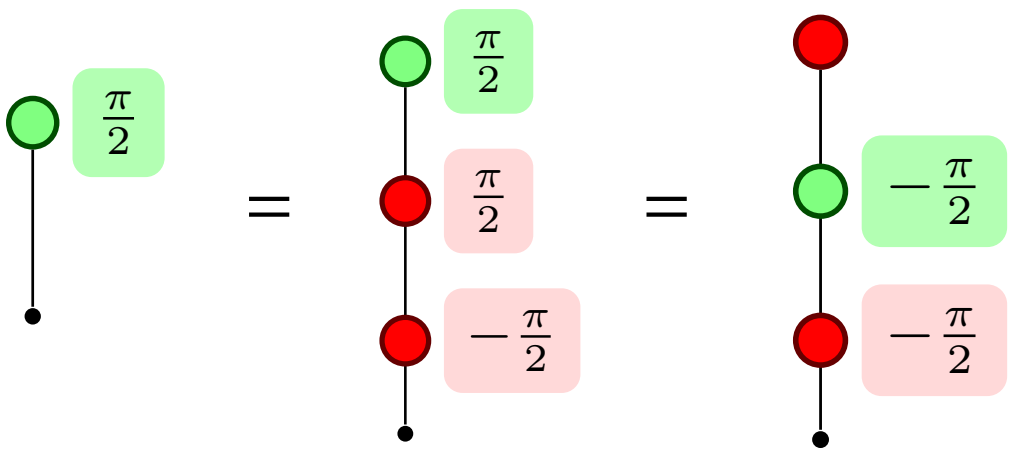
More on the Hadamard

Corollary 3:



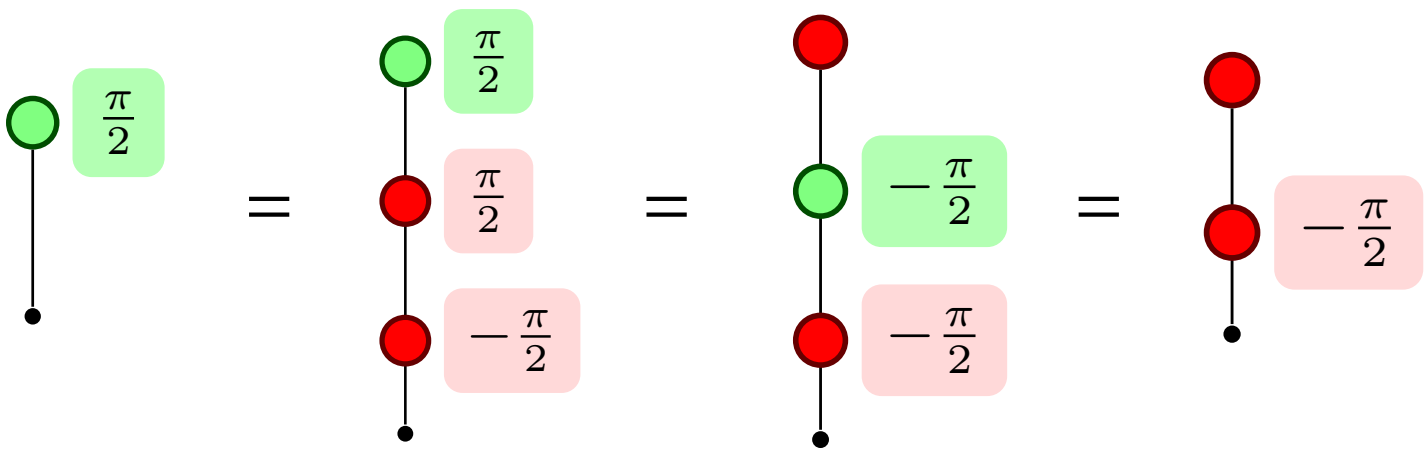
More on the Hadamard

Corollary 3:



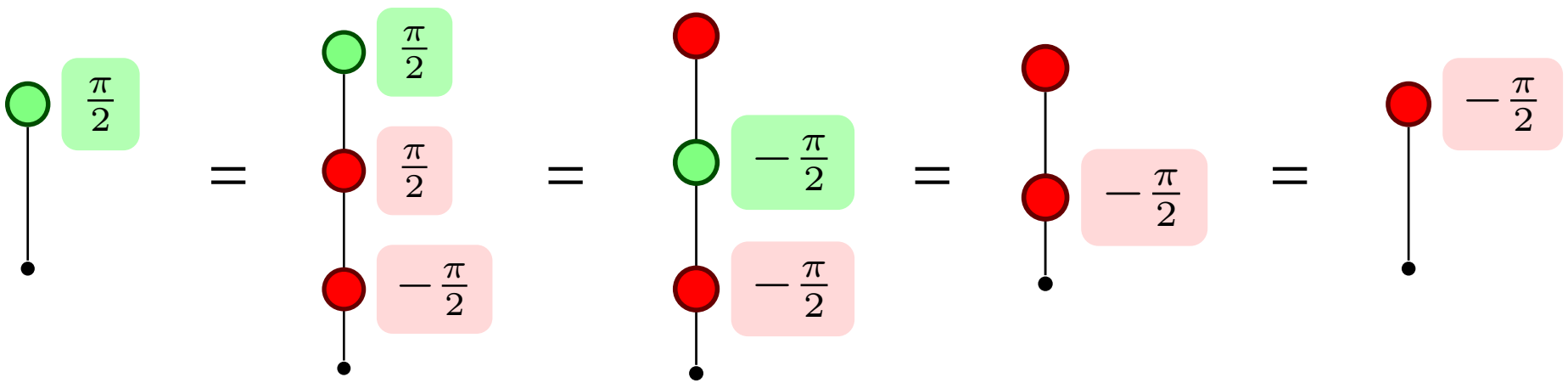
More on the Hadamard

Corollary 3:



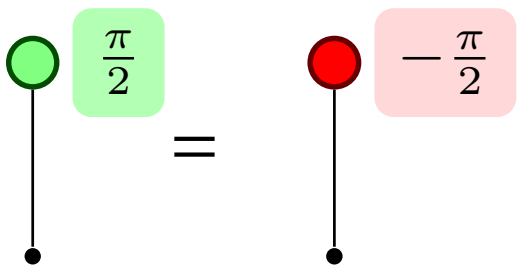
More on the Hadamard

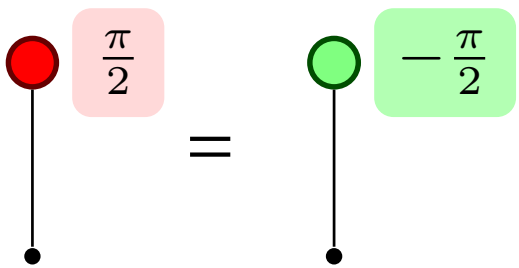
Corollary 3:



More on the Hadamard

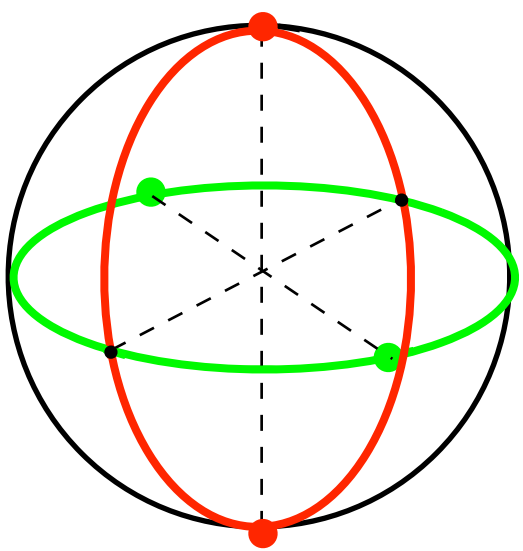
Corollary 3:





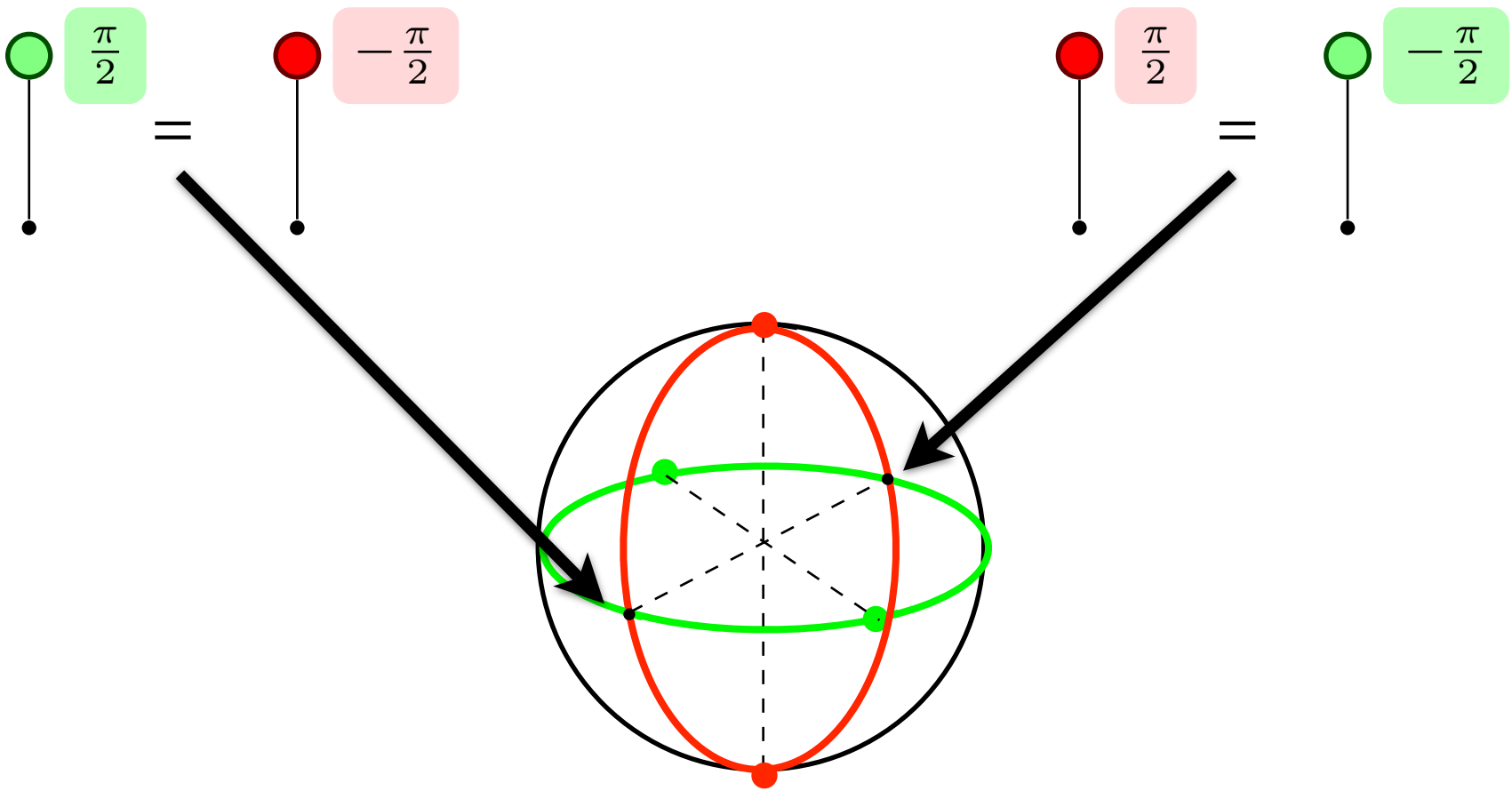
More on the Hadamard

Corollary 3:



More on the Hadamard

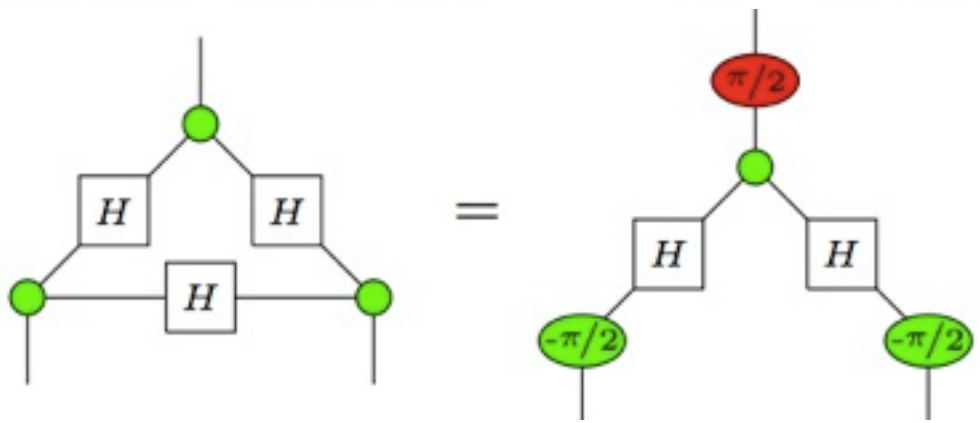
Corollary 3:



More on the Hadamard

Corollary 4: Van den Nest's theorem

$$R_x(\pi/2)^{(u)} R_z(-\pi/2)^{(N_G(u))} |G\rangle = |G * u\rangle$$

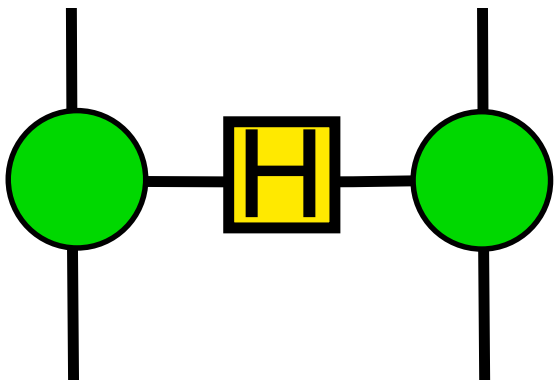


Van den Nest, M., Dehaene, J. and De Moor, B. "*Graphical description of the action of local clifford transformations on graph states*", Phys. Rev. A 69 (2004) 022316. arXiv:quant-ph/0411115

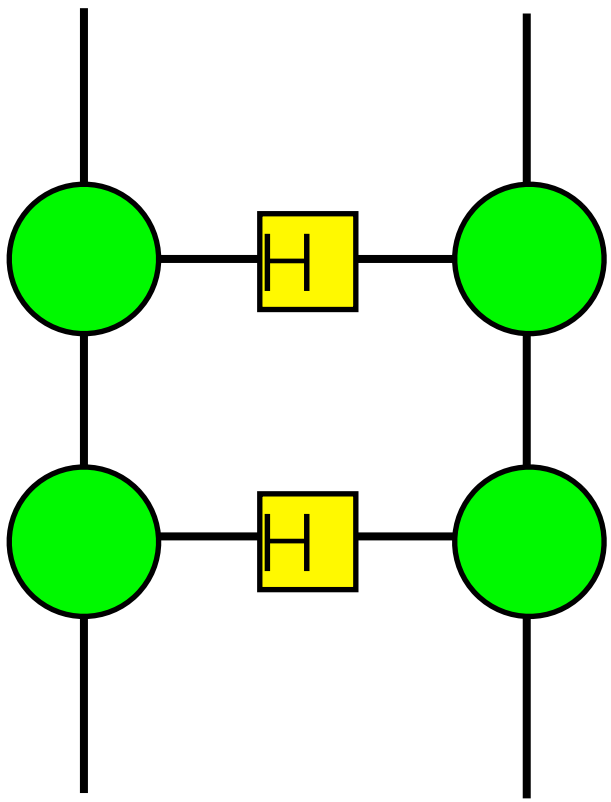
Duncan, R. and Perdrix, S. "*Graph States and the necessity of Euler Decomposition*", Proc. CiE09 (2009), LNCS 5635. arXiv:0902.0500

Representing Logic Gates

$$\wedge Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



Example: 2 CZs



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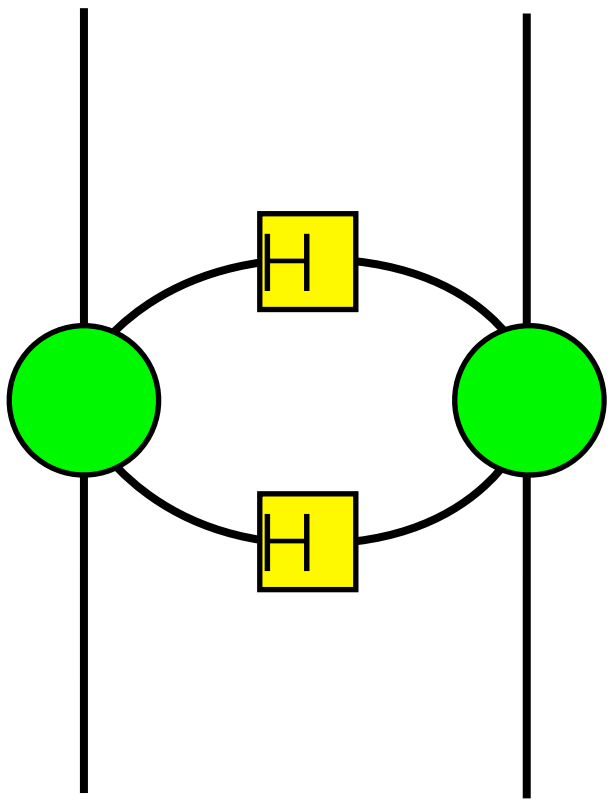

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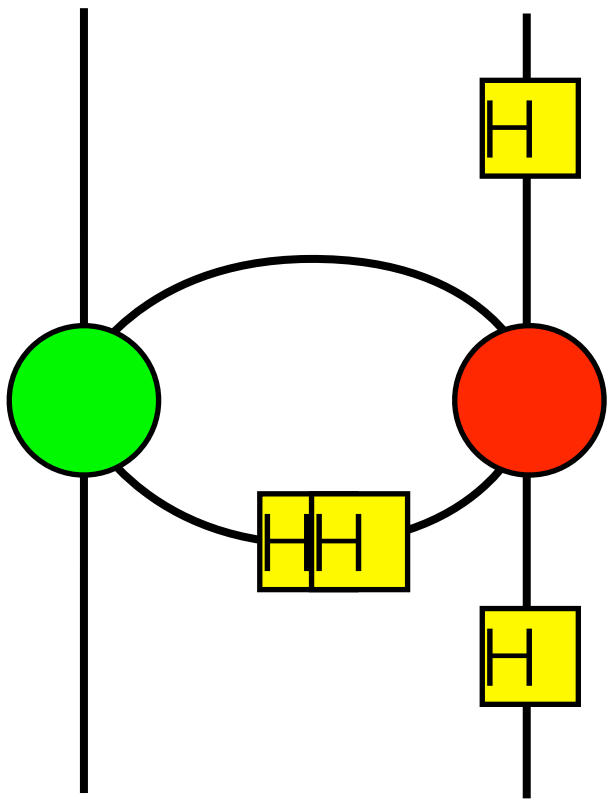

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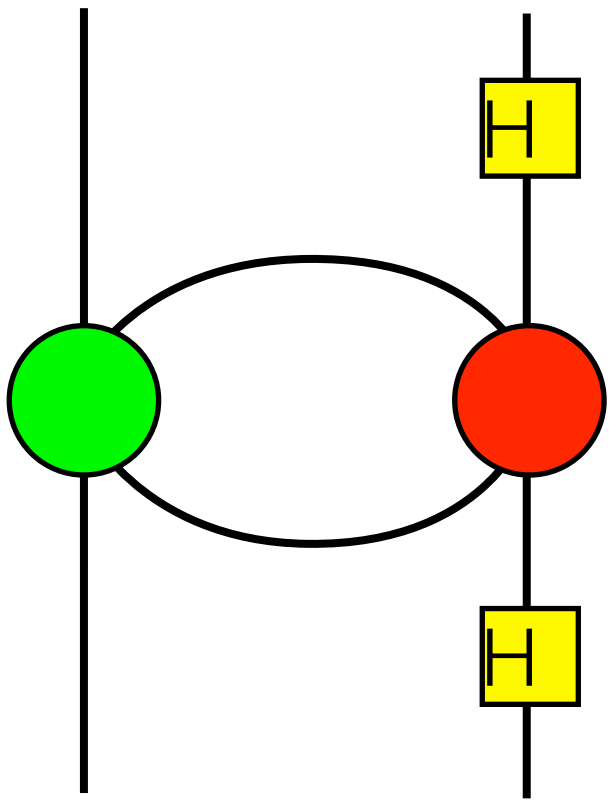

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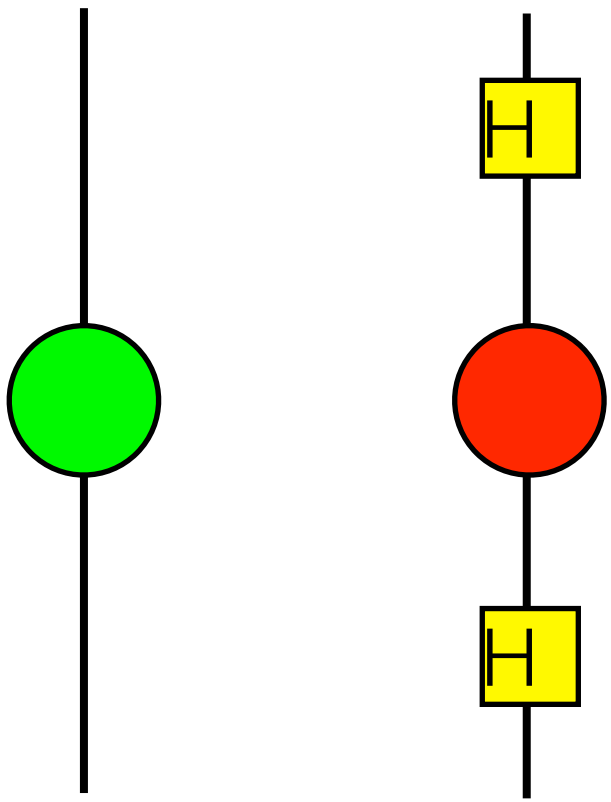

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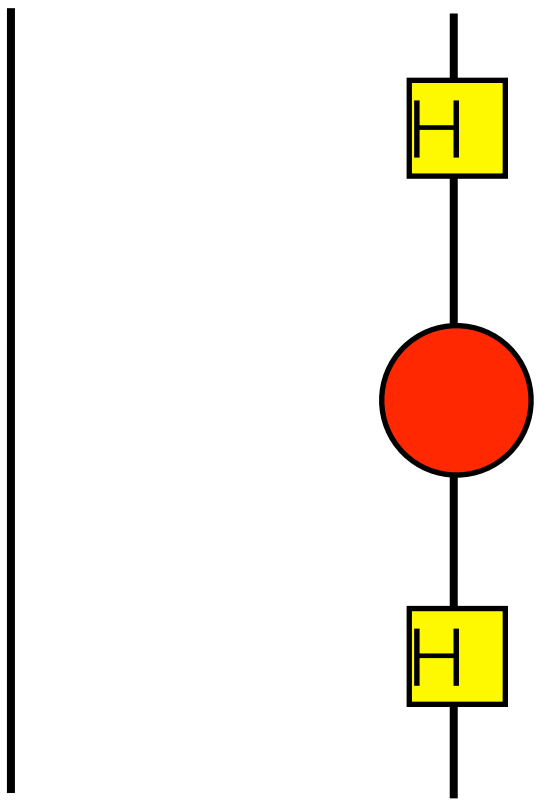

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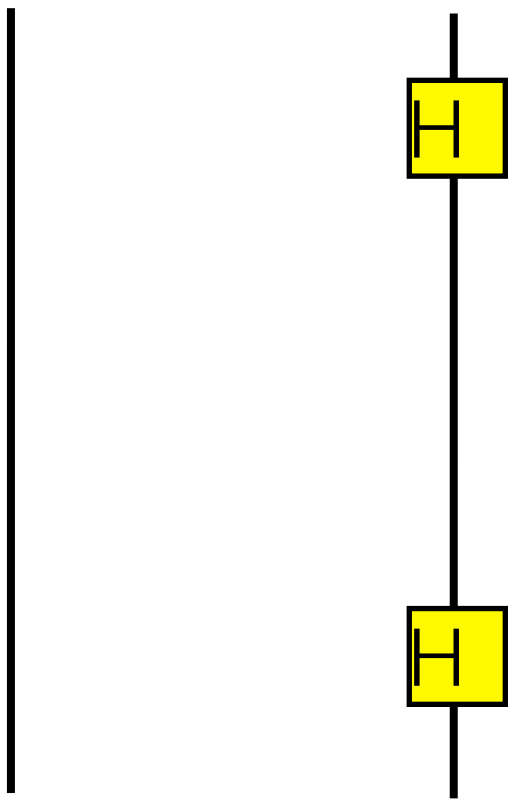

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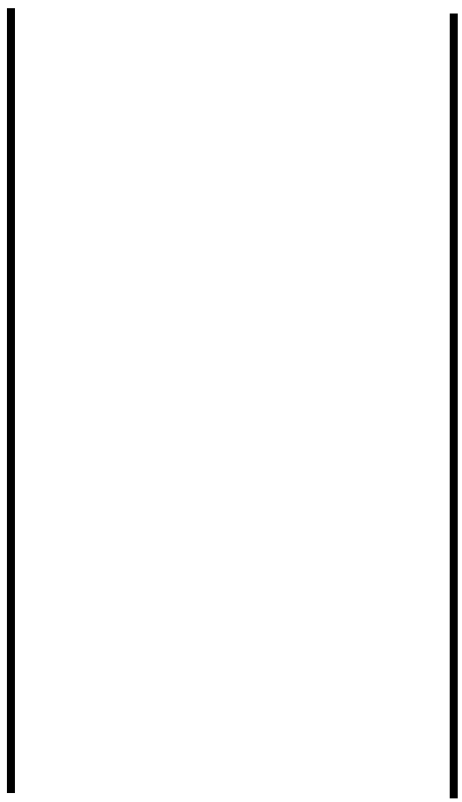
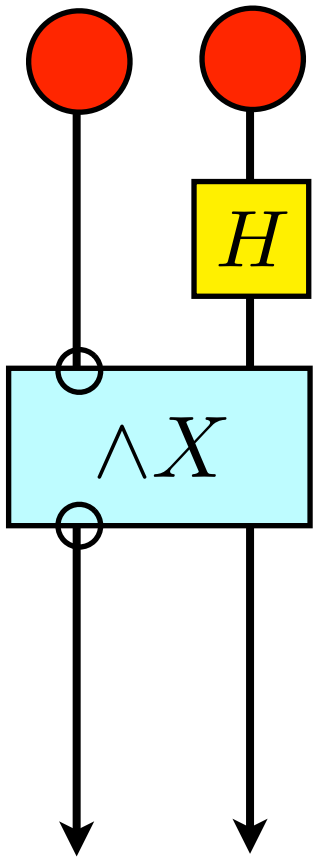


Example: Preparing a Bell state



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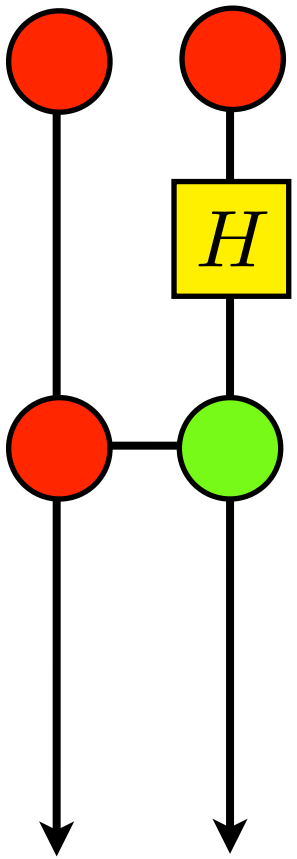
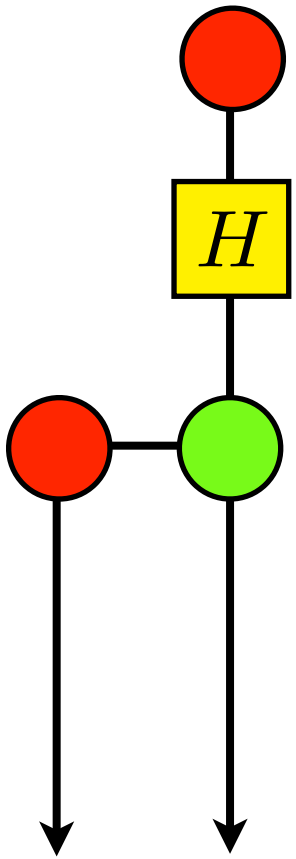
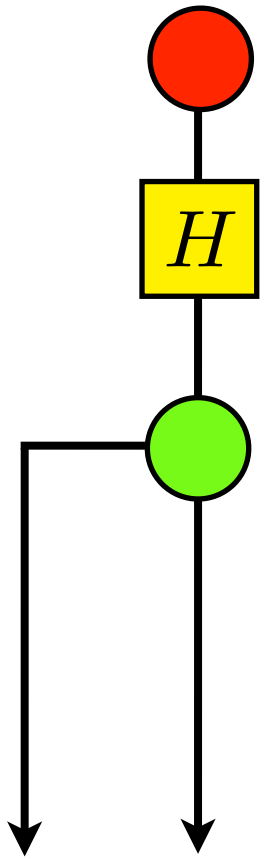


Example: Preparing a Bell state



Example: Preparing a Bell state



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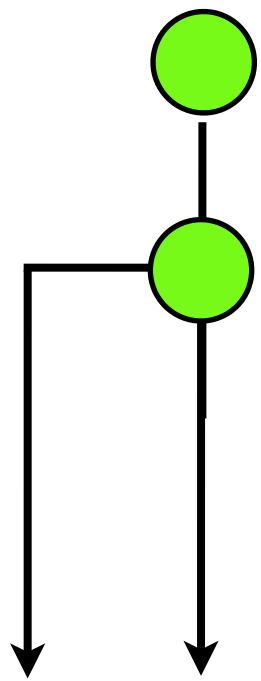

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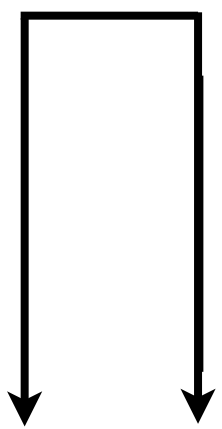
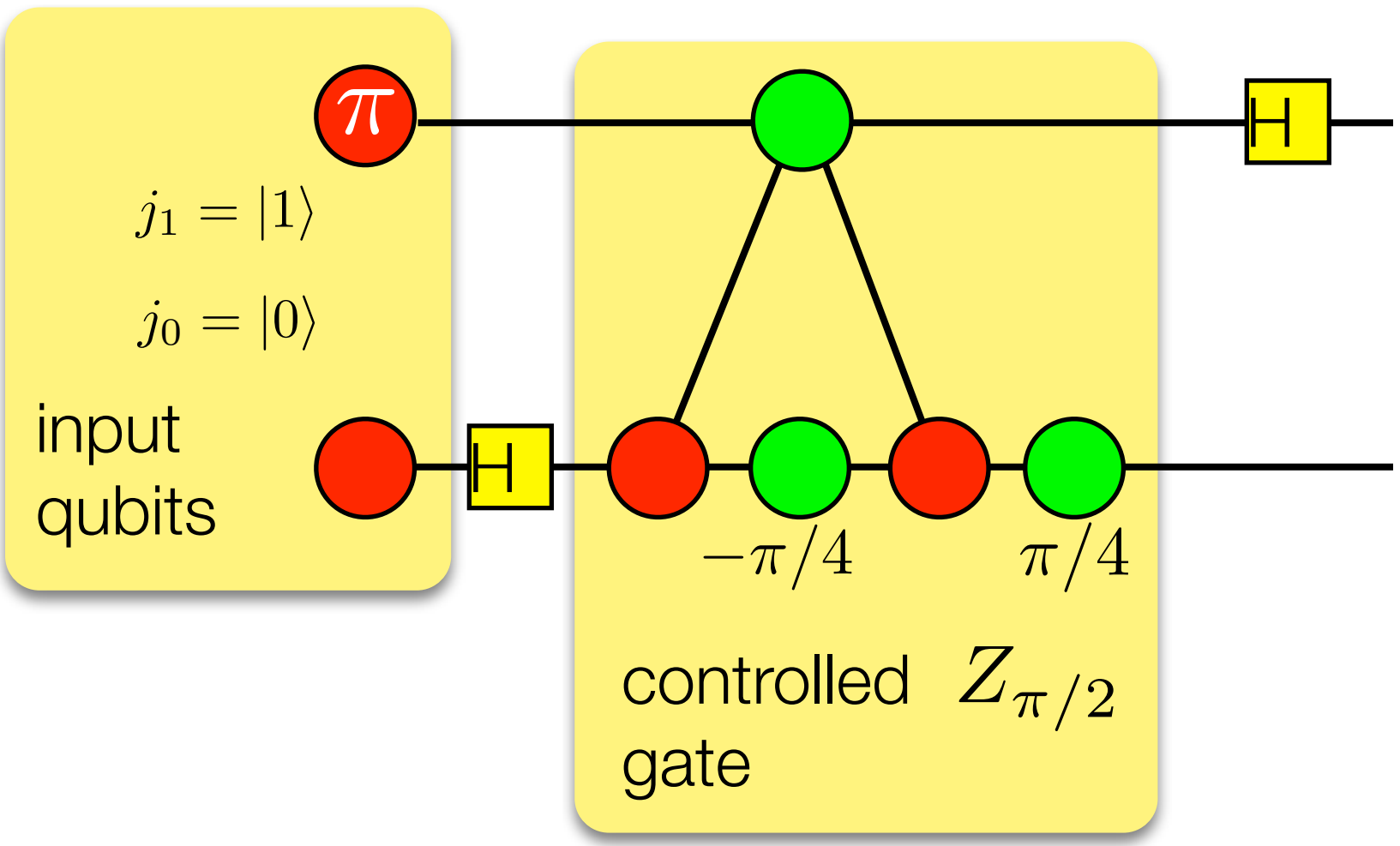


Example: 2-Qubit Quantum Fourier Transform



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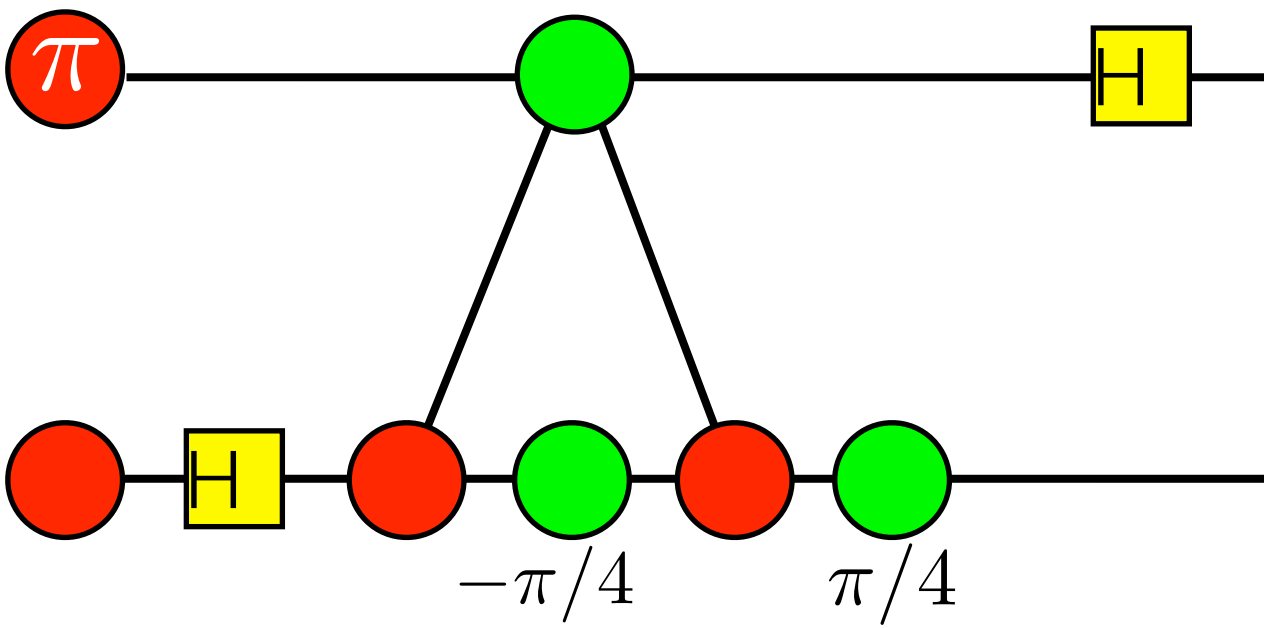

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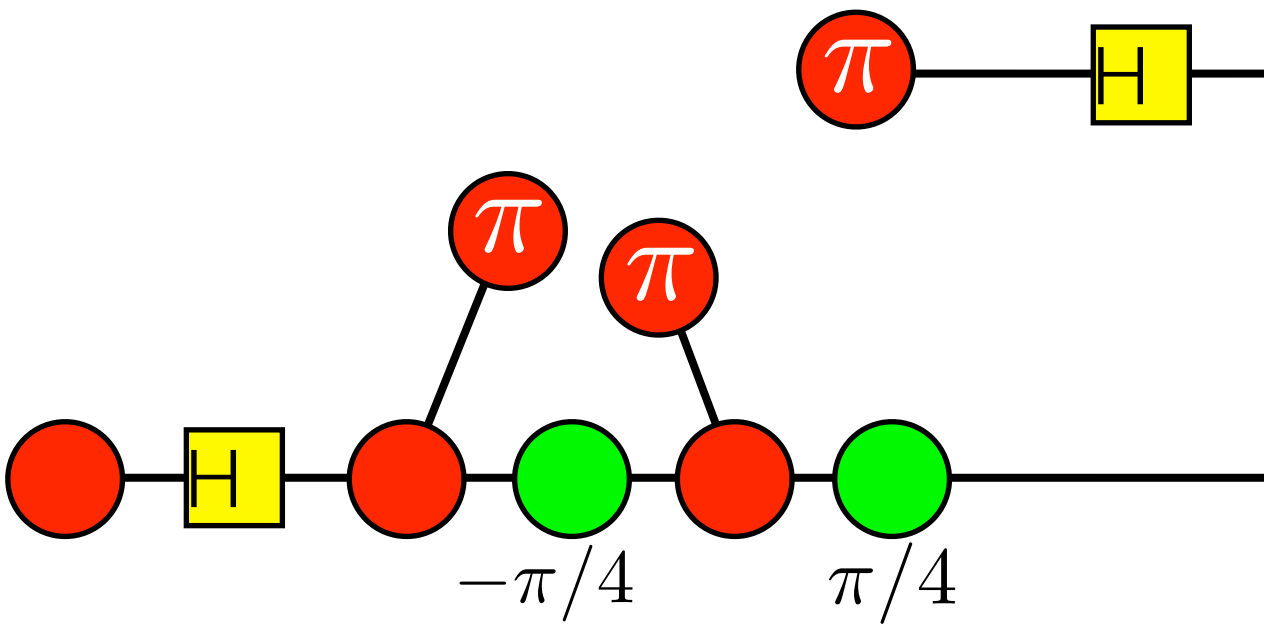

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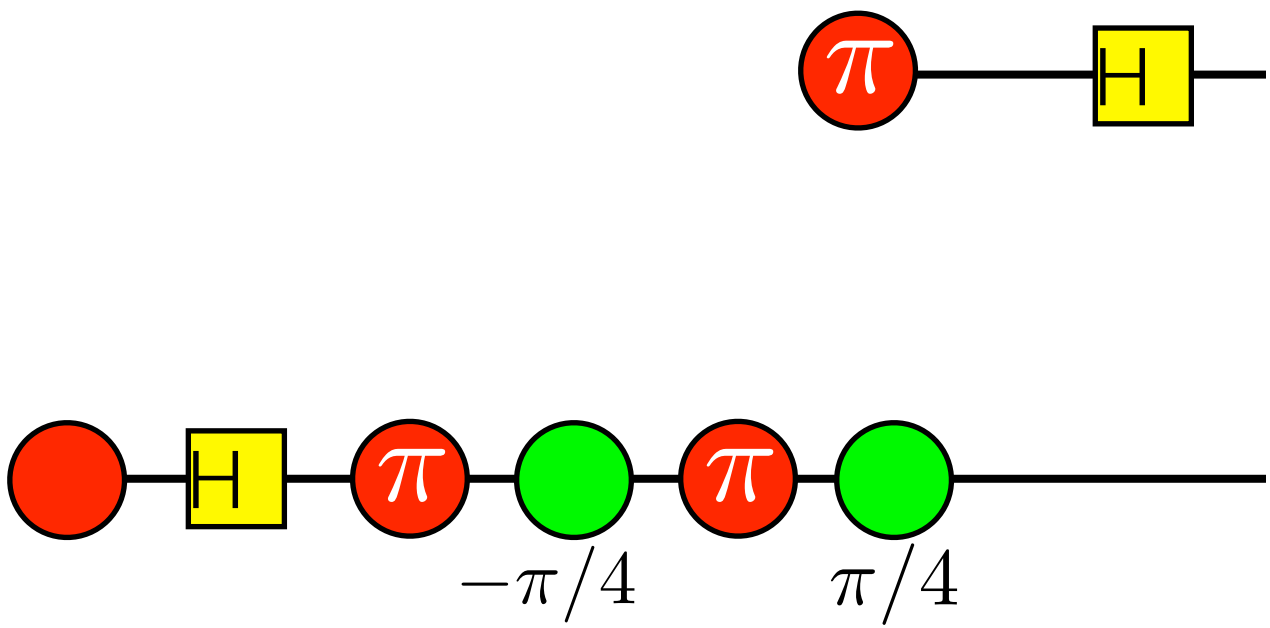

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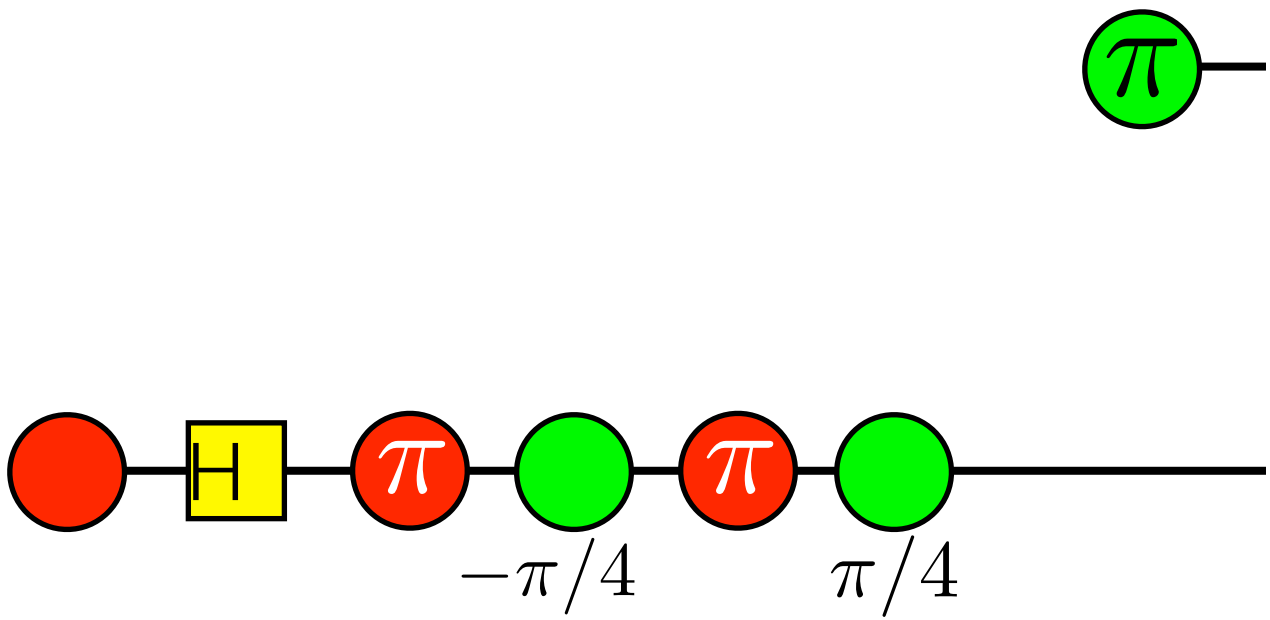

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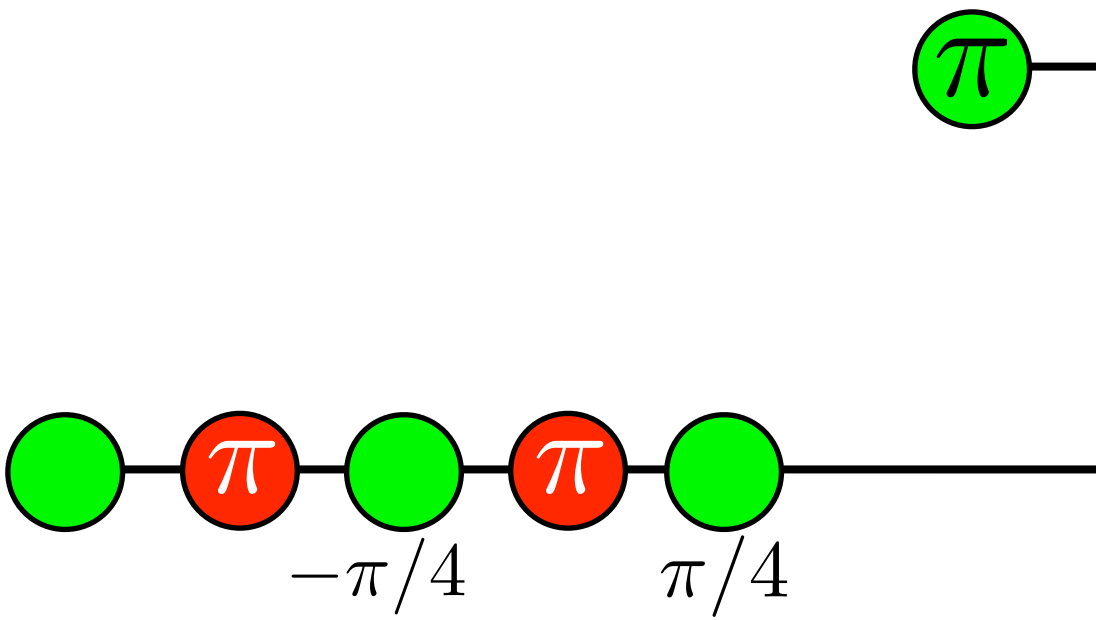

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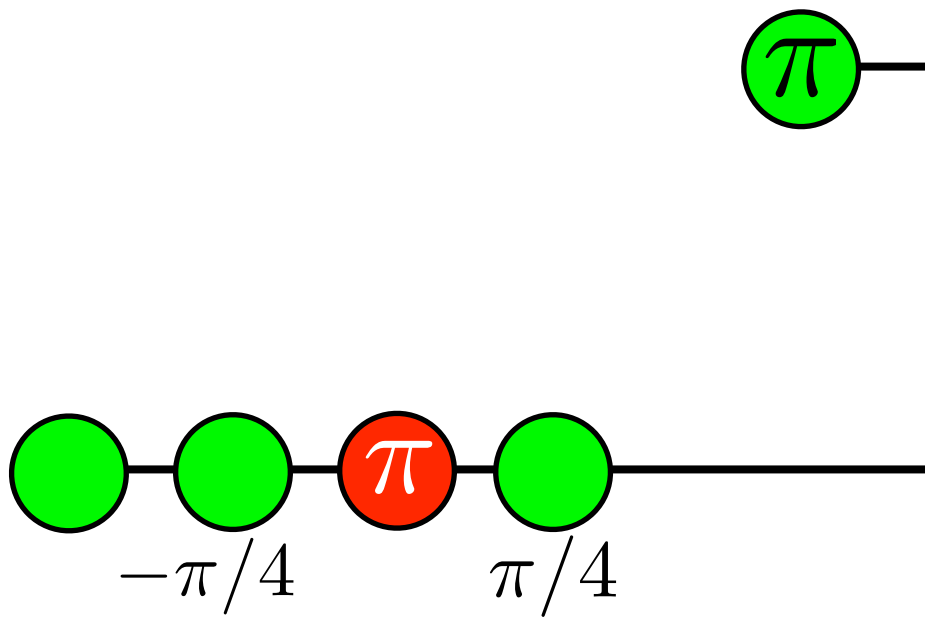

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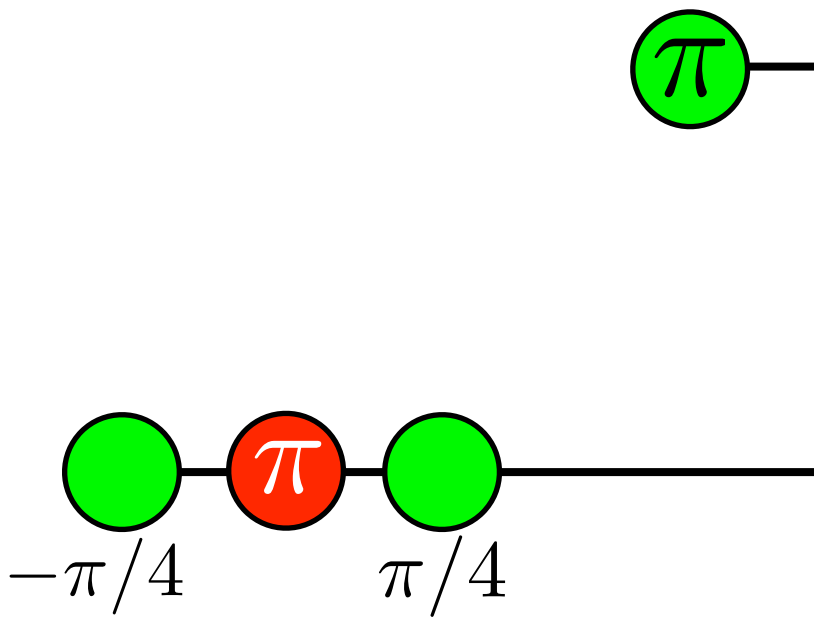

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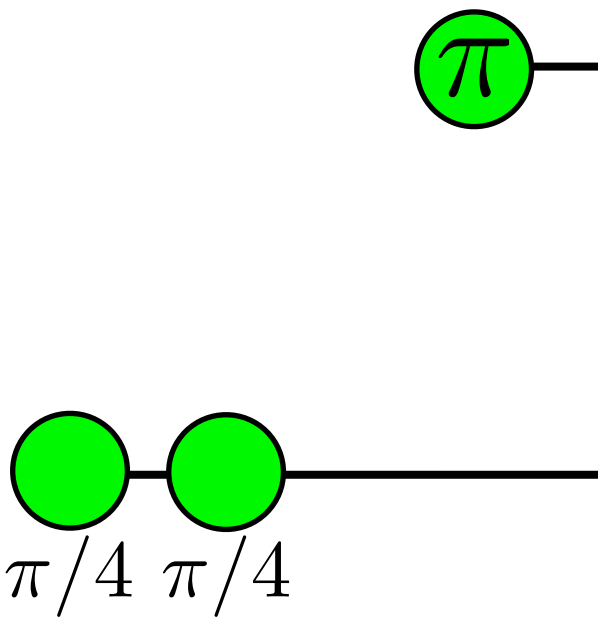

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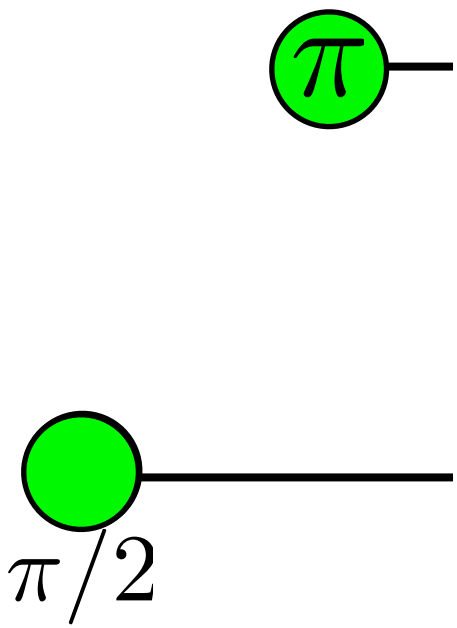

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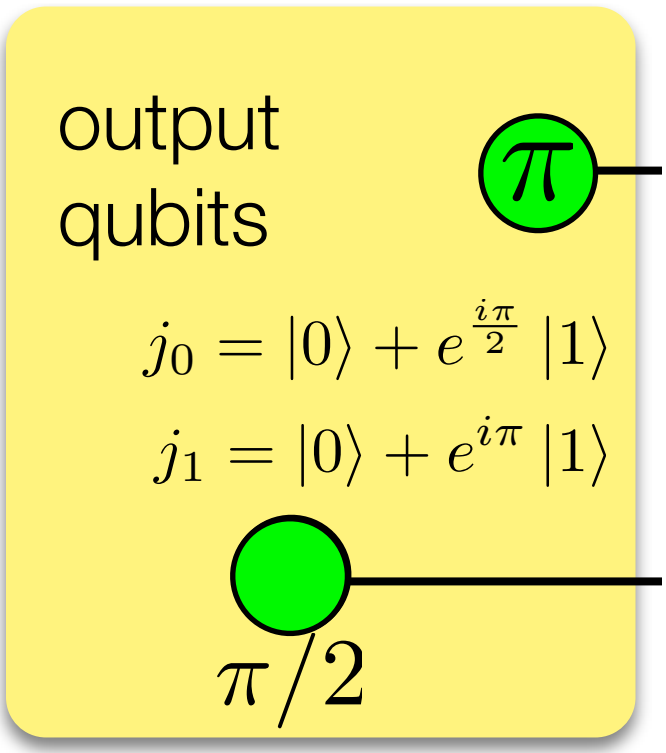
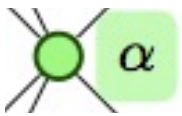
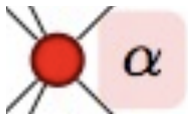


Which diagrams are circuits?

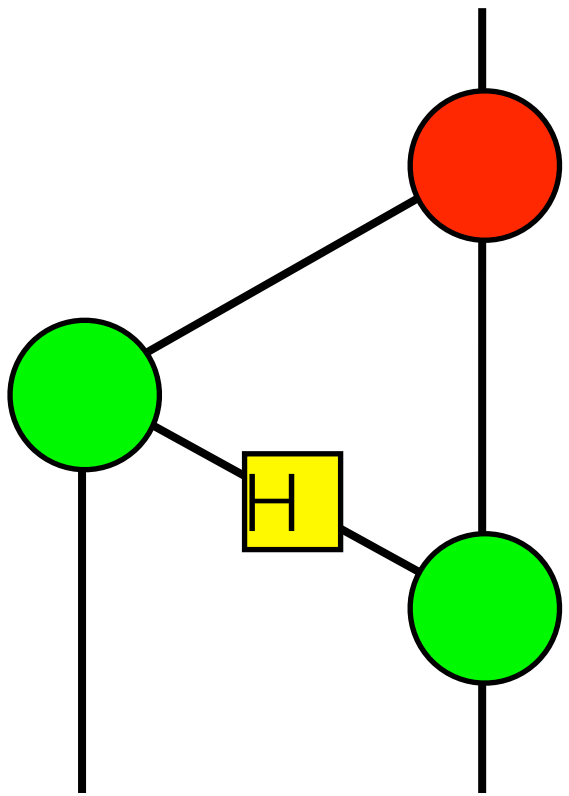
Defn: a diagram is called **circuit-like** if:

- All of its 

, 

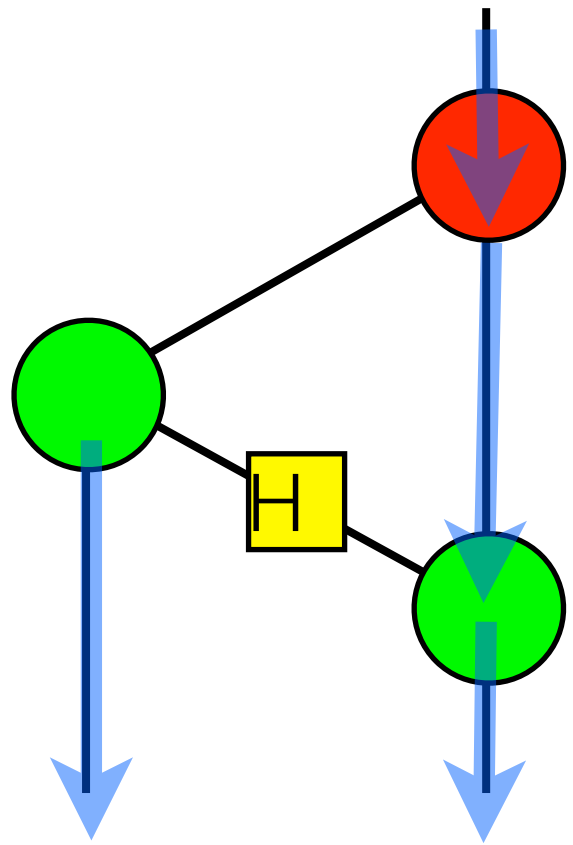
, and boundary vertices can be covered by set of disjoint paths P, each of which ends in an output.
- Every cycle in the diagram which overlaps with 2 paths in P traverses an edge in the opposite direction to P.
- It is 3-coloured.

Which diagrams are circuits?



Which diagrams are circuits?

YES!



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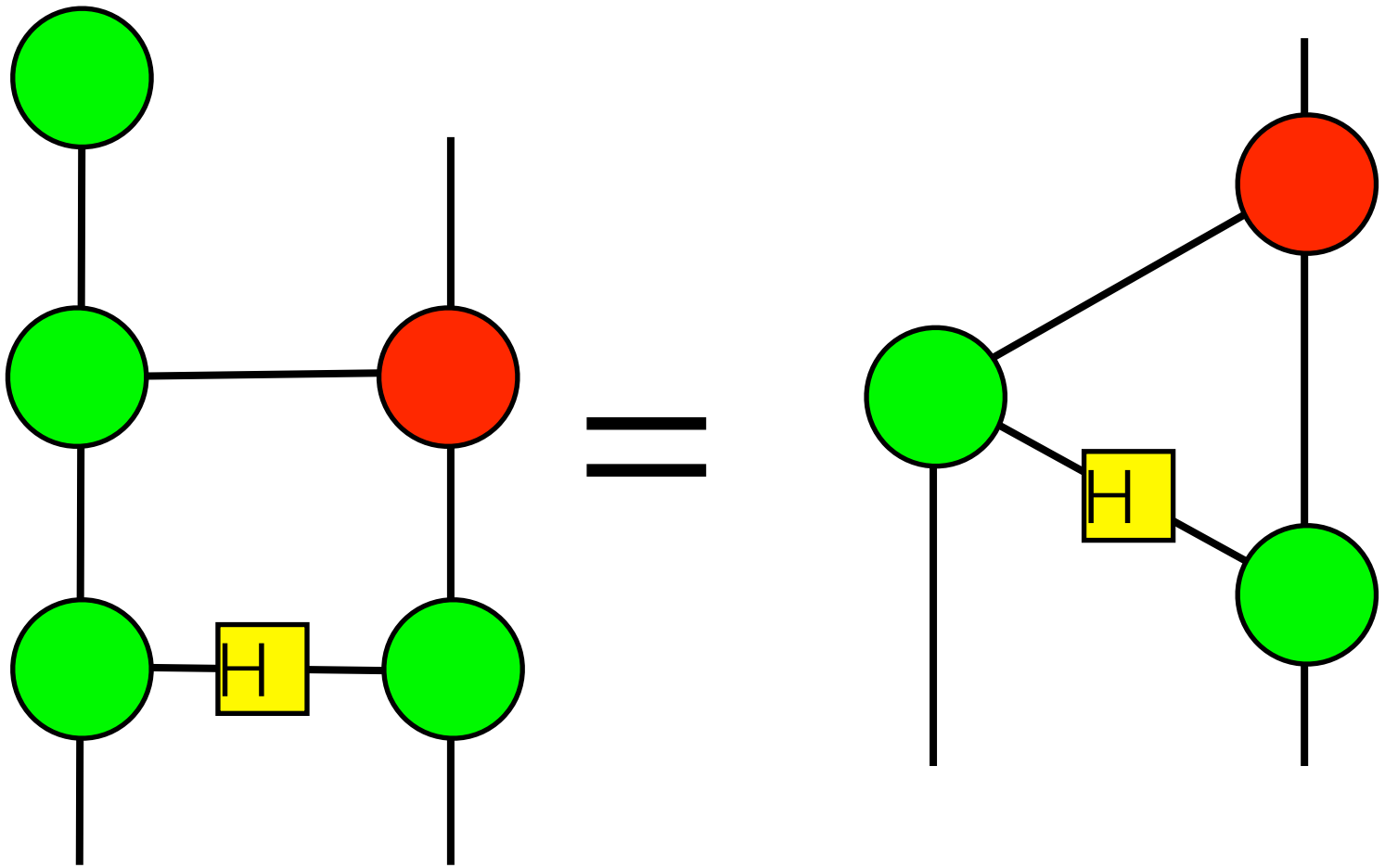

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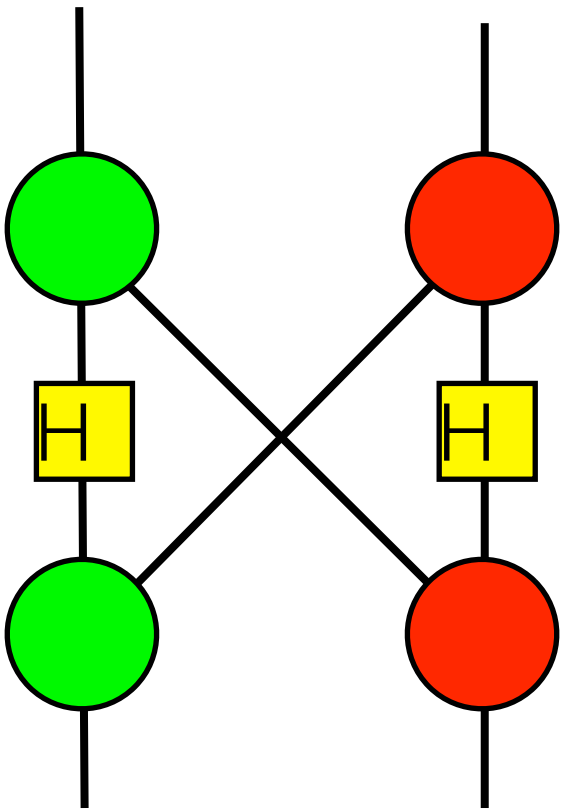
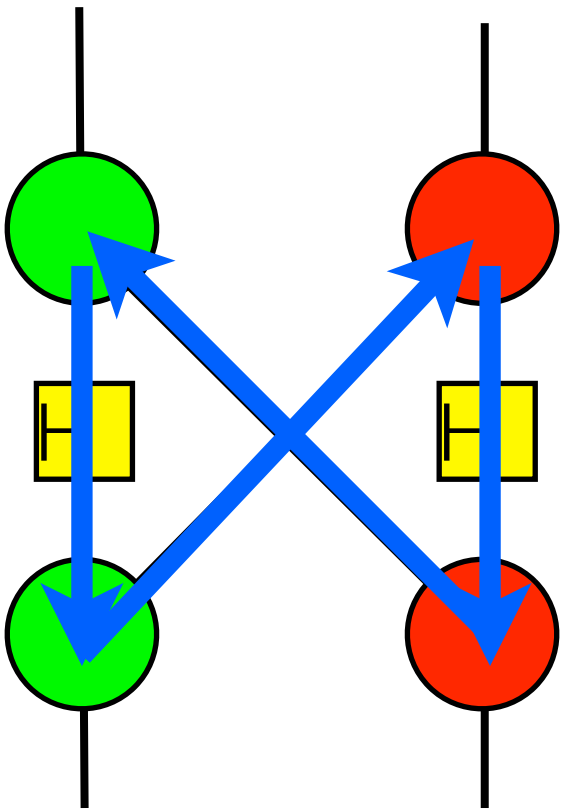


Which diagrams are circuits?

NO :(

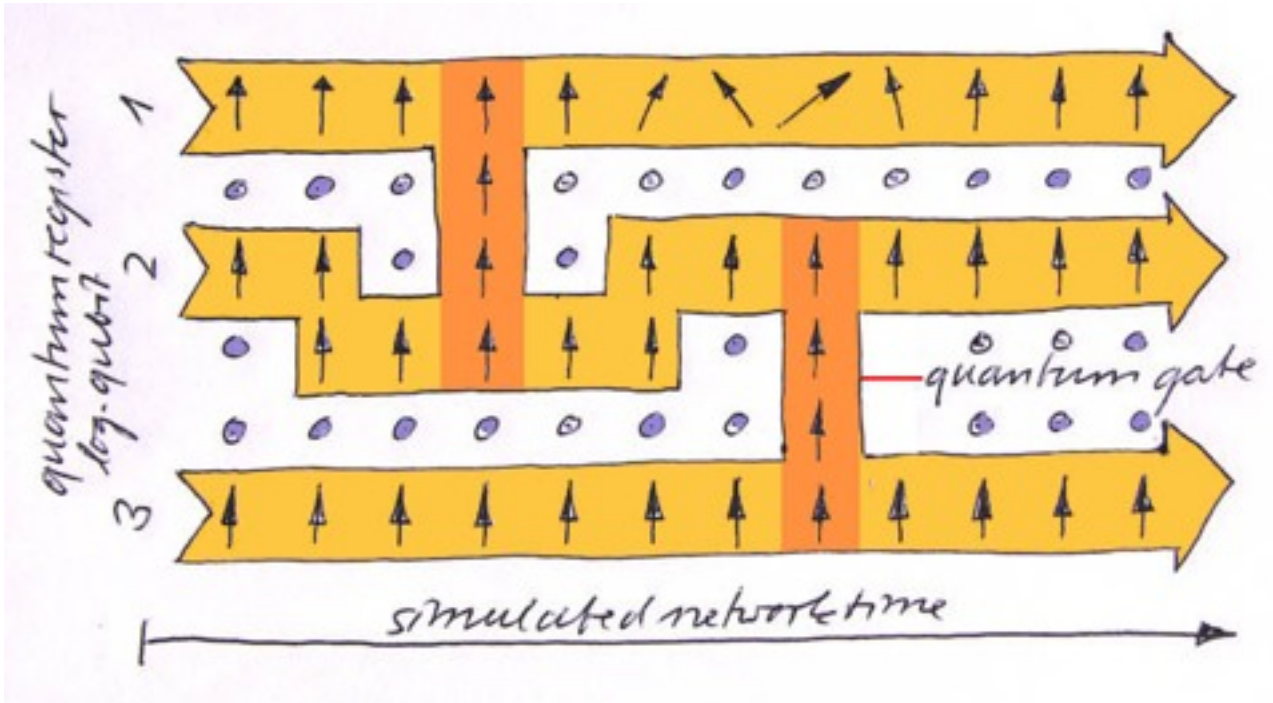


Which diagrams are circuits?

Thm: if a diagram D is circuit-like then $[\![D]\!]$ is a unitary embedding.

Being circuit-like is a little stronger than being a unitary embedding: we require a particular gate set, and a particular encoding of the gate set as diagrams, and the minimality w.r.t. to spider. Up to these conditions, the theorem is *if and only if*.

Information Flow in the 1-Way Quantum Computer

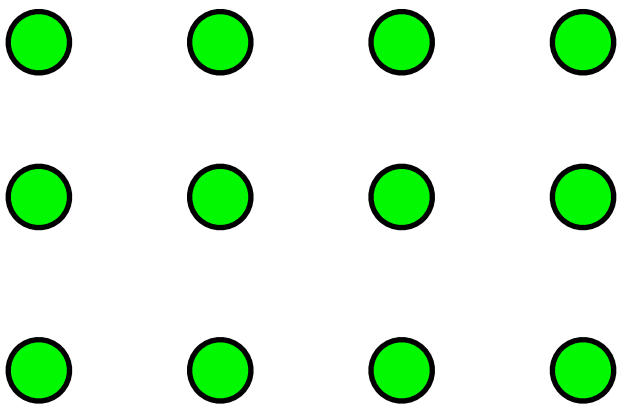


Using measurements to compute

Raussendorf and Briegel, PRL (86) 2001
Raussendorf, Browne and Briegel, J. Mod. Opt. (46) 2003

The One-Way Model

Some qubits, each initialised in the state $|+\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



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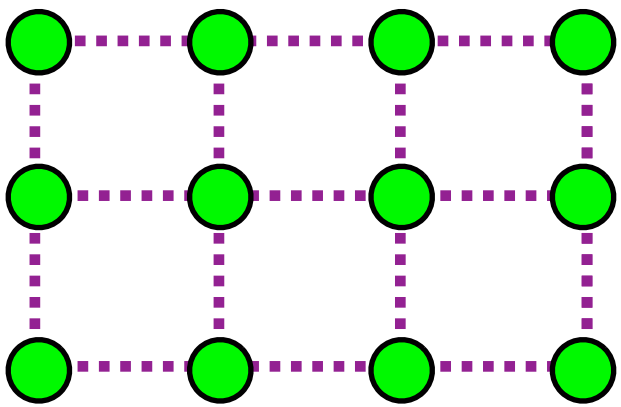

Entangling operation on pairs of qubits to make a **graph state.**

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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Raussendorf, Browne, Briegel J. Mod. Opt. (46) 2003

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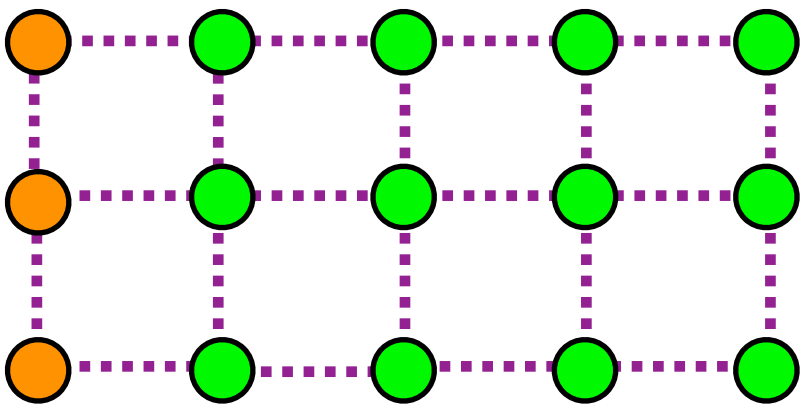
A graph state, coupled to some input qubits:



The only operation is to measure single qubits in the basis:

$$|+_\alpha\rangle := \frac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle)$$

$$|-_\alpha\rangle := \frac{1}{\sqrt{2}}(|0\rangle - e^{i\alpha}|1\rangle)$$

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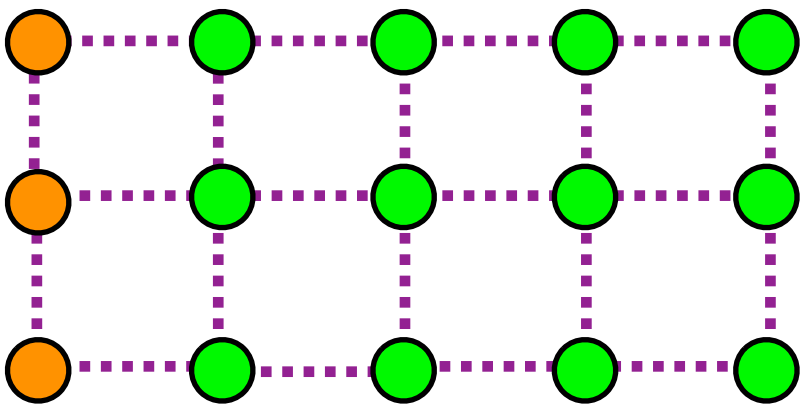

* Measured qubits are removed from the cluster;
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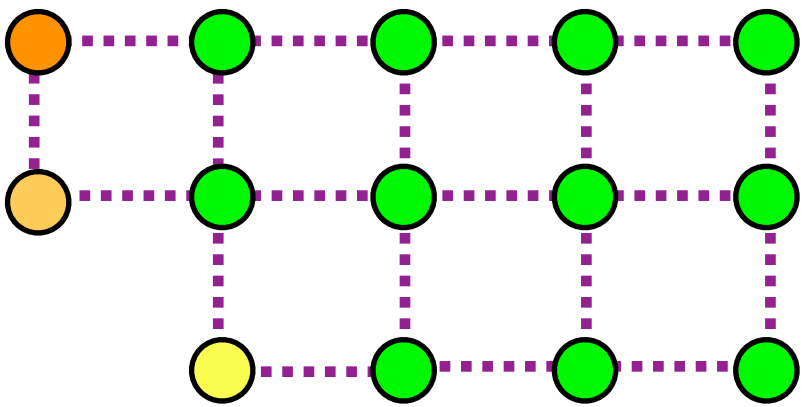

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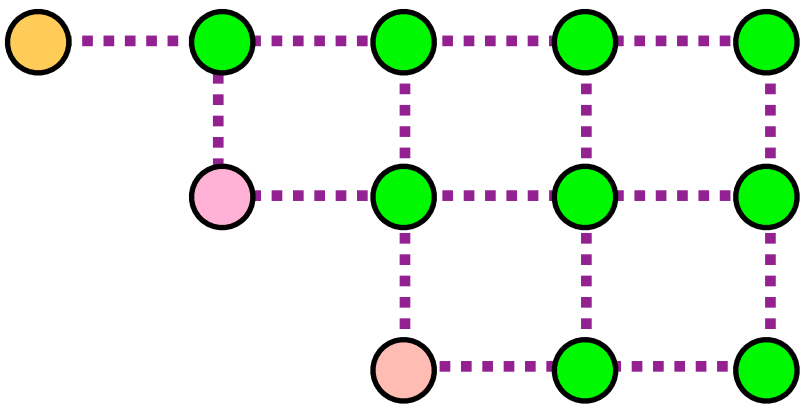

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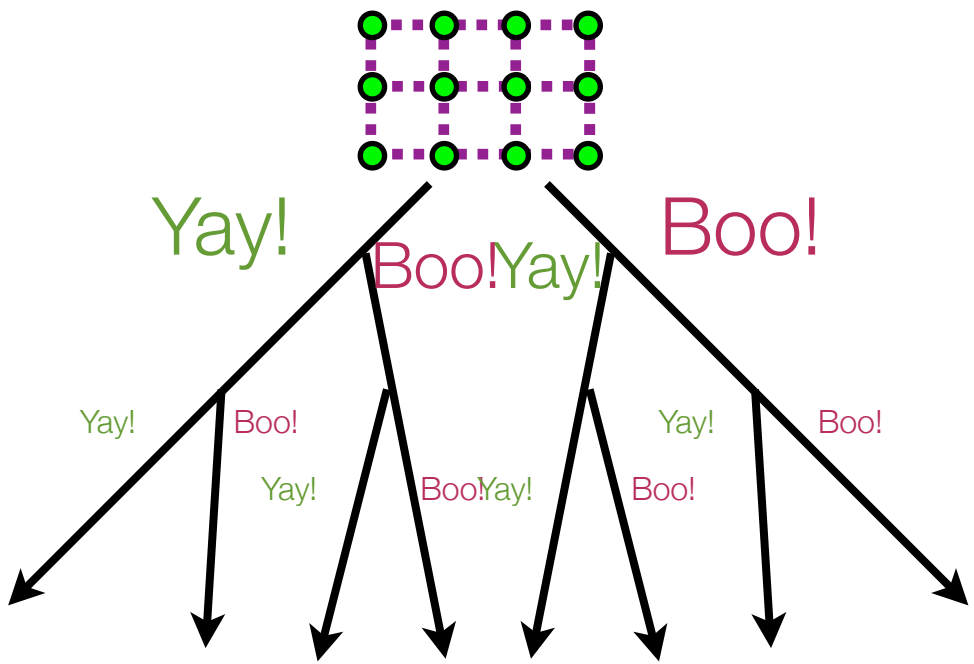
Non-determinism of measurements leads to probabilistic



Attempt to control branching by using *adaptive measurements*:

(choice of measurements depend on the outcome of earlier ones)

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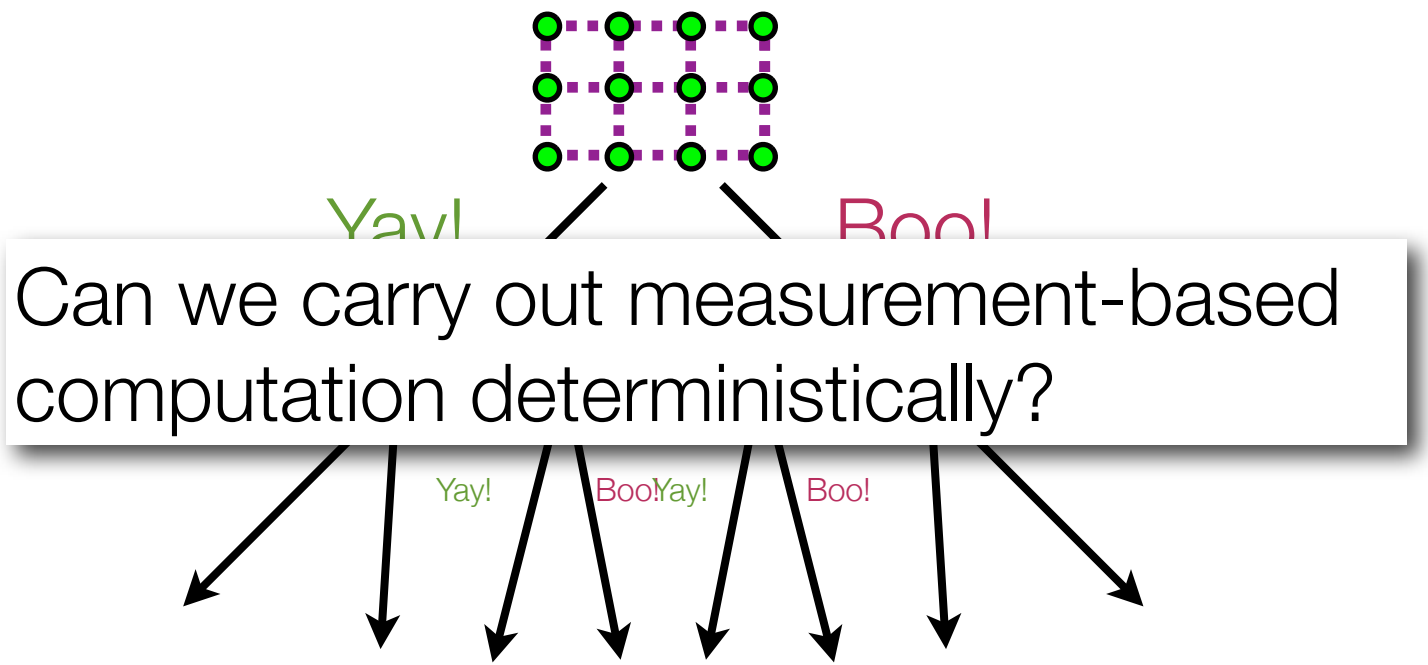

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Measurement Calculus

The measurement calculus is a formal syntax for programming the one-way model.

- N_i : initialise qubit i in the $|+\rangle$ state
- E_{ij} : entangle qubits i and j using a CZ operation
- ${}^{s}[M_i^{\alpha}]^{t}$: measure qubit i in the basis $|0\rangle \pm e^{(-1)^s \alpha + t\pi} |1\rangle$
- X_i^s, Z_i^s: apply a Pauli X or Z operator to qubit i

The boolean variables s, t are *signals*; their values are determined by the results of measuring the corresponding qubit.

- Standard form: operations occur in the order above
  - Every pattern can be standardised

How to Measure

Suppose we have a measuring device for the standard 1-qubit basis:

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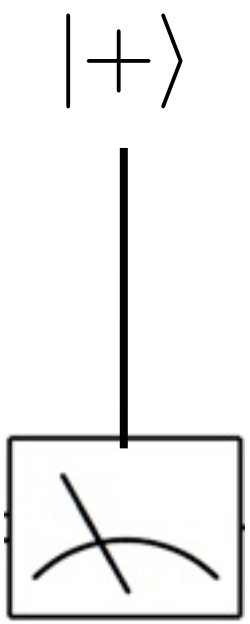
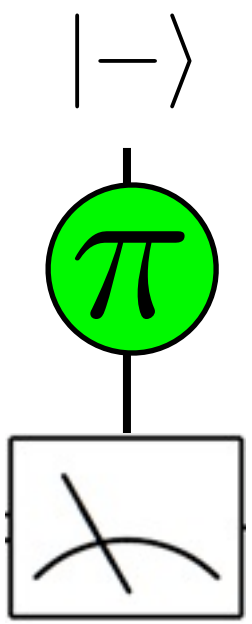
Should be this:

Yay!





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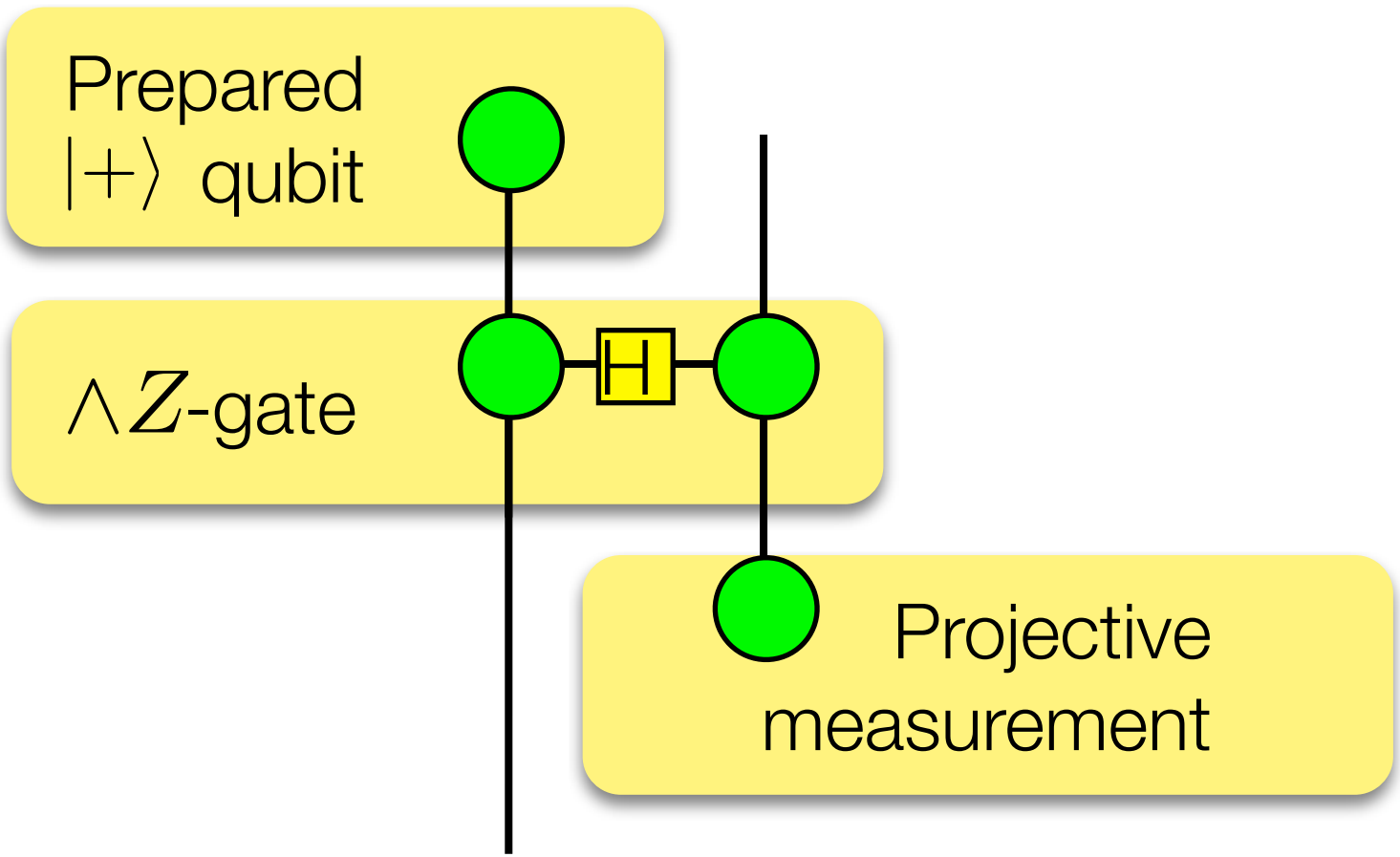
Example: Hadamard



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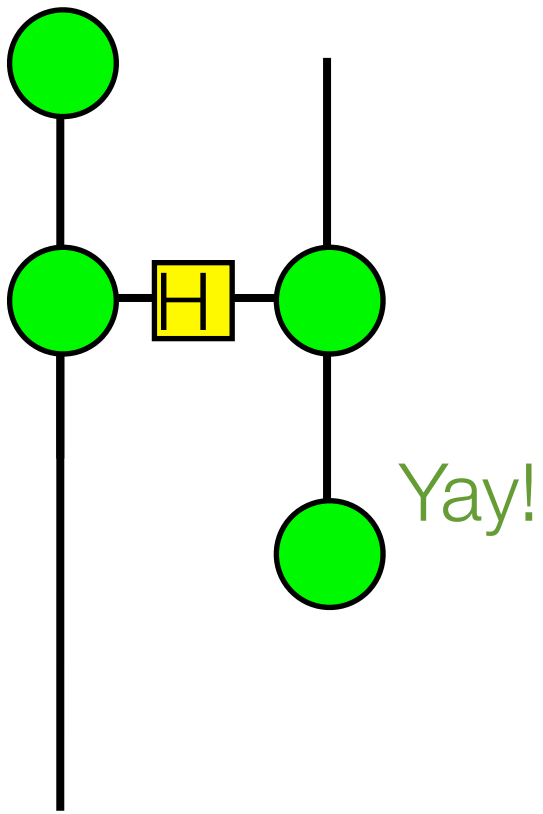

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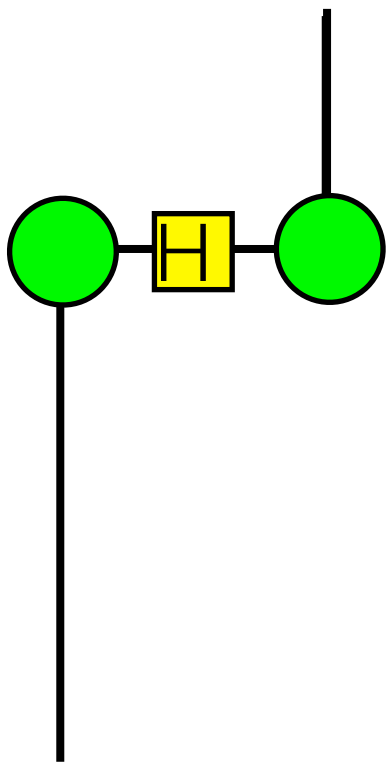

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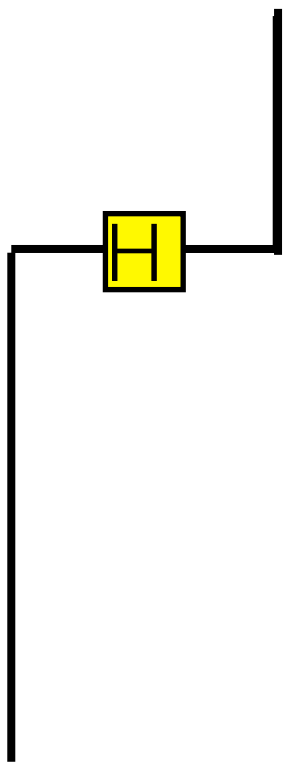

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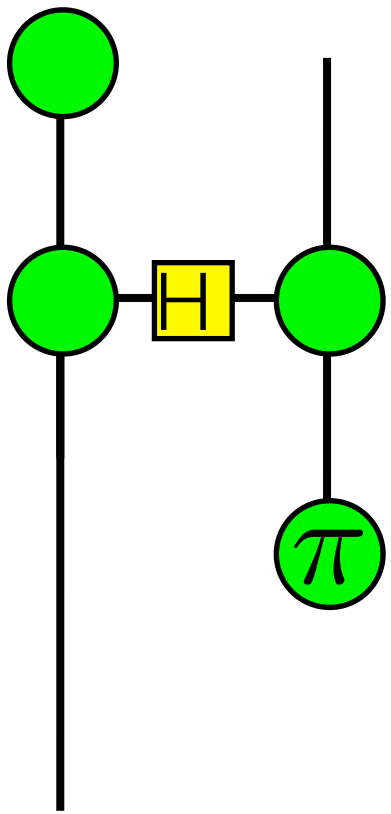

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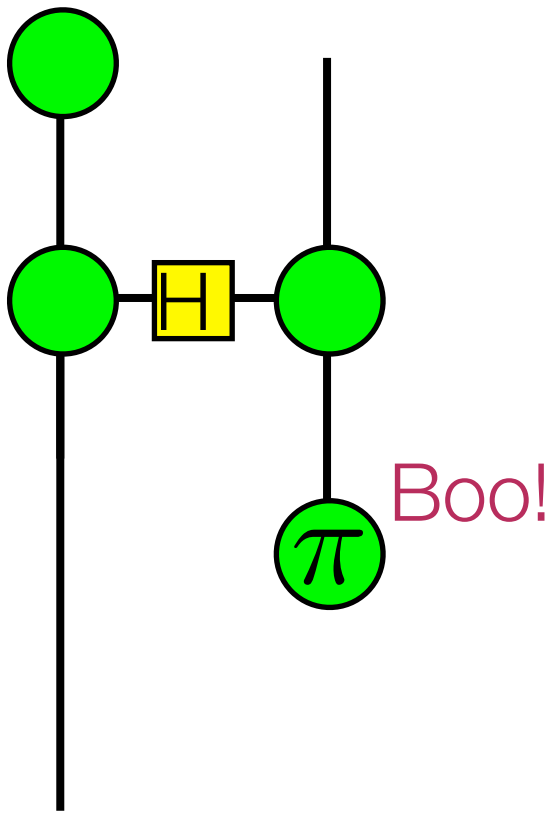

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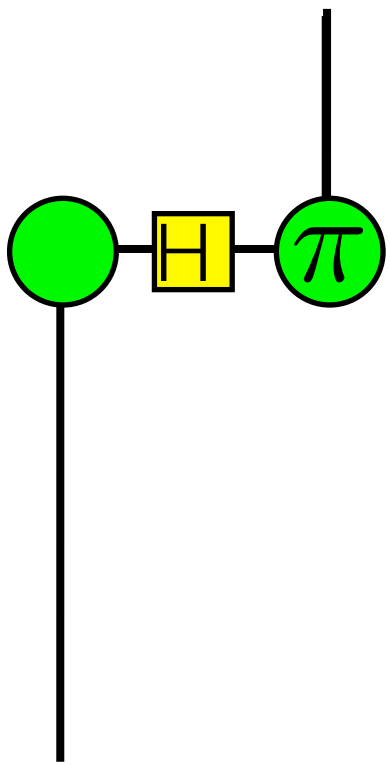

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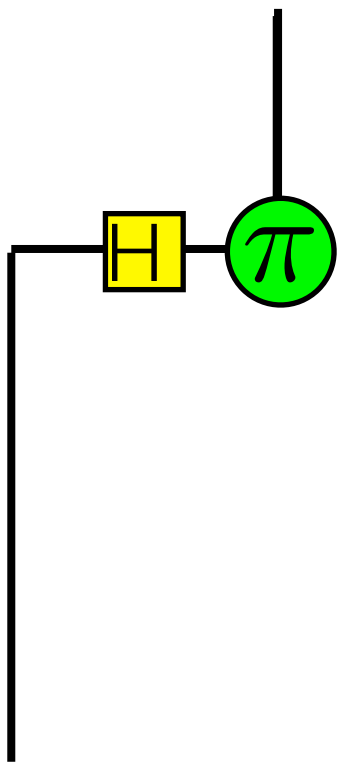

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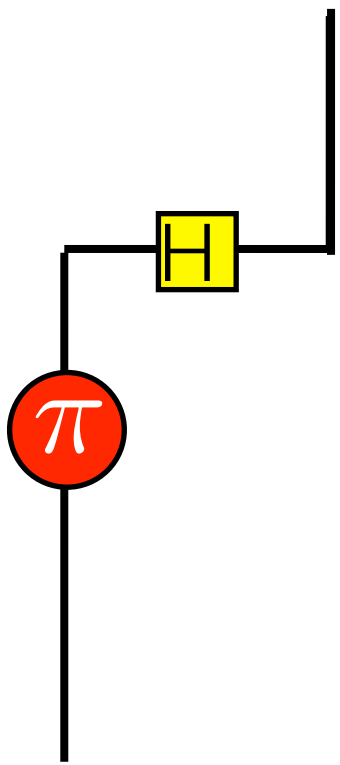

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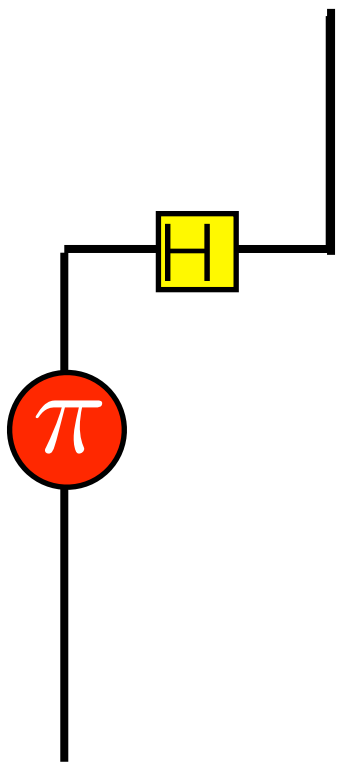

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Add correction here:

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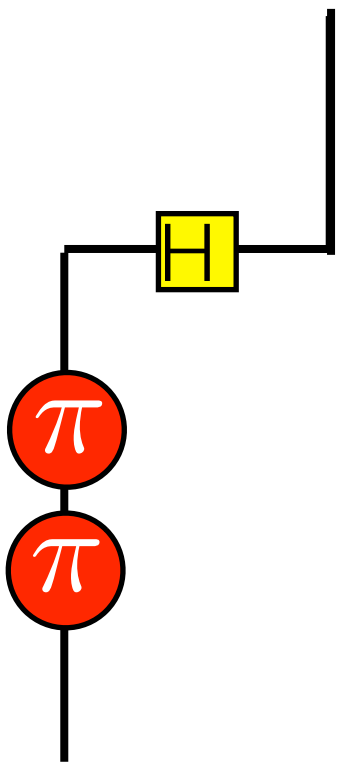

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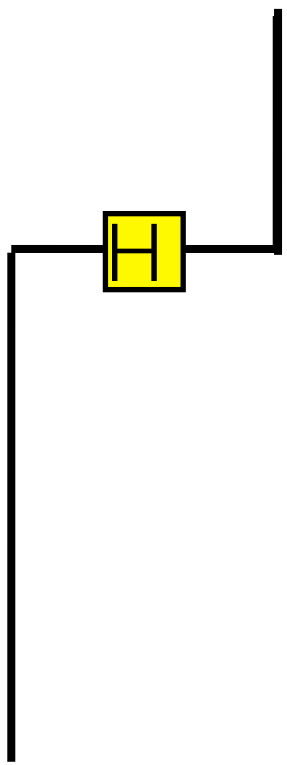

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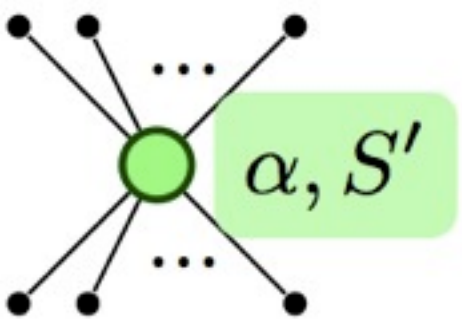
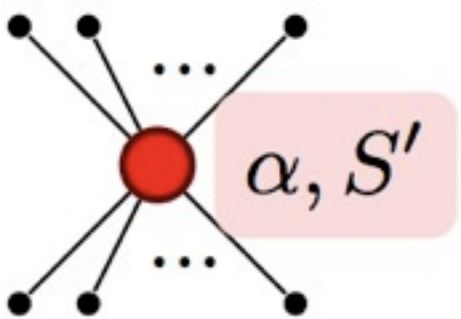
Add correction here:

Conditional Operations

Defn: Let S be a set of variables ("signals"); define $\mathbb{D}(S)$ to be the category of diagrams generated by:





where $S' \subseteq S$

Semantics

Defn: a *valuation* of S is a function $v : S \to \{0, 1\}$. For each valuation we can define a map $\hat{v} : \mathbb{D}(S) \to \mathbb{D}$ by replacing the angle alpha with 0 if

$$\prod_{s \in S'} v(s) = 0$$

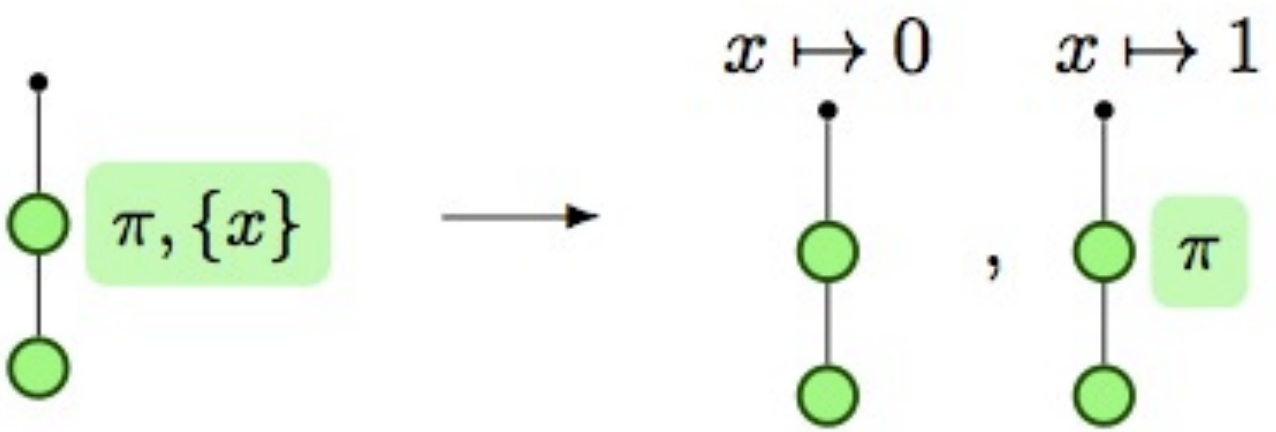
and leaving if unchanged otherwise.

Defn: The **denotation** of a diagram D in $\mathbb{D}(S)$ is a superoperator:

$$\rho \mapsto \sum_{v \in 2^S} [\![\hat{v}(D)]\!] \rho [\![\hat{v}(D)]\!]^\dagger$$

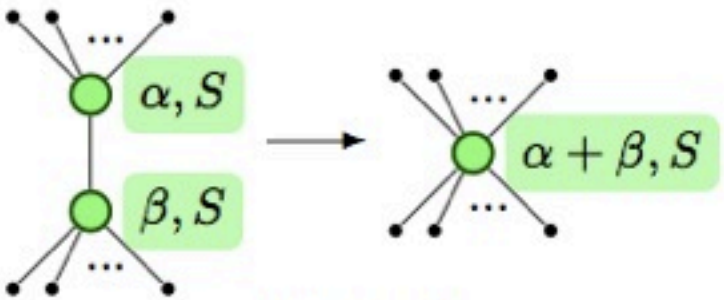
Example

A measurement in the basis: $|+\rangle, |-\rangle$



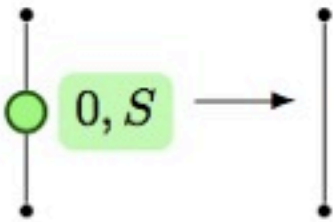
$$\rho \mapsto \langle +|\,\rho\,|+\rangle + \langle +|\,Z\rho Z\,|+\rangle$$
$$= \langle +|\,\rho\,|+\rangle + \langle -|\,\rho\,|-\rangle$$

Equations for conditional diagrams

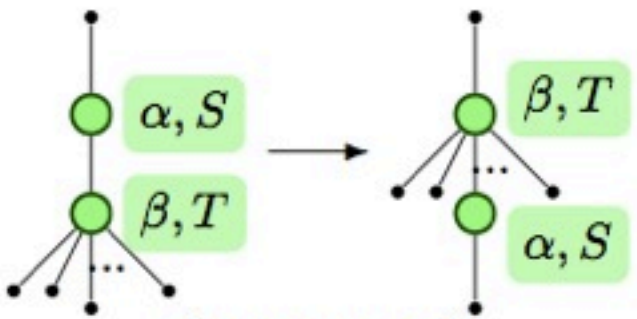


(spider)

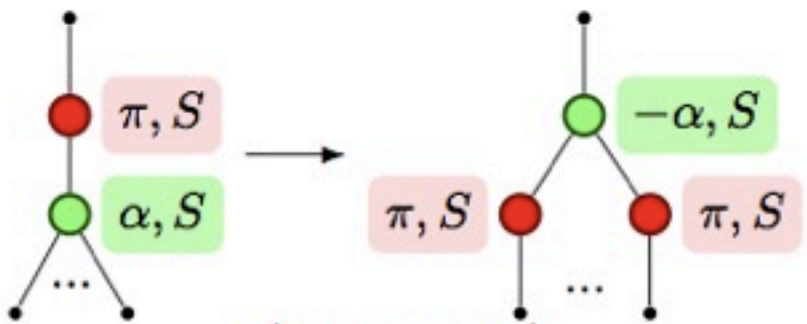
(anti-loop)



(identity)

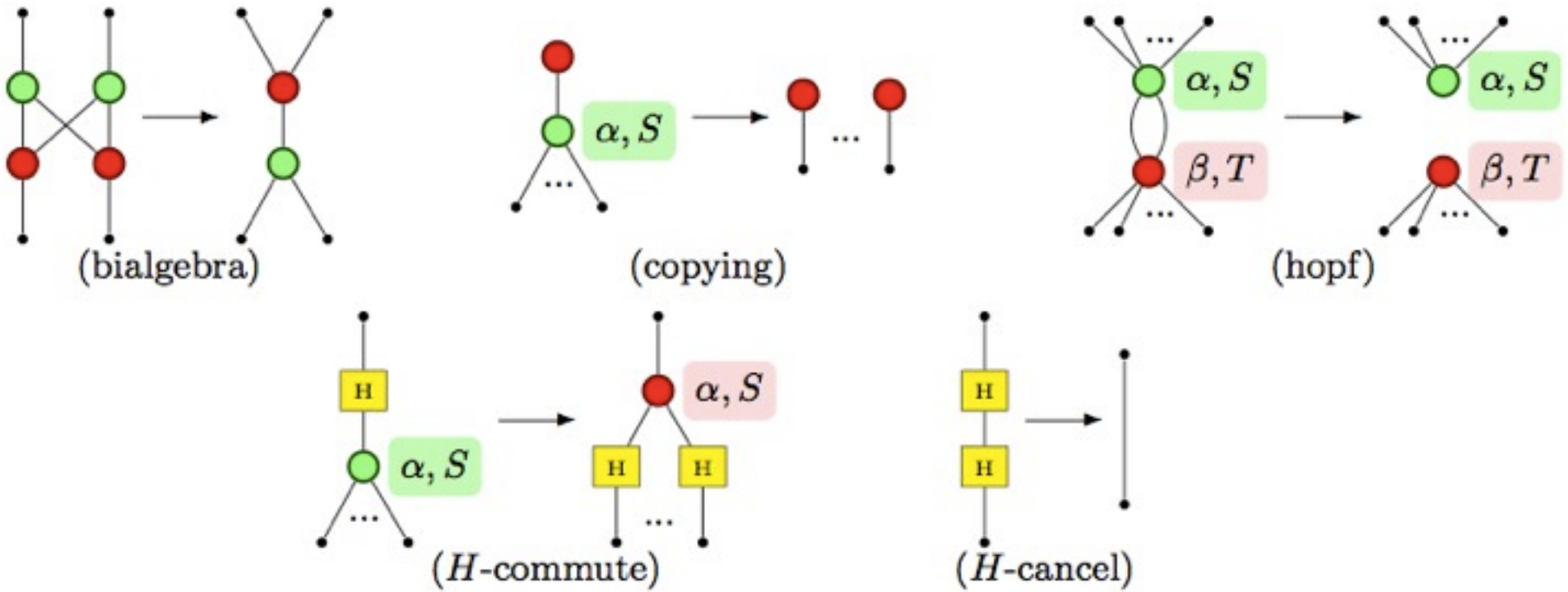


(α-commute)

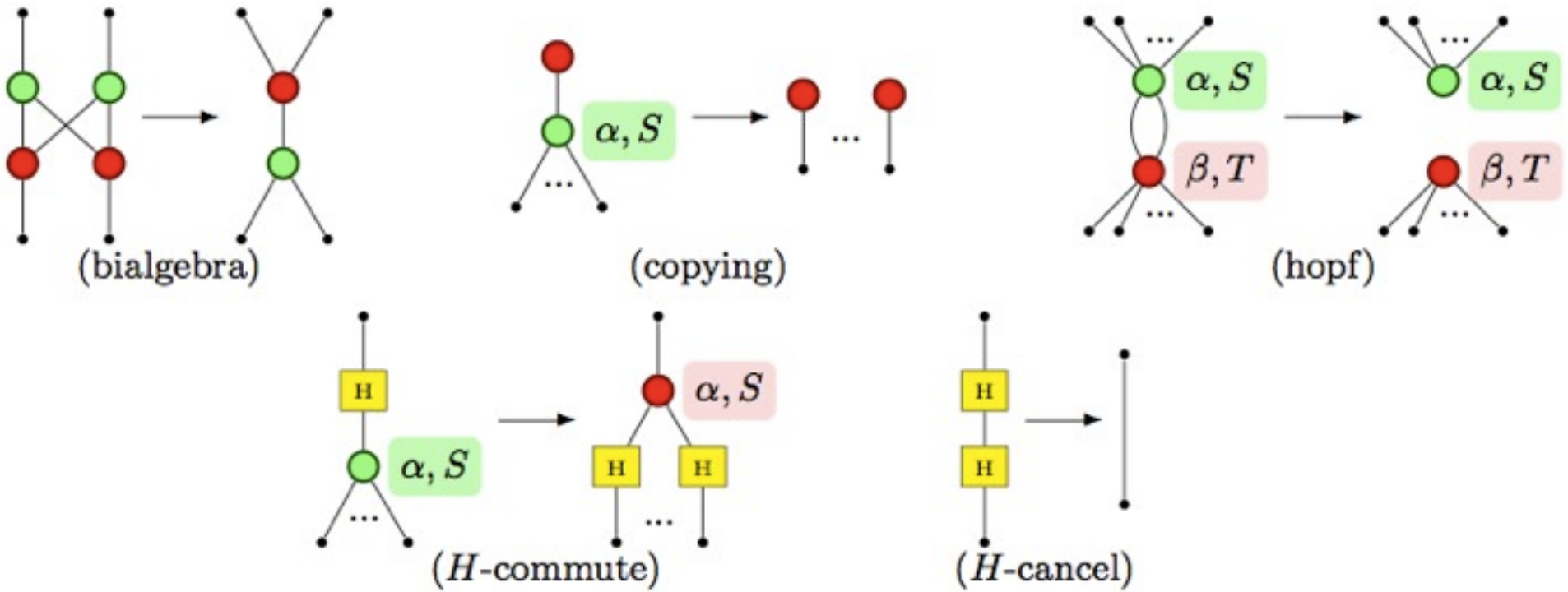


(π-commute)

Equations for conditional diagrams



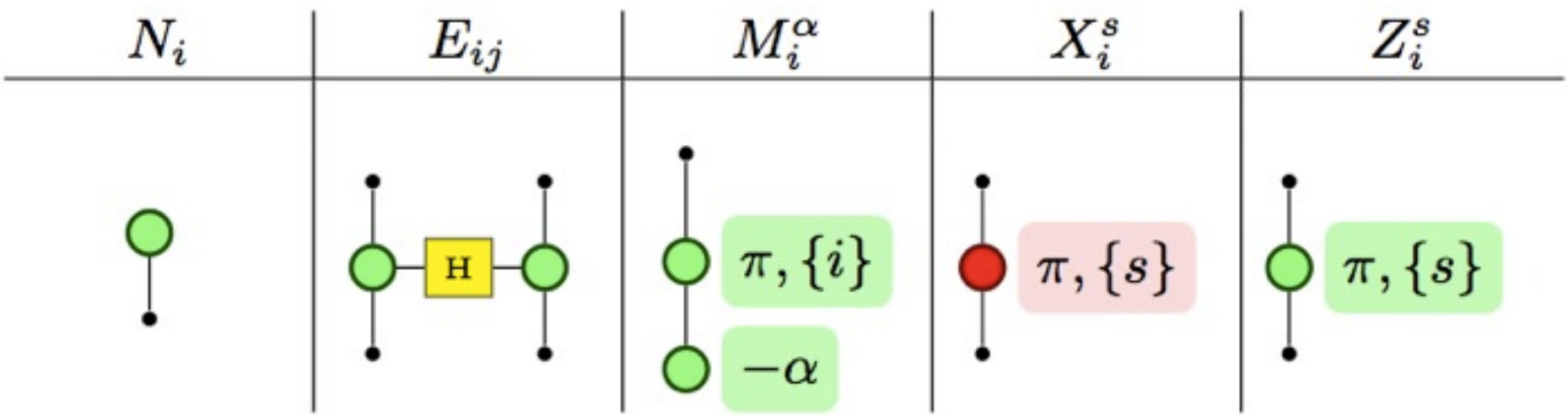
Equations for conditional diagrams



.... plus the same again with the colours exchanged

Translation from MC to ZX-calculus

We can translate any measurement pattern to diagram using the table below:



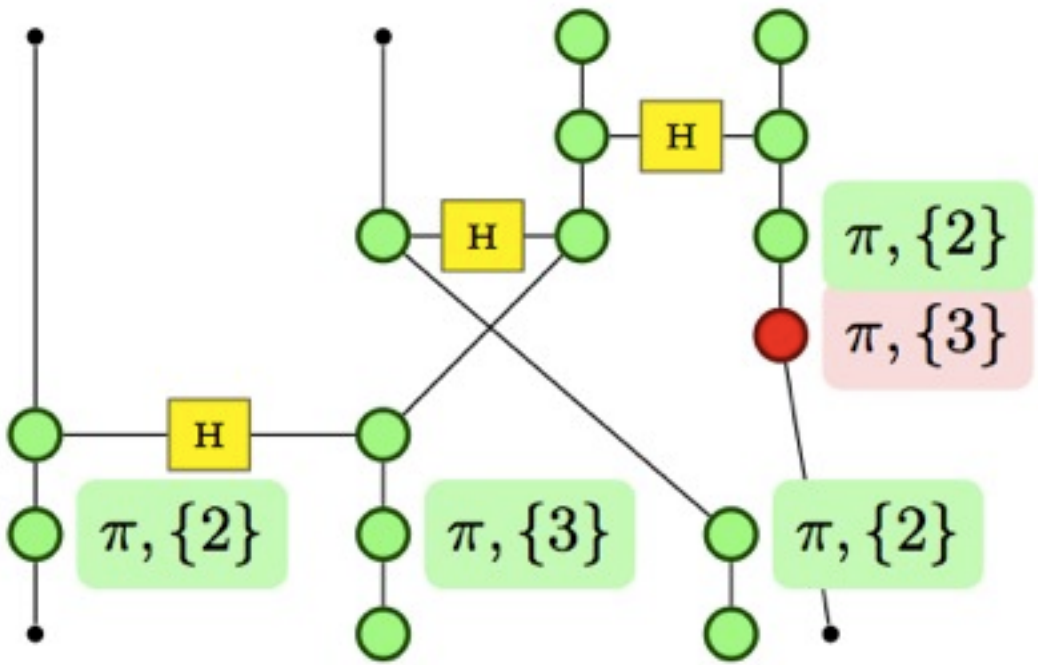
Example

The CNOT is computed by the pattern:

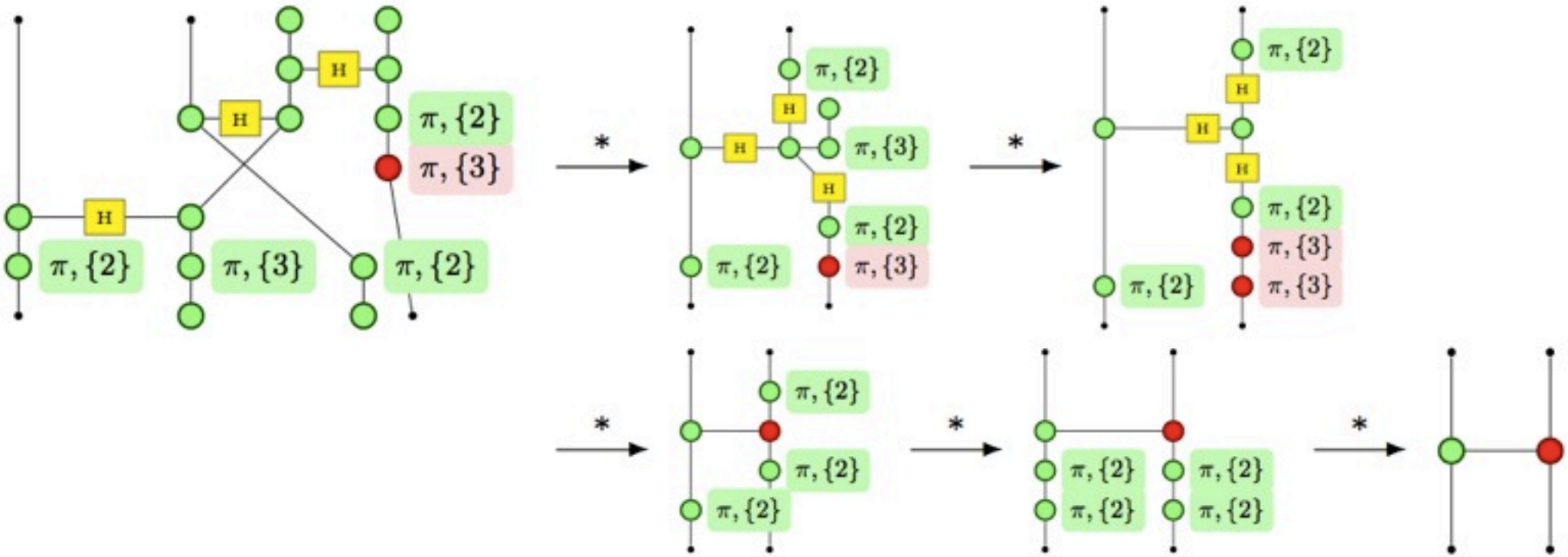
$$\mathfrak{P} = X_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 E_{13} E_{23} E_{34} N_3 N_4$$

Which yields the diagram:

$D_{\mathfrak{P}} =$



Example Rewrite



We removed all the conditional operations, therefore this **pattern** is deterministic.

Can we do this in general?

Geometry of a pattern

The following pattern computes the CNOT gate:

$$\mathfrak{P} = \bar{X}_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 \bar{E}_{13} E_{23} E_{34} N_3 N_4$$

Each pattern implicitly defines a *geometry* (G, I, O)

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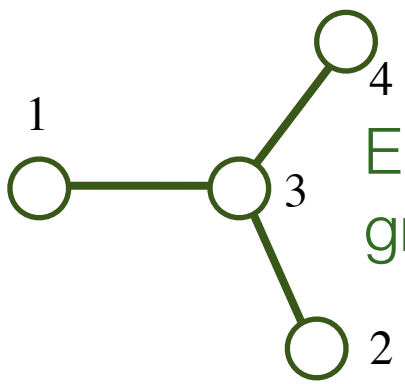

Entanglement graph

Geometry of a pattern

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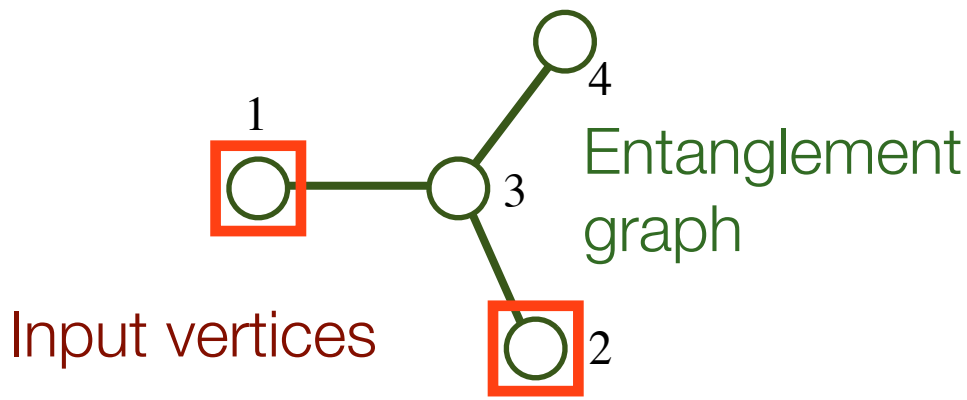

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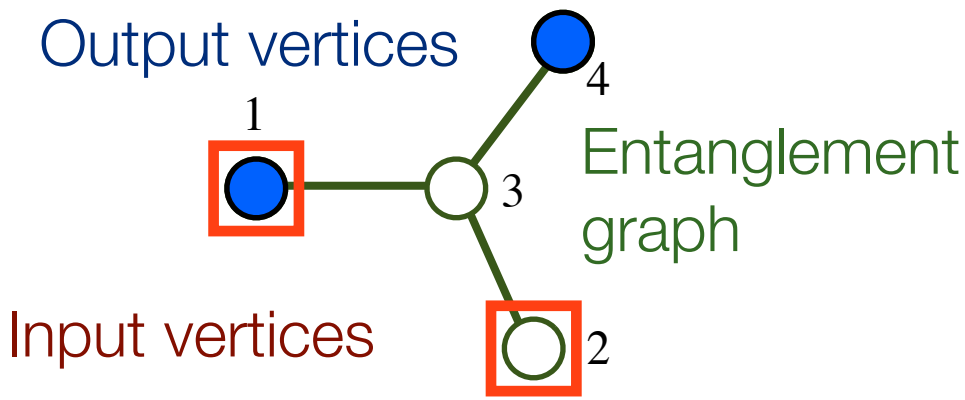


Flow and determinism

Defn: let (G, I, O) be a geometry; a *flow* on G is a pair $(f, <)$ where $f : O^C \to I^C$ is a function and $<$ is a partial order s.t.

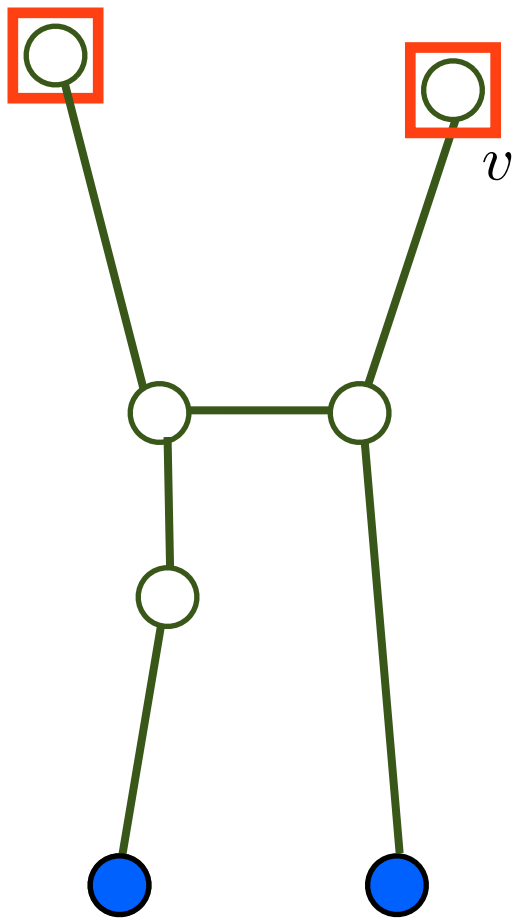
- $f(u) \sim u$
- $u < f(u)$
- If $f(u) \sim v, v \neq u$ then $u < v$

Intuitively $f(u)$ is the qubit which must correct the error produced by measuring u. The partial order guarantees that this is consistent with causality.

Thm: if a geometry has a flow there exists a uniformly deterministic pattern on it.

Flow

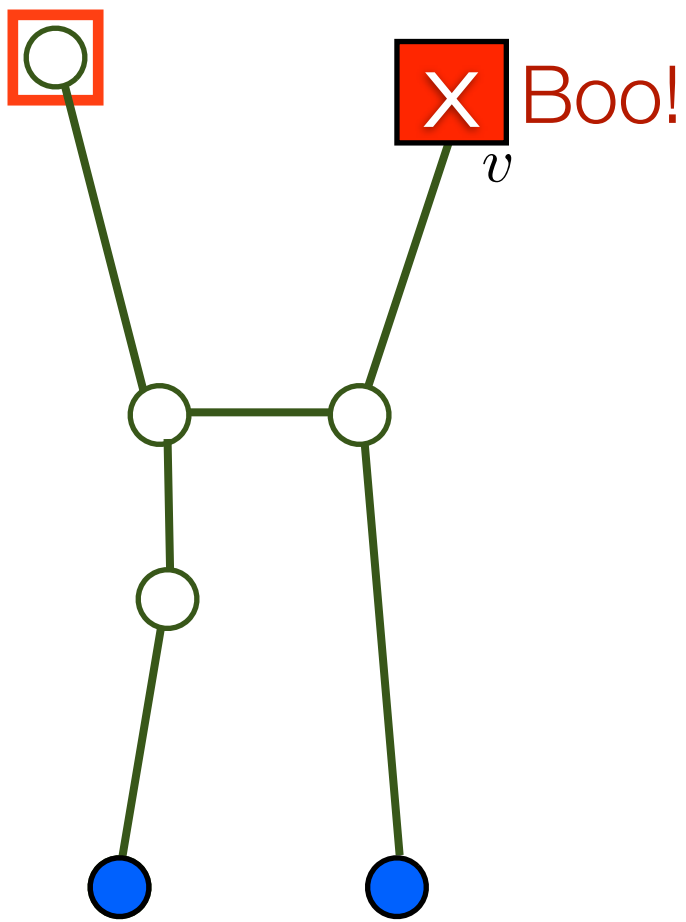
Input vertices



Output vertices

Flow

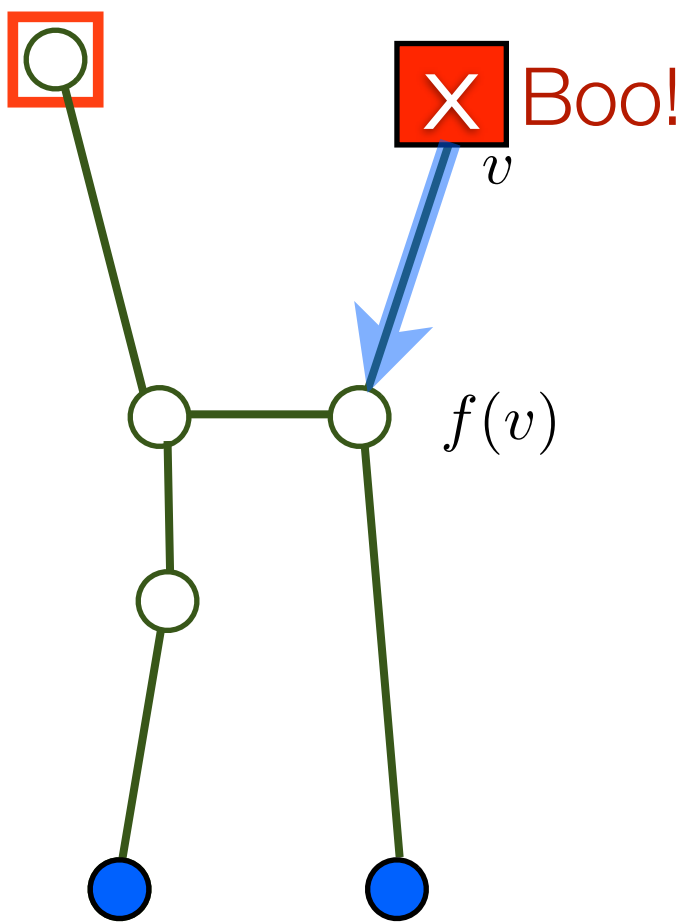
Input vertices



Output vertices

Flow

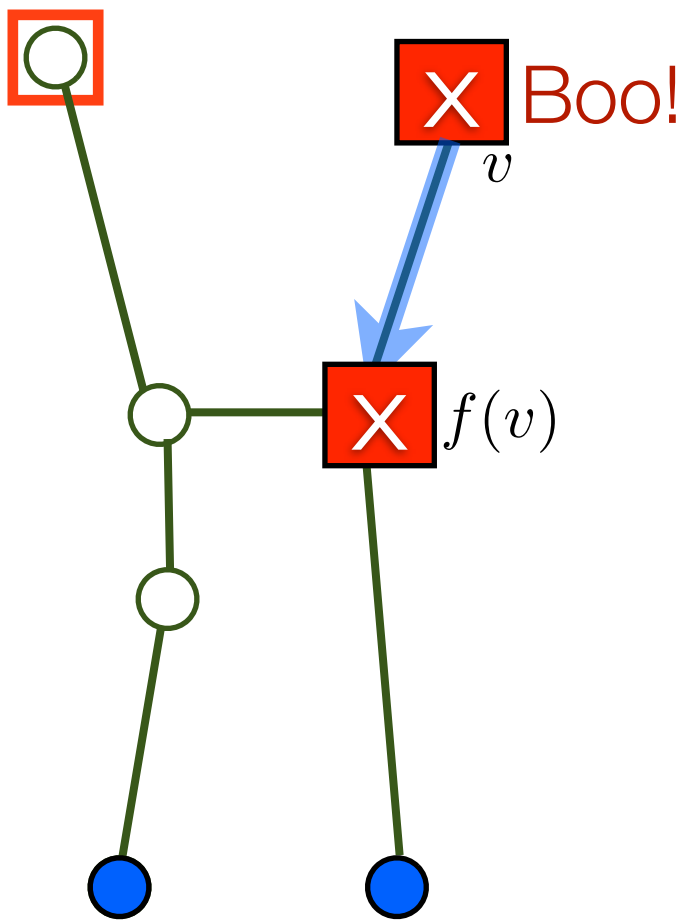
Input vertices



Output vertices

Flow

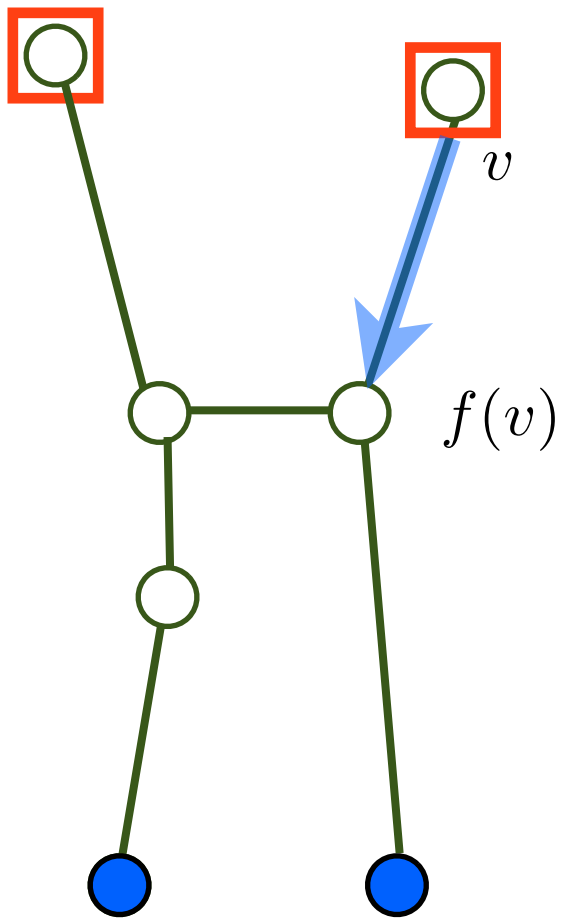
Input vertices



Output vertices

Flow

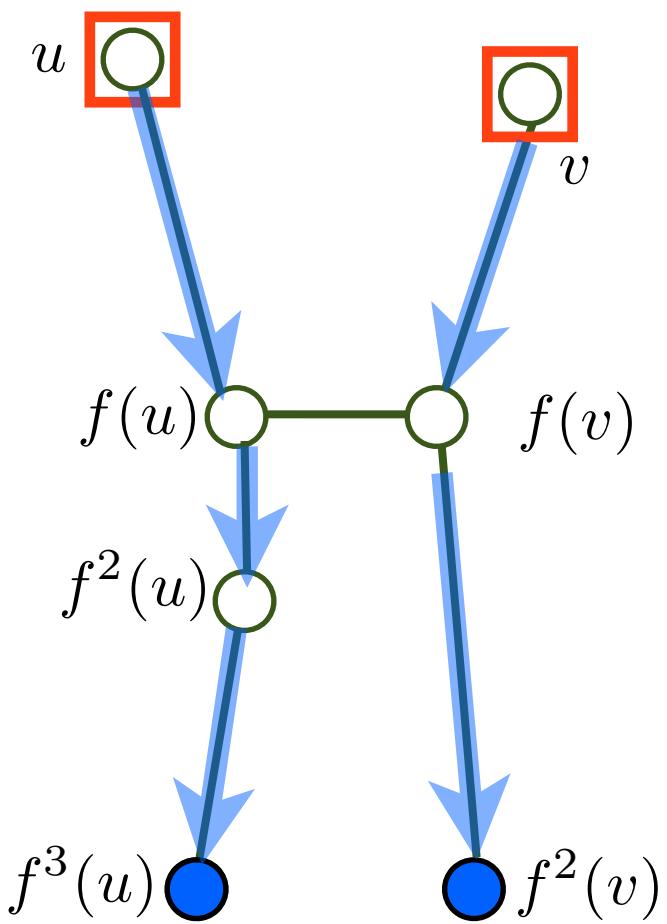
Input vertices



Output vertices

Flow

Input vertices



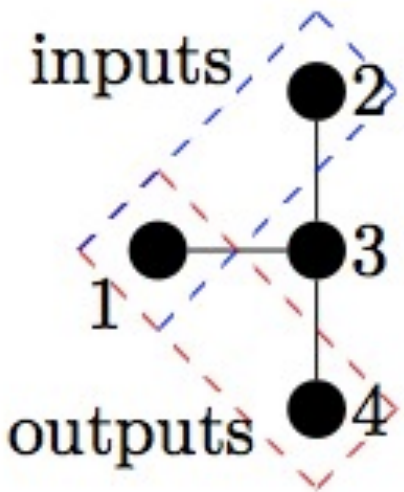
Output vertices

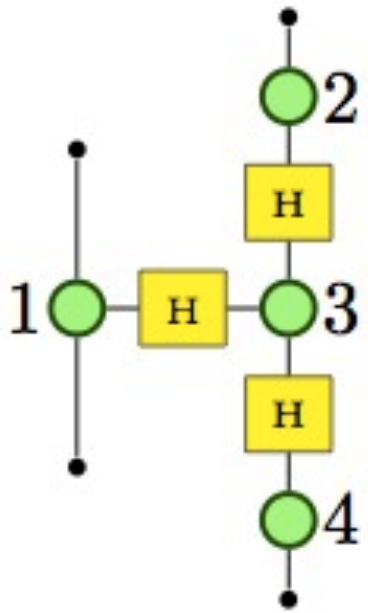
The sequence
$u, f(u), f^2(u), \ldots$
determines a path from inputs to outputs.

Effectively the flow path" is a "logical qubit" in a circuit equivalent to the pattern.

Geometry of a pattern

We can define a diagram directly from the geometry (G, I, O)

$\Gamma_{\mathfrak{P}_\wedge x} =$ 

$D(\Gamma_{\mathfrak{P}_\wedge x}) =$ 

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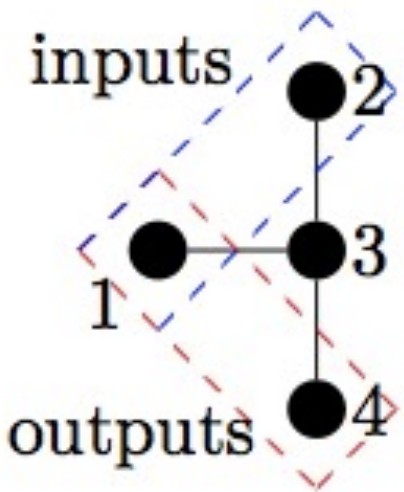
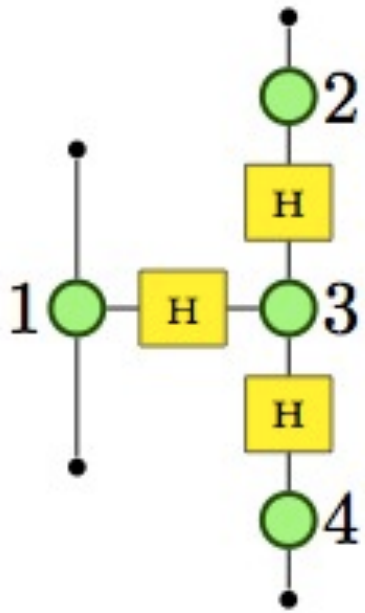
$\Gamma_{\mathfrak{P}_\wedge x} =$ 

$D(\Gamma_{\mathfrak{P}_\wedge x}) =$ 

Theorem: If geometry G has a flow then $D(G)$ is (equivalent to) a circuit-like diagram.

Getting back to the pattern

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We have $D(G(P))$ and $D(P)$. What is the relation?

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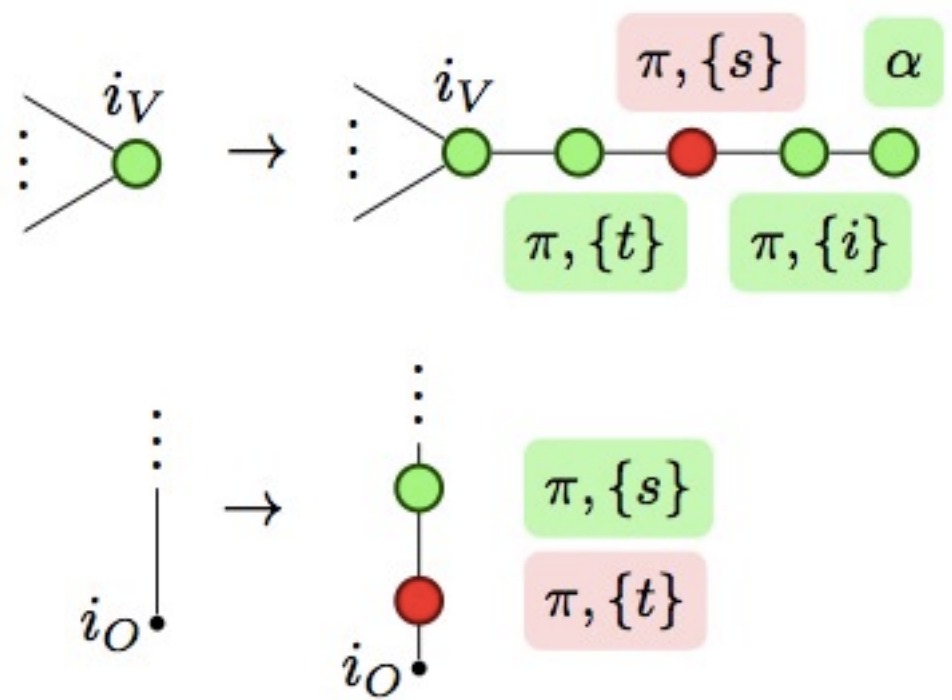

Getting back to the pattern

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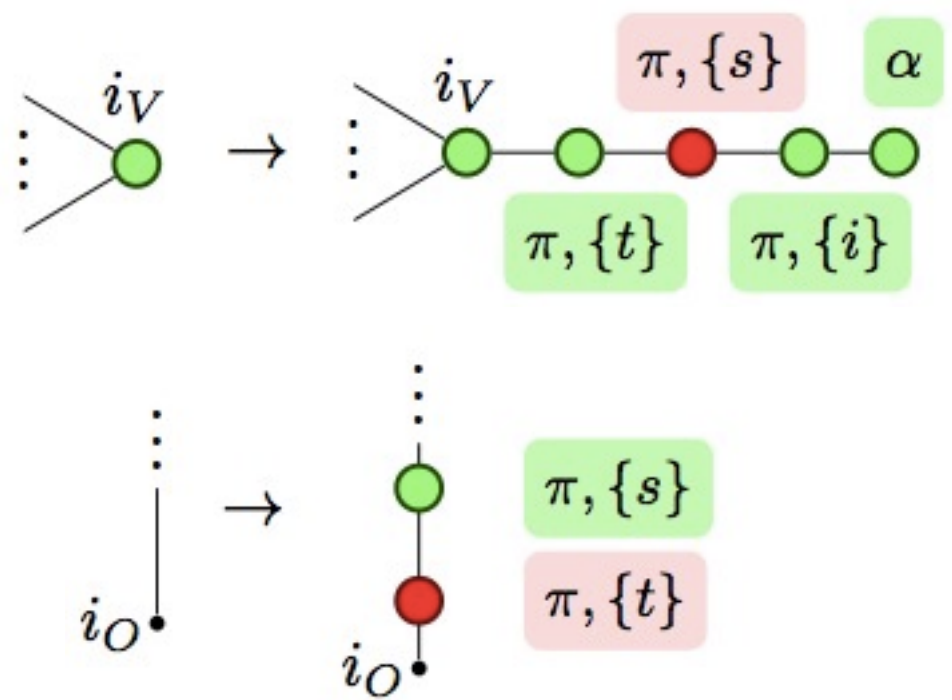
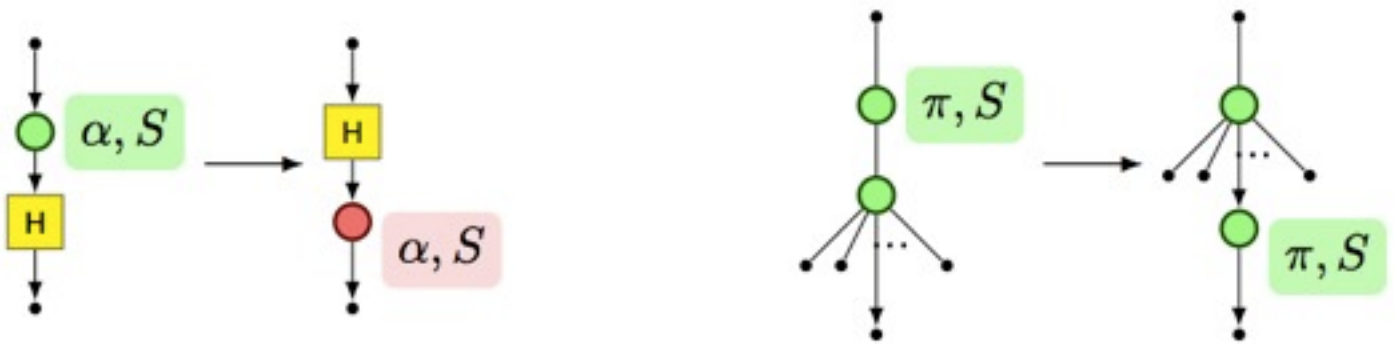


Prop: $D(P)$ rewrites to $D^*(P)$

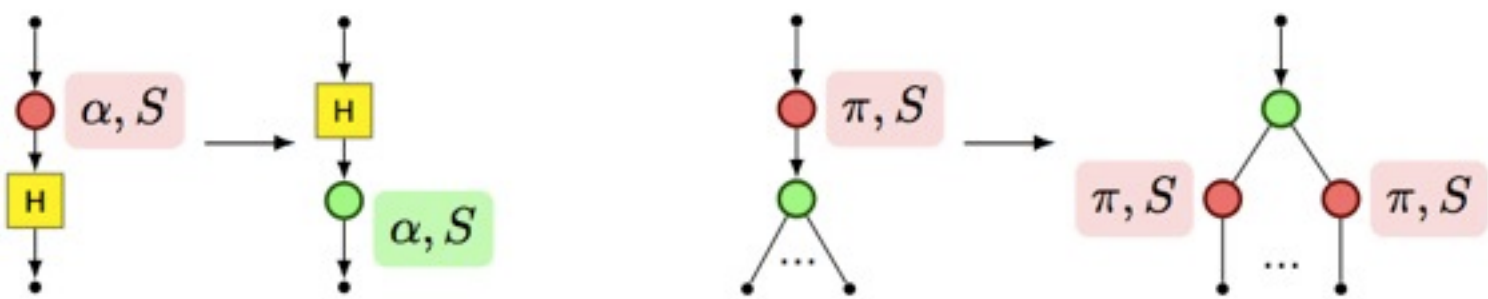
Flow Strategy

If the geometry has a flow we can define a rewrite strategy to propagate the “errors” forward:

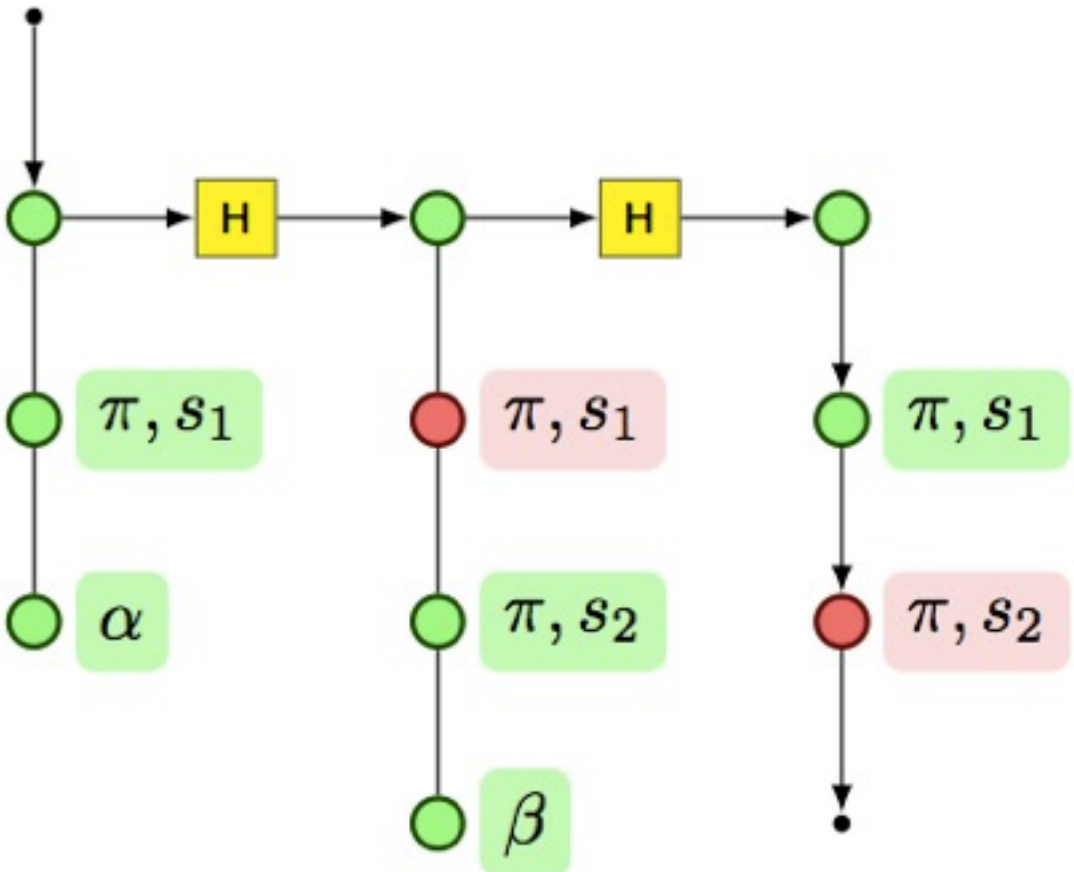
'π-green is attracted by the flow':



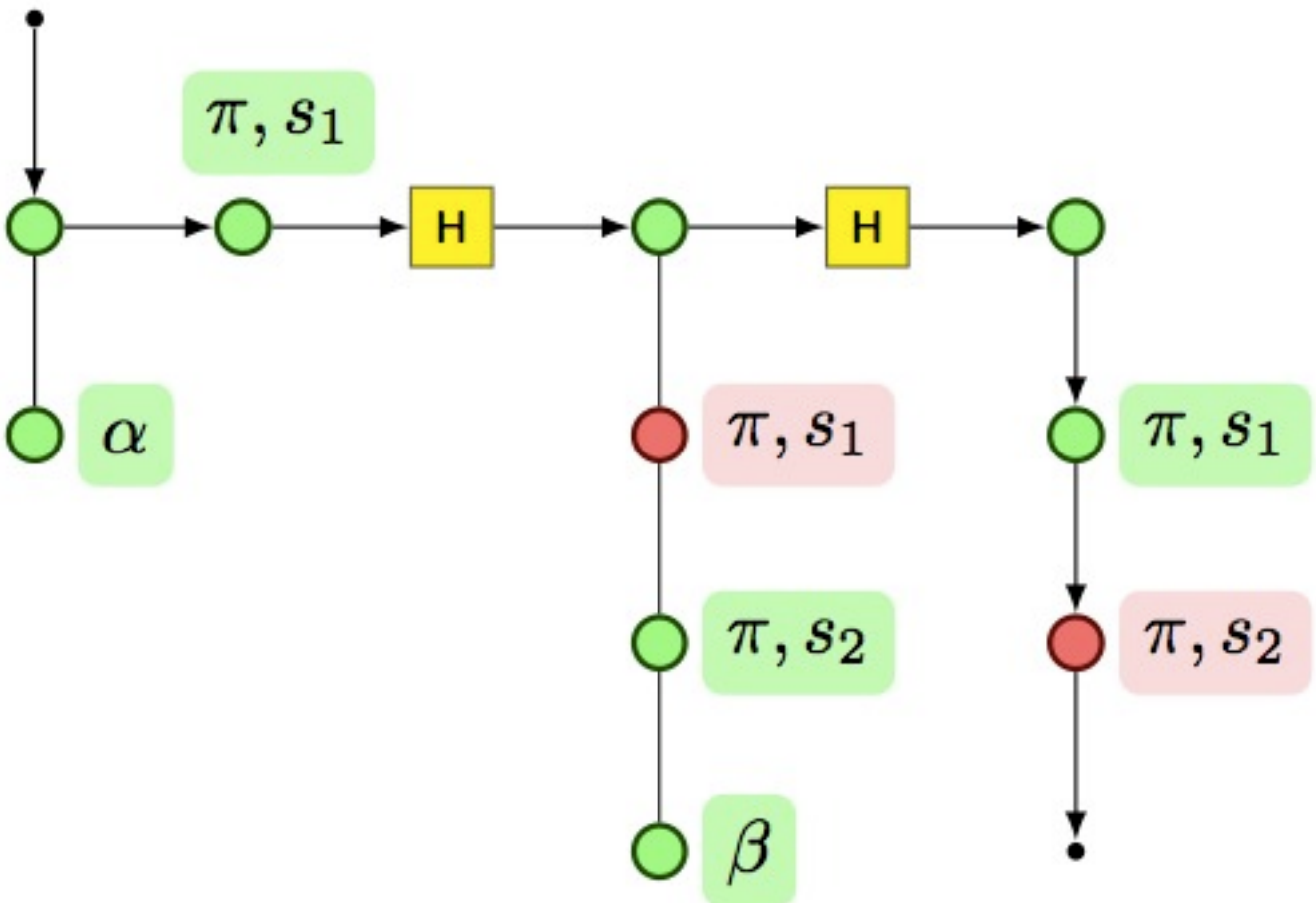
'π-red is pushed by the flow':



Flow Strategy



Flow Strategy



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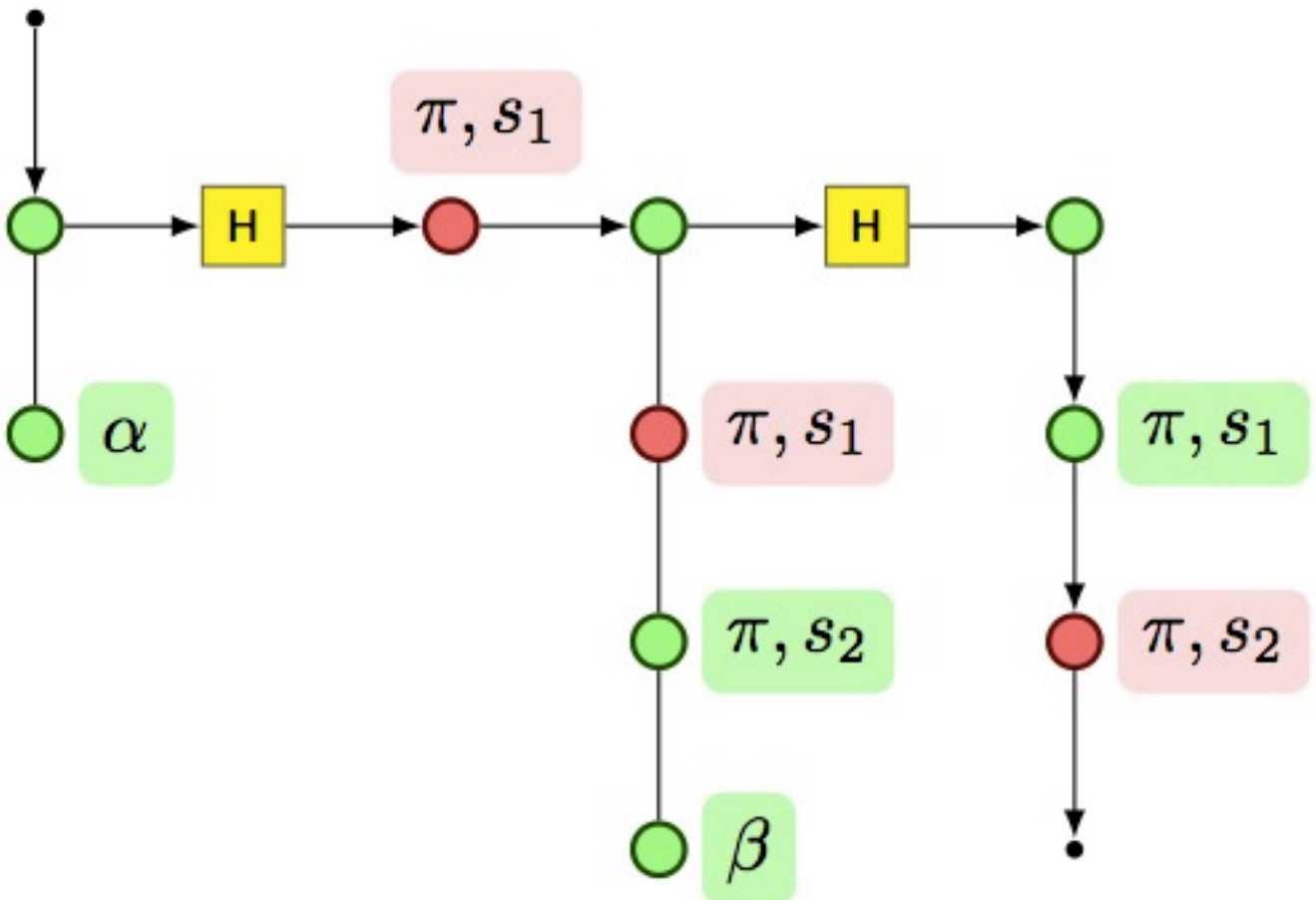
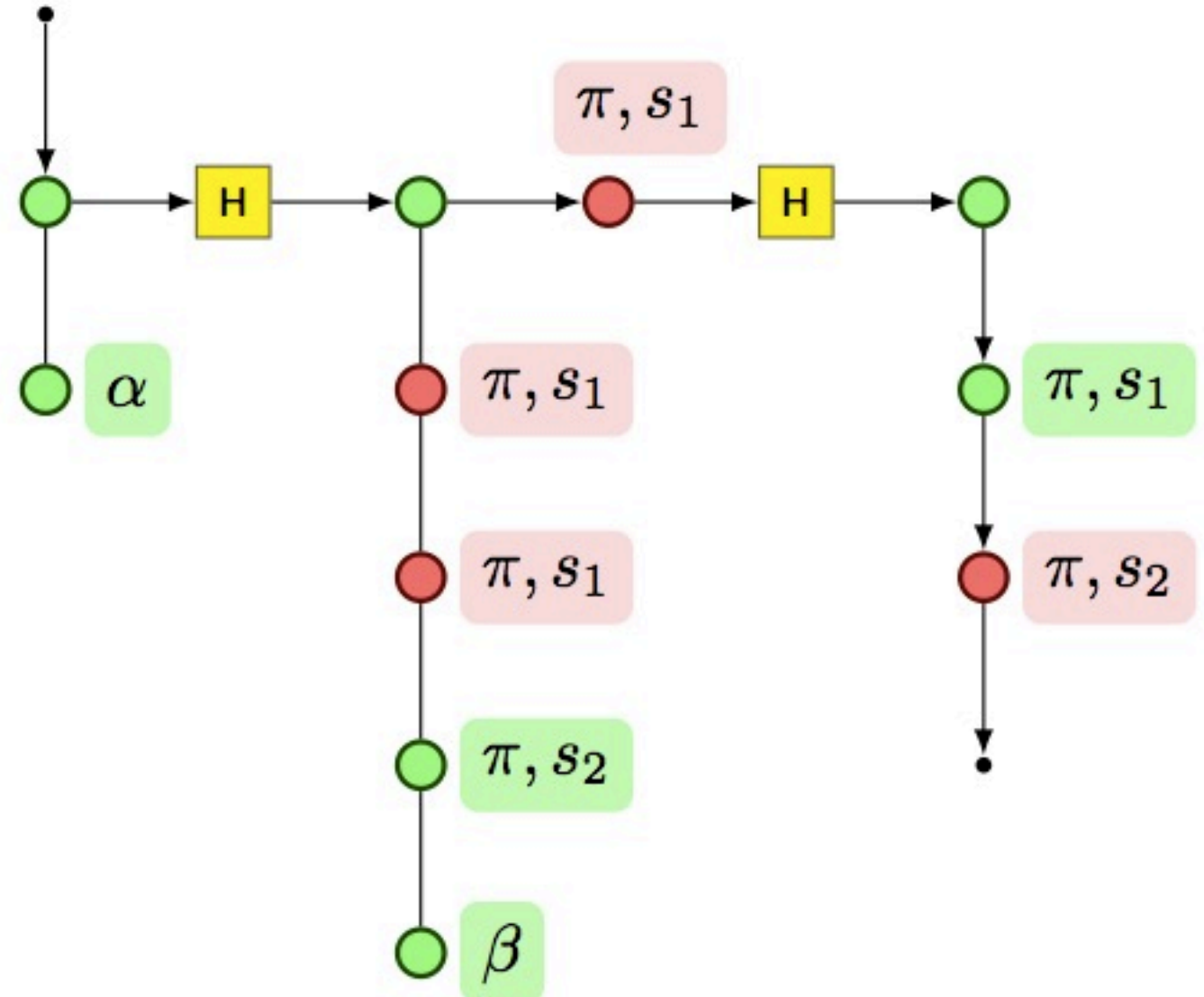
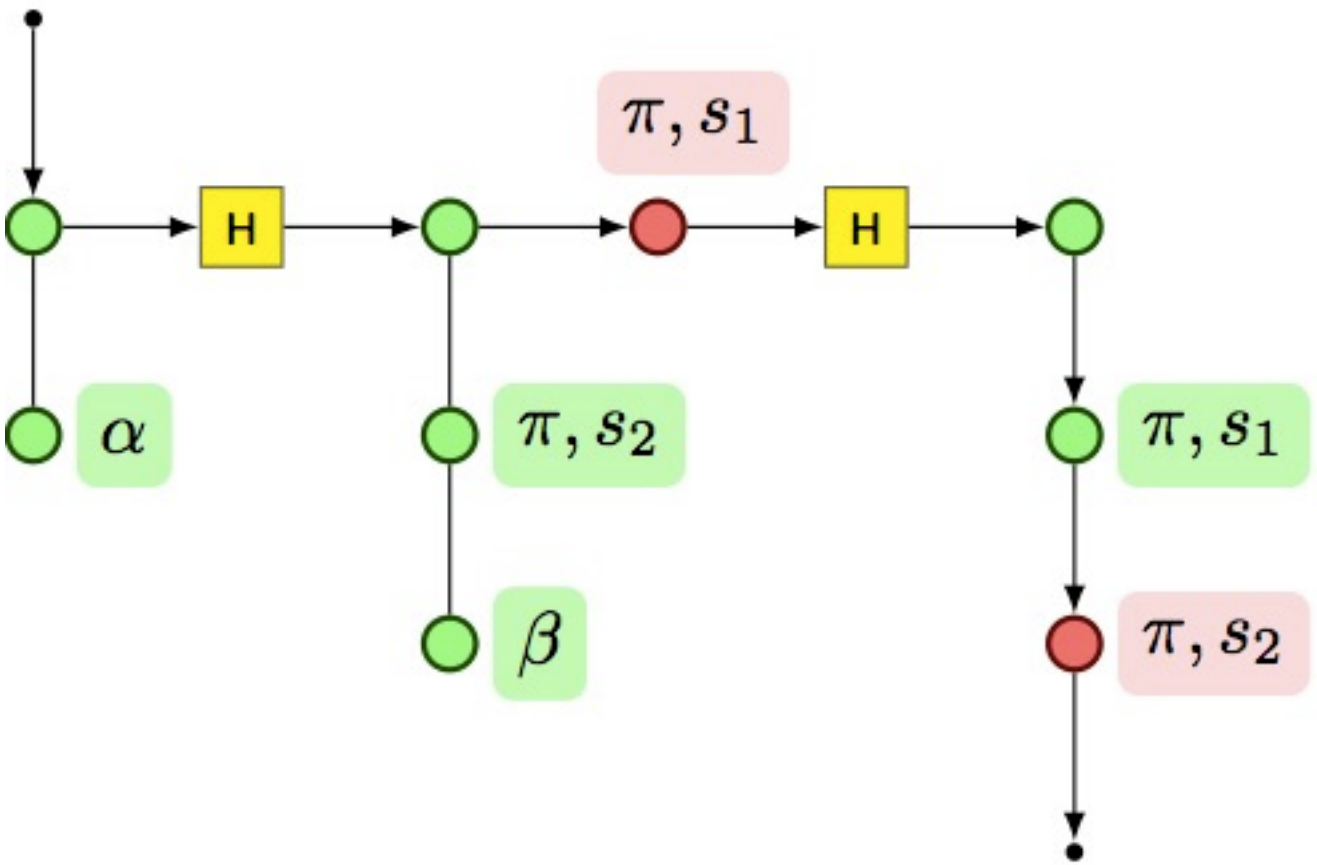


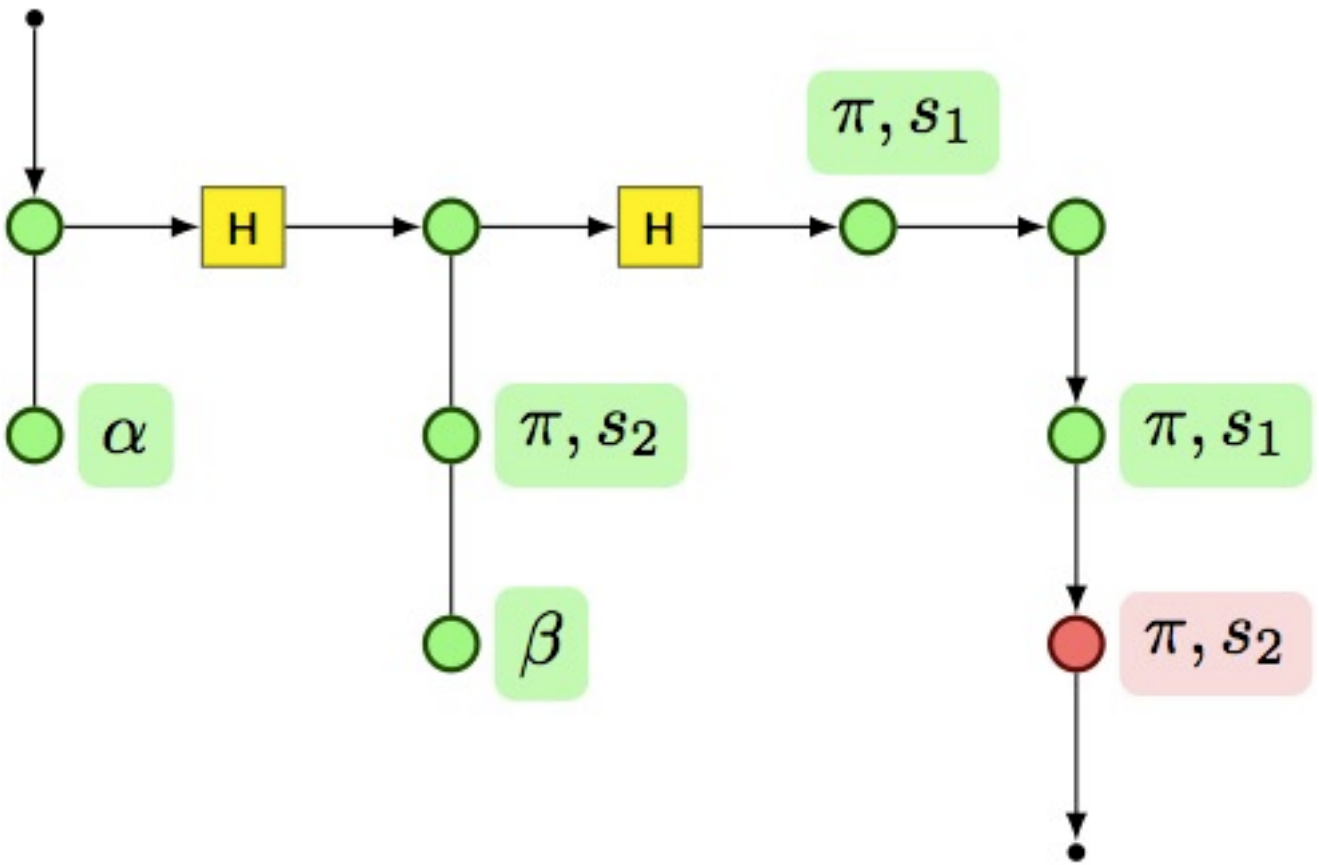
Flow Strategy



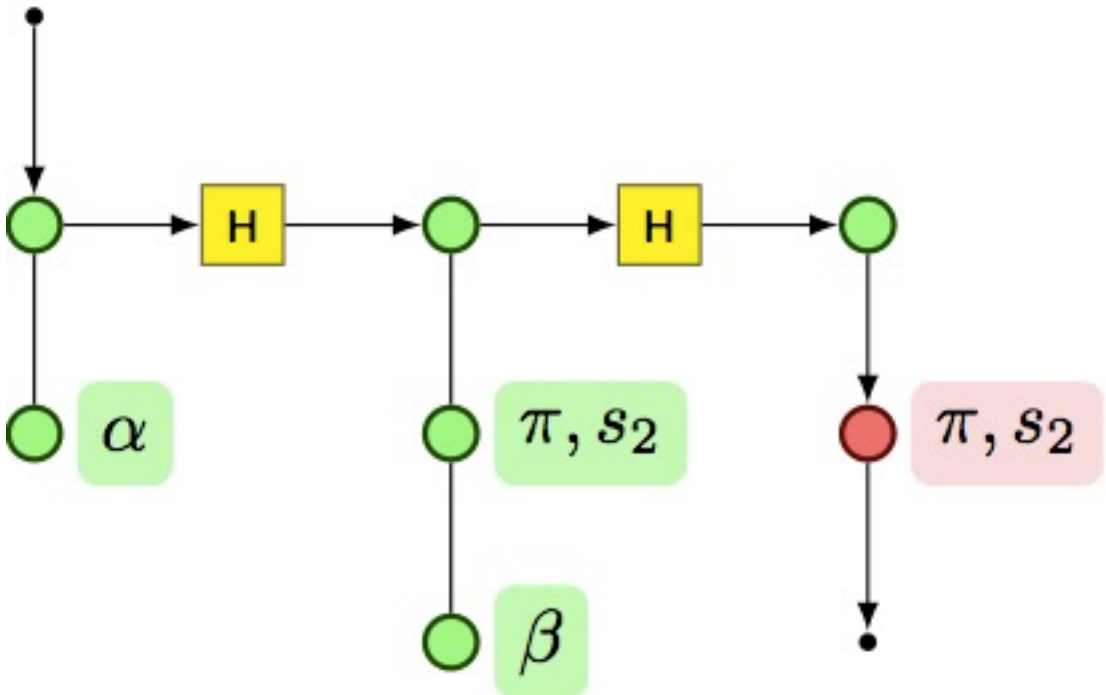
Flow Strategy



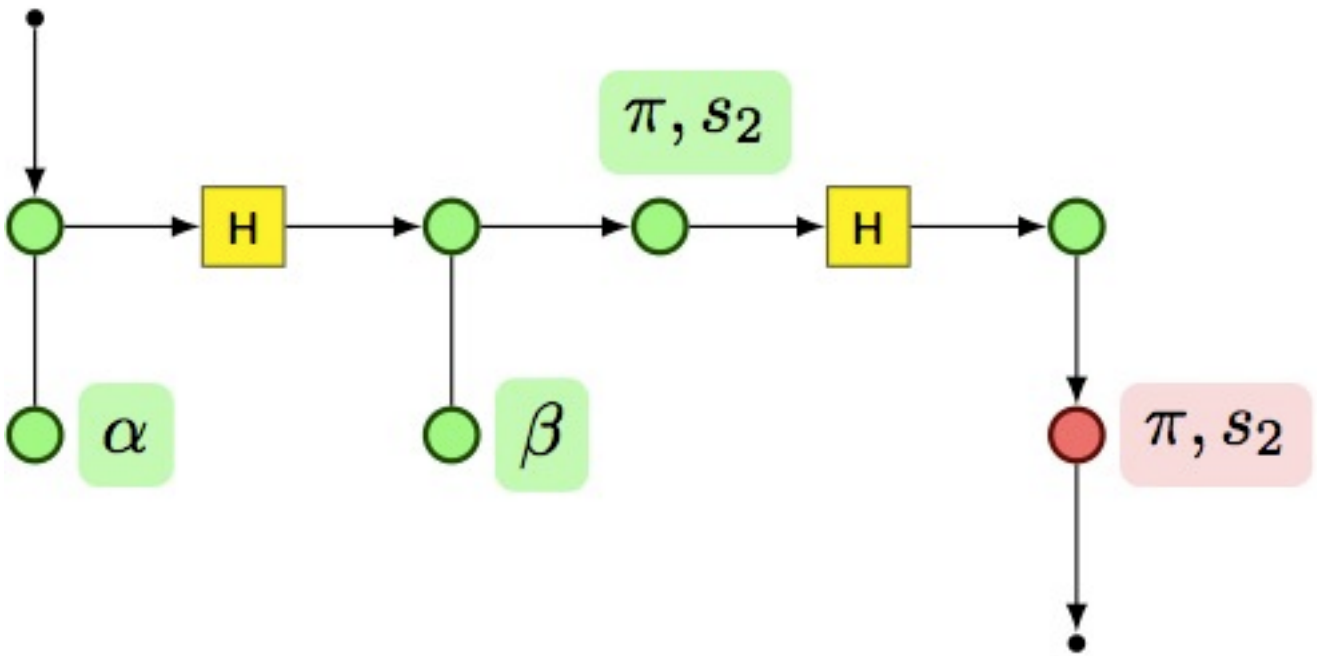
Flow Strategy



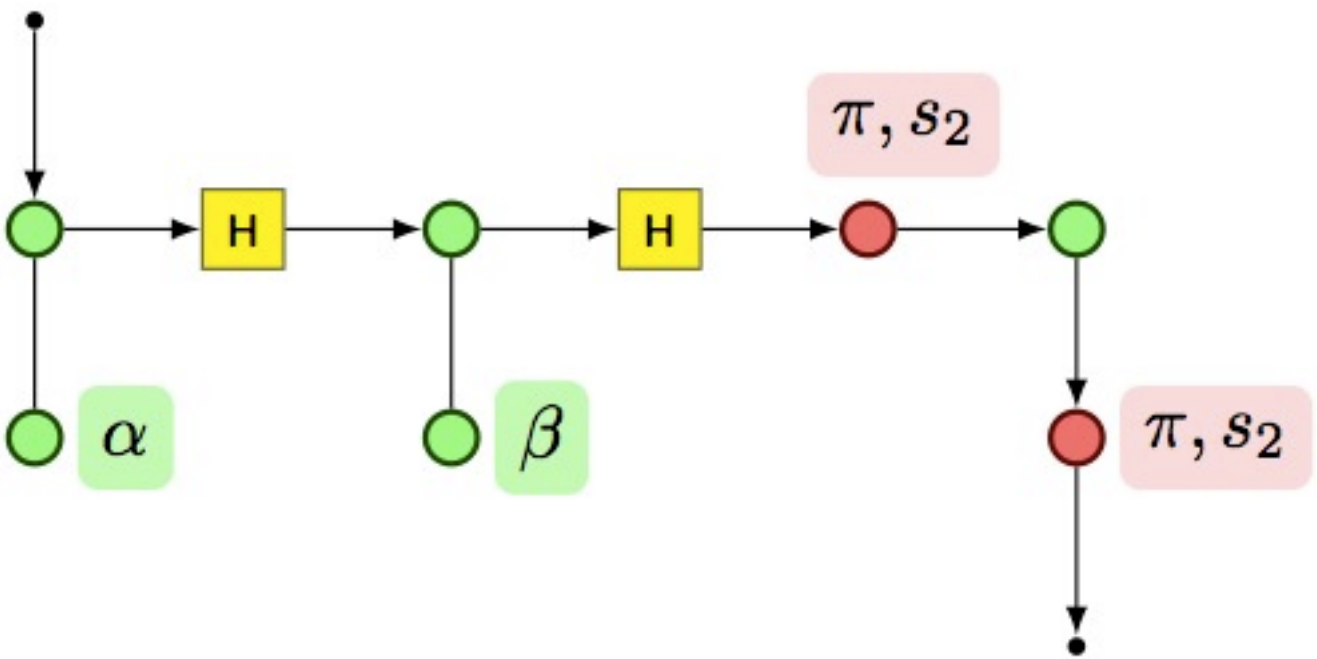
Flow Strategy



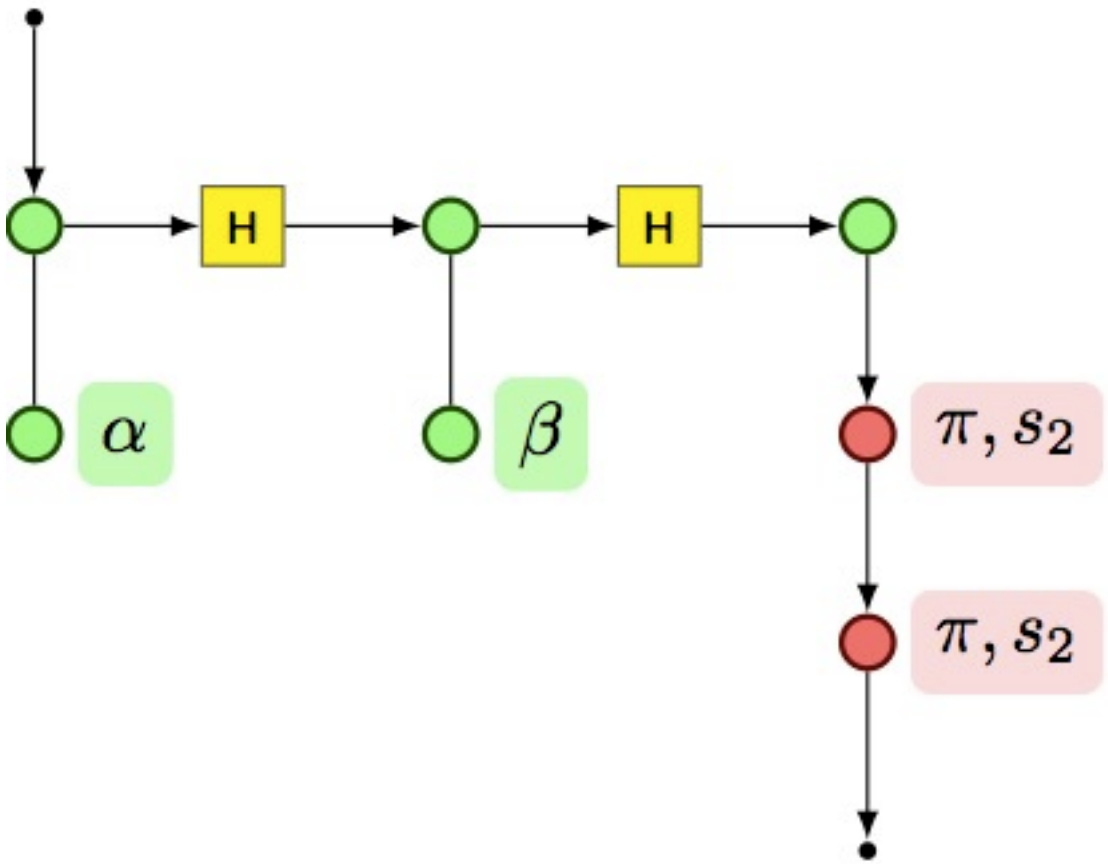
Flow Strategy



Flow Strategy



Flow Strategy



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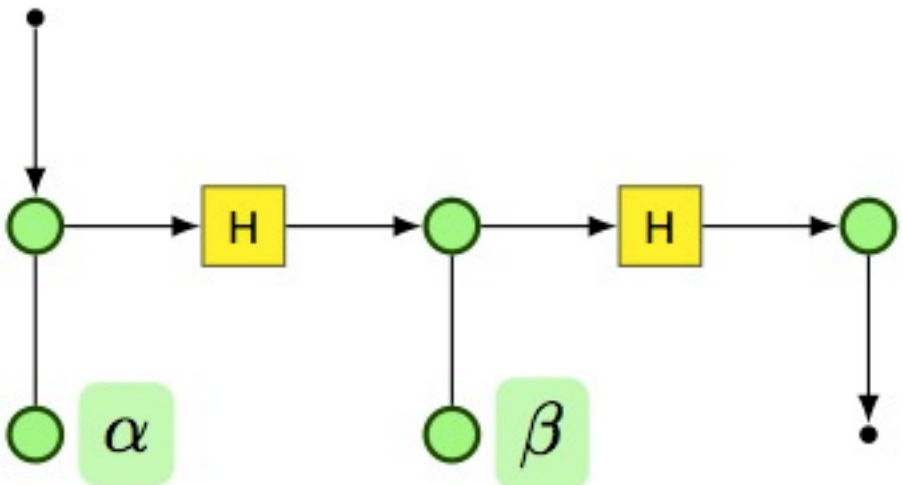
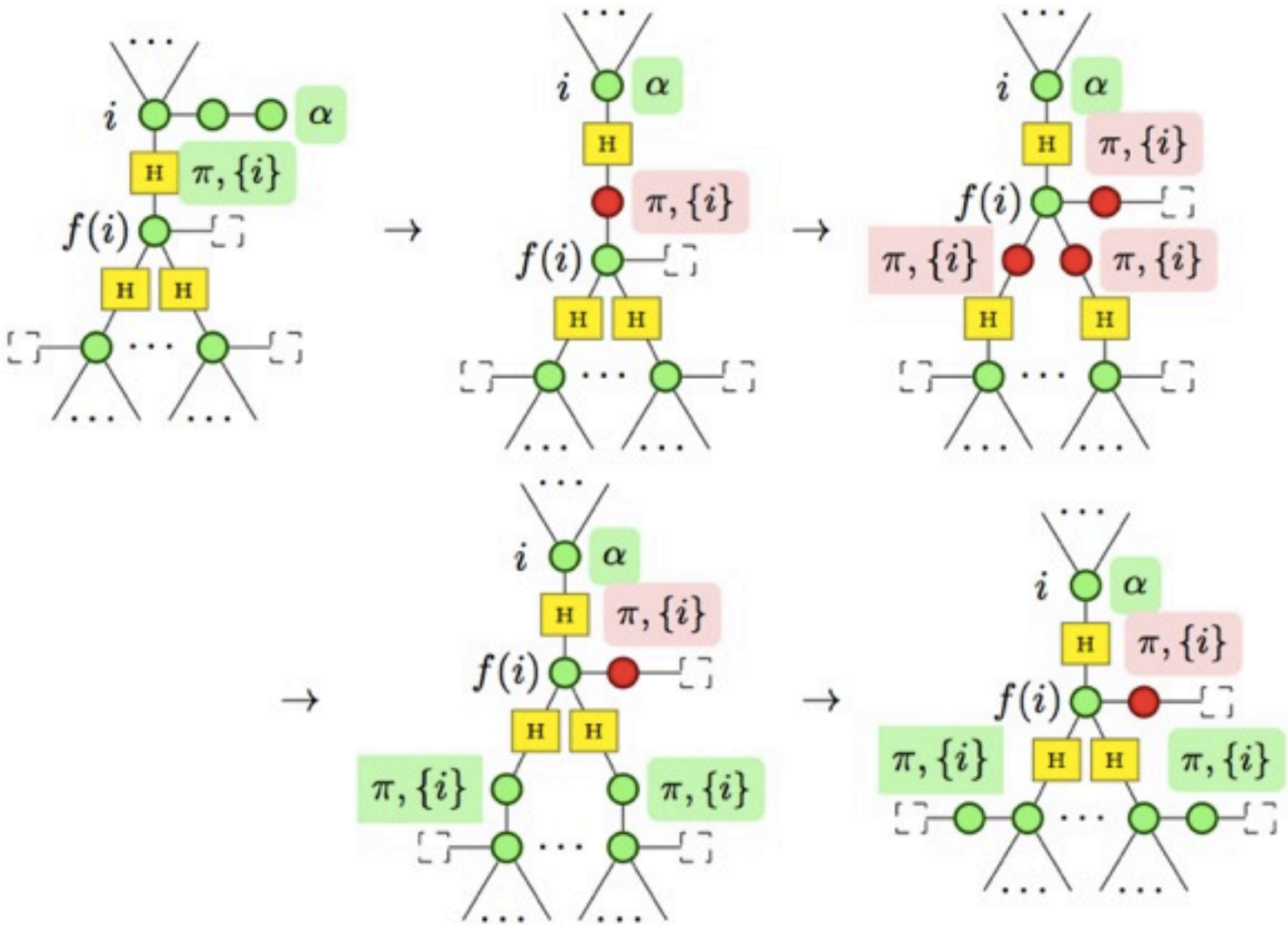


Flow Strategy

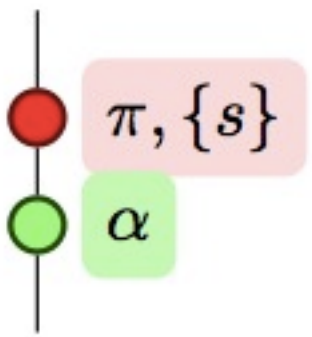
Thm: This strategy terminates in one of two states:

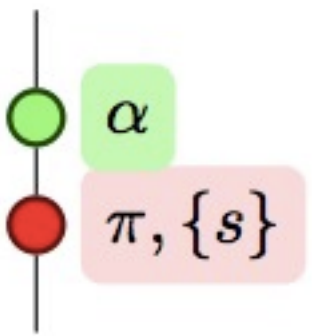
- All conditional operations removed => deterministic
- Not all removed, but their location reveals where extra corrections are needed.

Flow Strategy



Flow Strategy

$D_1 =$ 

$D_2 =$ 

$$[\![D_1]\!] \neq [\![D_2]\!] \text{ unless } \alpha = 0 \text{ or } \alpha = \pi$$

From here we can reconstruct the original pattern given by DKP as the uniformly deterministic pattern based on G.

Generalised Flow

Generalised Flow is the generalisation of flow where each qubit has a set of qubits.

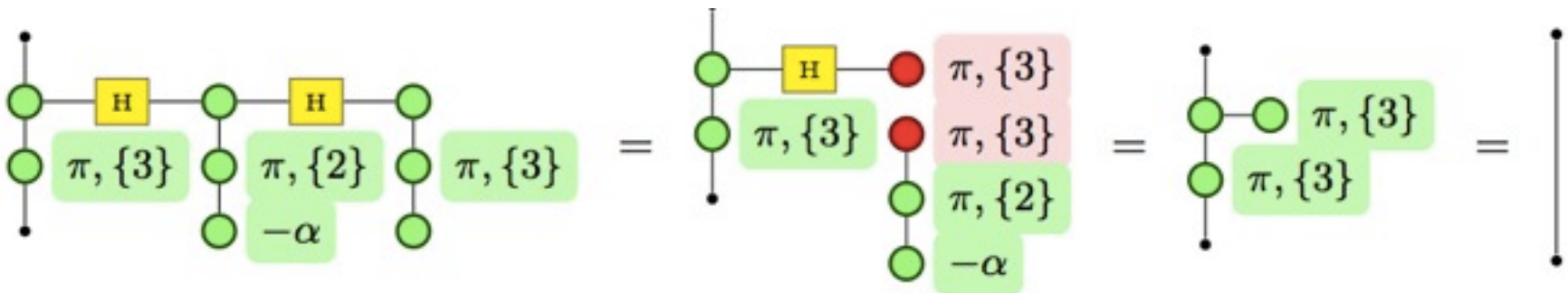
Theorem [BKMP 2007]: A geometry has generalised flow iff it supports a step-wise strongly uniformly deterministic pattern.

Theorem [DP 2010] If geometry has flow then the corresponding diagram can rewrite to a circuit-like diagram.

(No time to discuss => relies very strongly upon the bialgebra rule)

Non-uniform determinism

Note the ZX-calculus works with the pattern, not the geometry, so it can also treat non-uniform cases. E.g.:



Does the ZX-calculus decide the word problem for the Clifford group?

Ross Duncan

Laboratoire d'Information Quantique
Université Libre de Bruxelles

The Pauli Group

Pauli Group:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We don't care about phase so Y = XZ

$$P_i = 1 \otimes \cdots \otimes 1 \otimes P \otimes 1 \otimes \cdots \otimes 1$$

The Clifford Group

The Clifford group is the stabilizer of the Pauli group

$$CP = P'C$$

The Clifford Group

The Clifford group is generated by the following matrices:

$$Z_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The Clifford Group

The Clifford group is generated by the following matrices:

$$Z_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad X_{\pi/2} = \begin{pmatrix} \omega & \bar{\omega} \\ \bar{\omega} & \omega \end{pmatrix}$$

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The Clifford Group

- The Clifford group preserves the the eigenstates of the Pauli operators -- the stabilizer formalism.
- The Clifford group is the basis of quantum error correcting codes
- Clifford circuits can be classically simulated in linear time
- Measurement-based quantum computations with only Pauli measurements can be performed in one layer (i.e. no feed-forward)
- Add almost any gate to the Clifford group to get (approximate) universal QC

The Clifford Group

The Clifford group gets big fast...

$$|\mathcal{C}_1| = 24$$

$$|\mathcal{C}_2| = 11520$$

$$|\mathcal{C}_3| = 92827280$$

$$|\mathcal{C}_n| = \prod_{j=1}^{n} 2(4^j - 1)4^j = 2^{n^2+2n} \prod_{j=1}^{n} (4^j - 1).$$

The Word Problem

Given two Clifford circuits, are they equal?

- or: given one circuit, does it compute the identity?

Obviously this is decidable by just working out the matrix of the circuit, but this may be very large.

- No presentation by generators and relations is known.

The Completeness Question

How powerful is the ZX-calculus?

The Completeness Question

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Conjecture: the ZX-calculus decides the word problem for C_n

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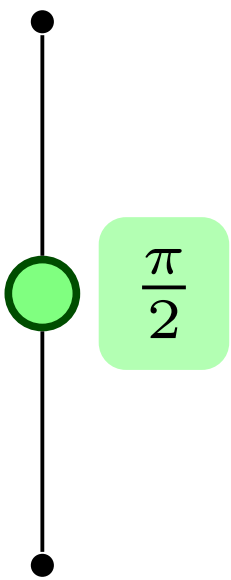
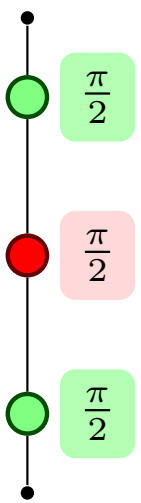
Conjecture: the ZX-calculus decides the word problem for $SU(2^n)$

The Clifford Group C_1

The *Clifford group* on 1 qubit (aka C_1) is generated by the unitaries:

$$Z_{\frac{\pi}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

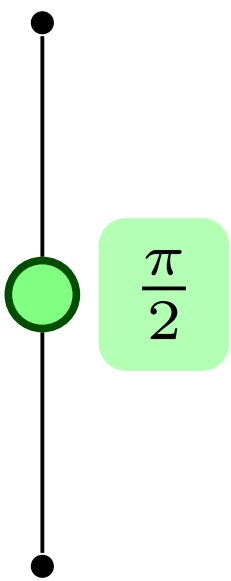


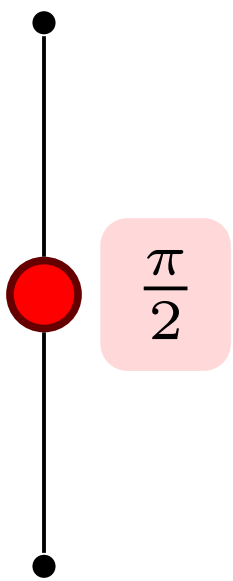


The Clifford Group C_1

Or equivalently:

$$Z_{\frac{\pi}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad X_{\frac{\pi}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega & \bar{\omega} \\ \bar{\omega} & \omega \end{pmatrix}$$

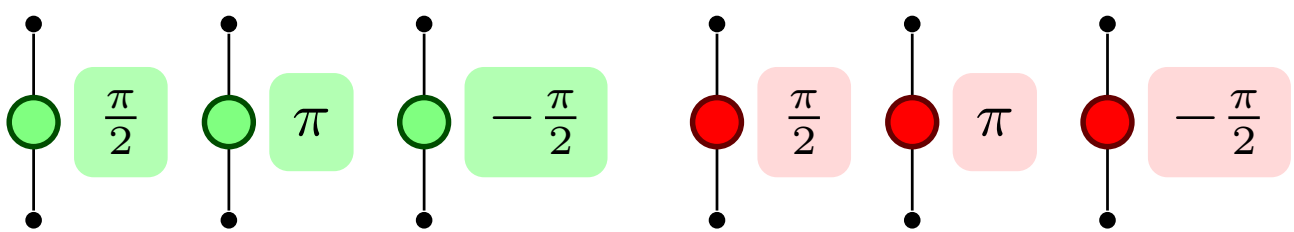




The Clifford Group C_1

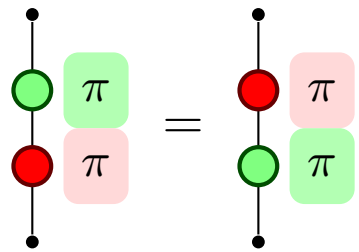
0 vertices:

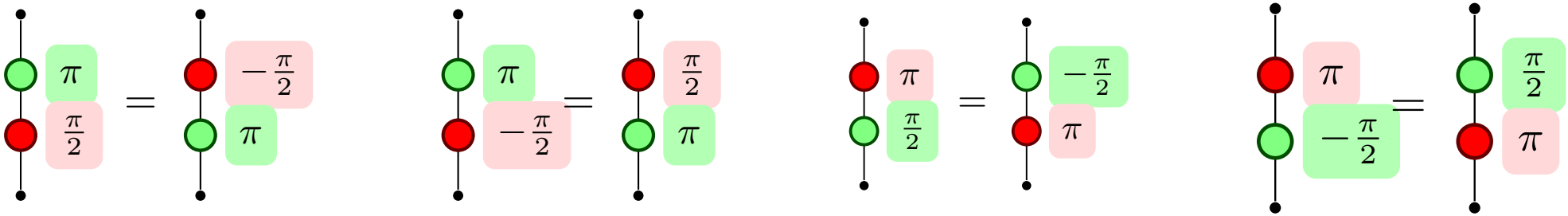
1 vertex:

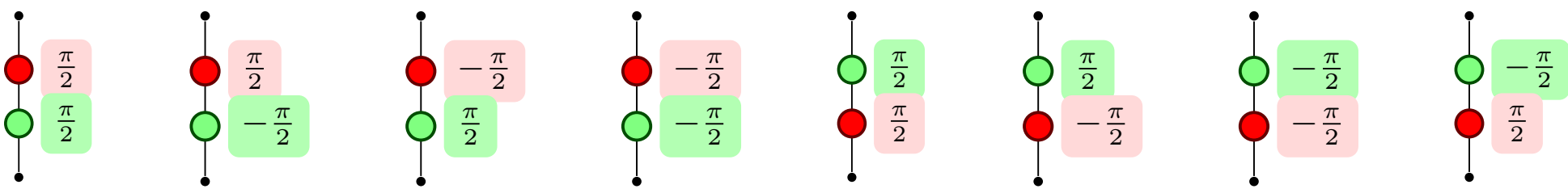


The Clifford Group C_1

2 vertices:

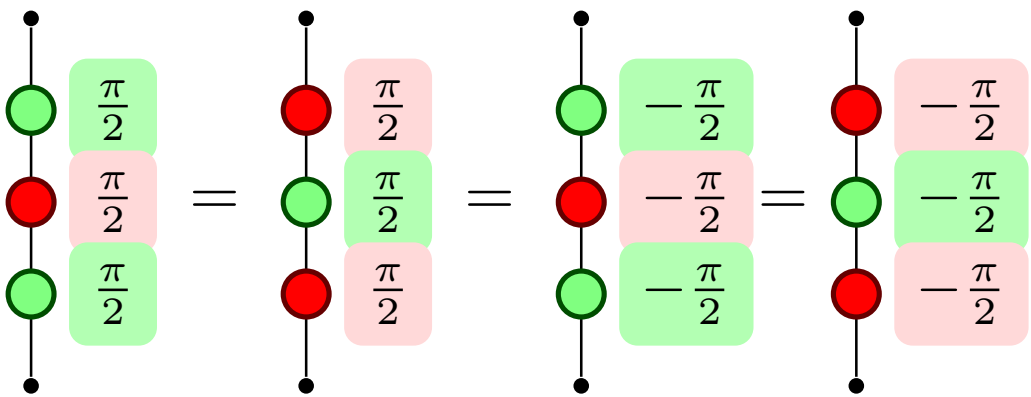


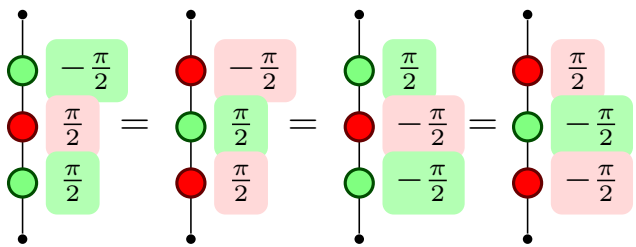


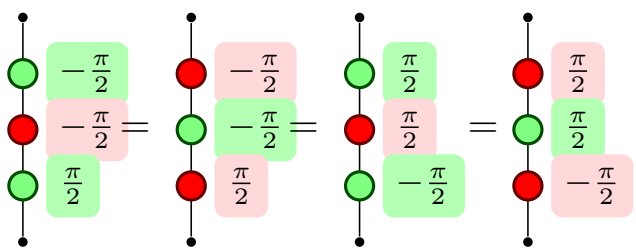


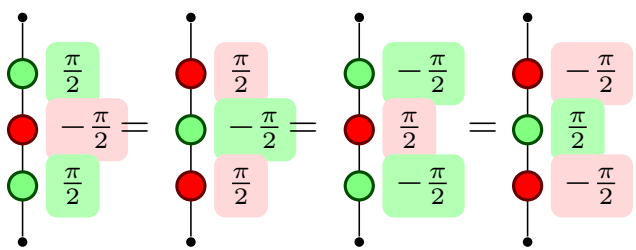
The Clifford Group C_1

3 vertices:







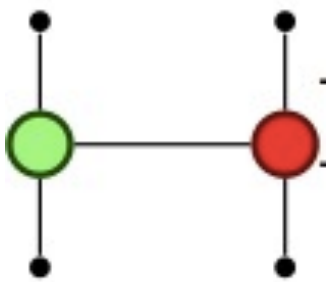


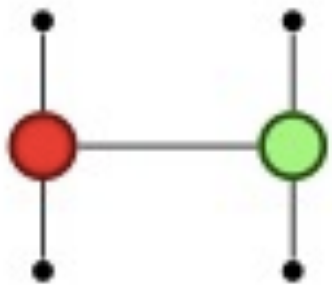
The Clifford Group C_1

Theorem: the ZX-calculus decides the word problem for C_1

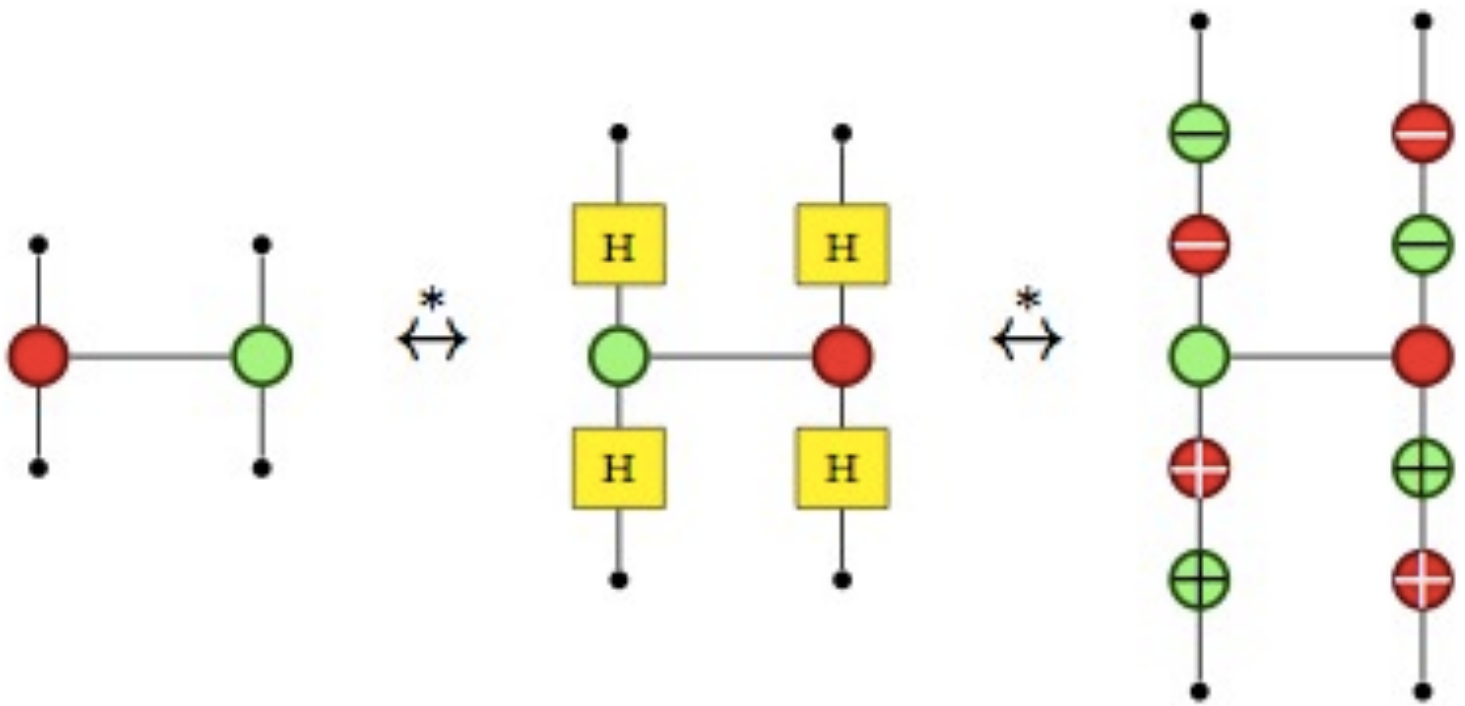
Proof: the 1 + 6 + 13 + 4 = 24 elements shown are all distinct; any combination of n > 3 vertices can be reduced to smaller number via the spider rule and commutation with Paulis.

CNOT and TONC

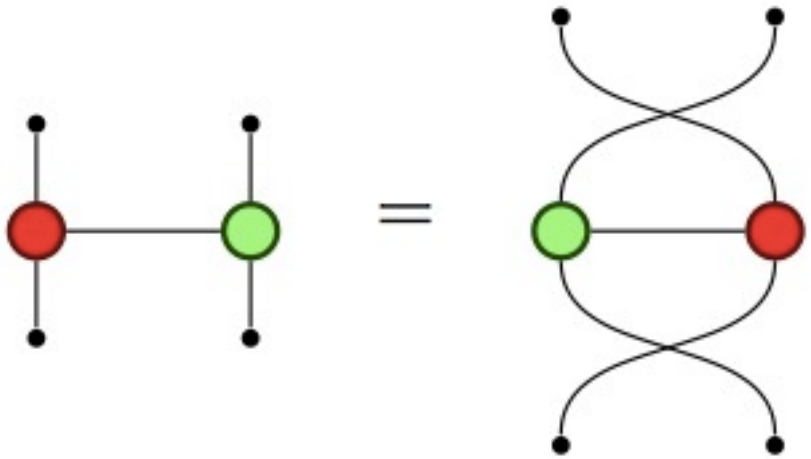
$\mathrm{CNOT} =$ 

$\mathrm{TONC} =$ 

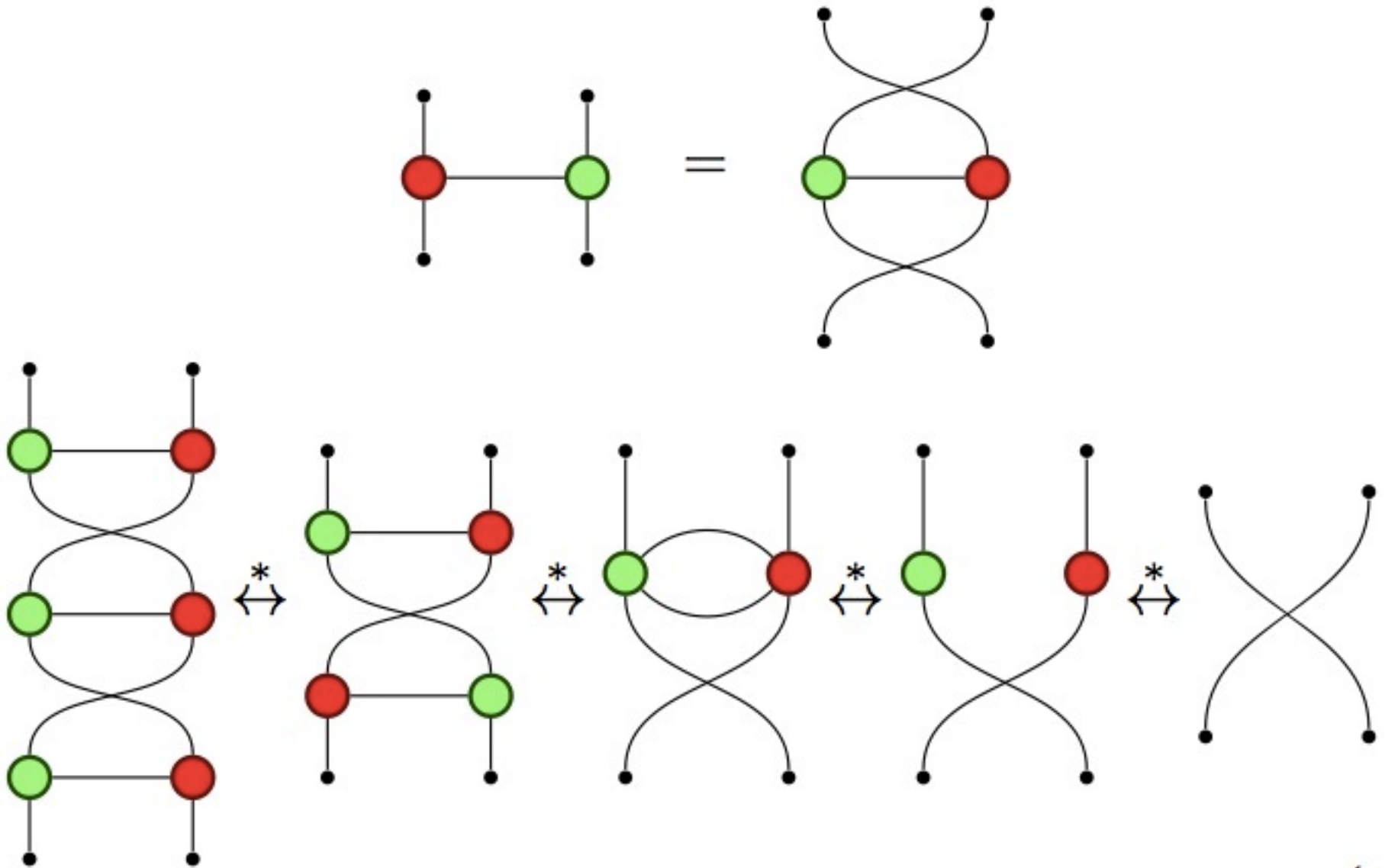
CNOT and TONC



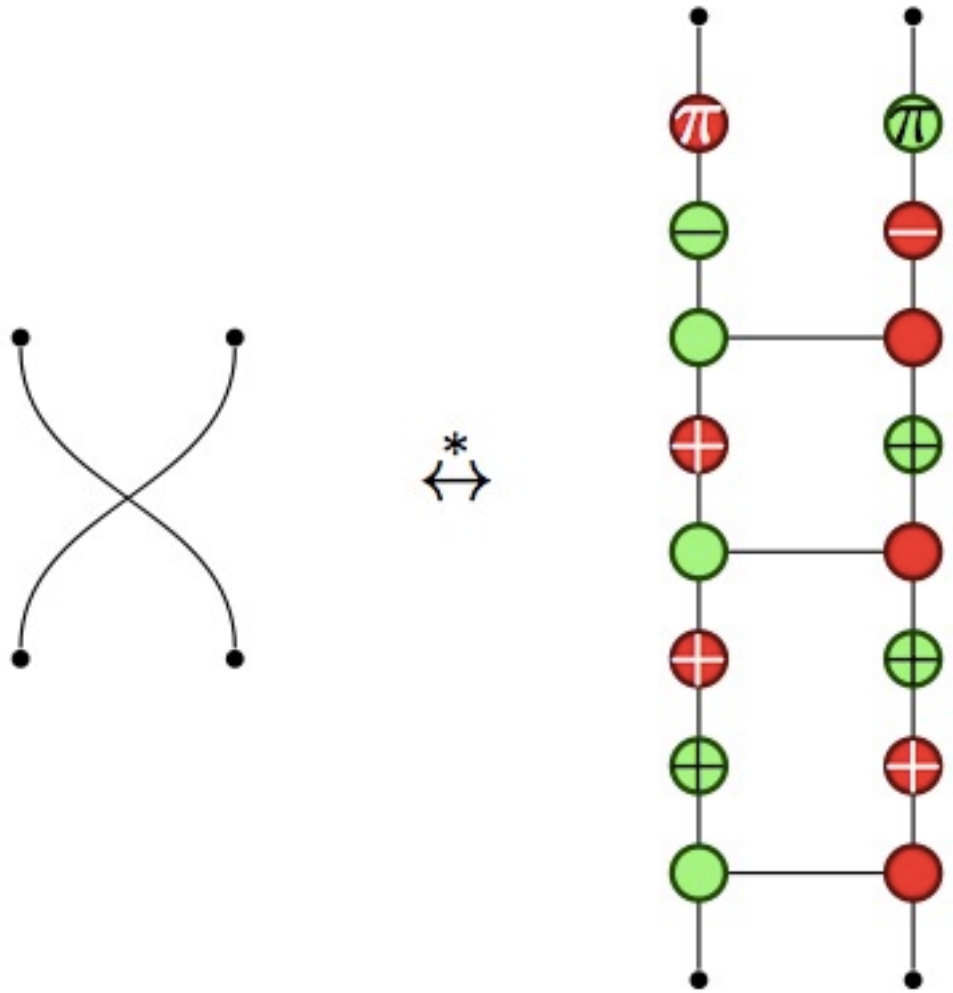
TONC and Swap



TONC and Swap

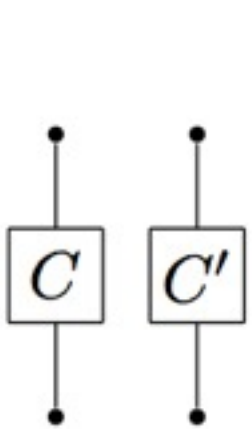


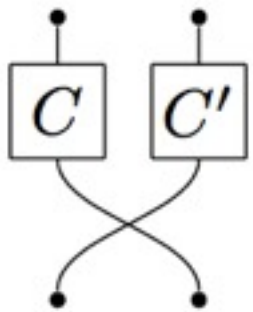
TONC and Swap

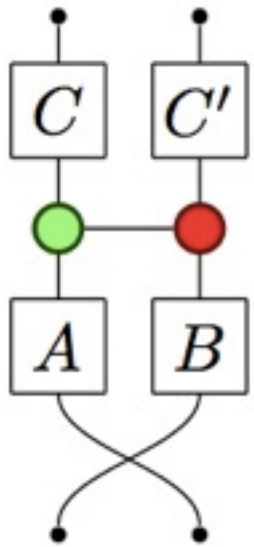


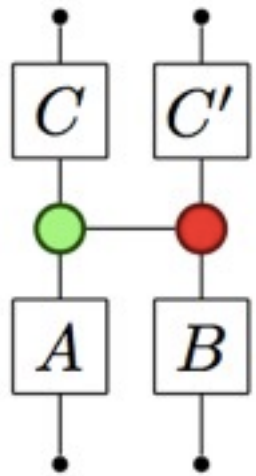
The Clifford Group C_2

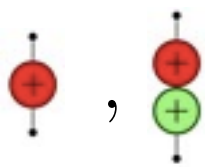
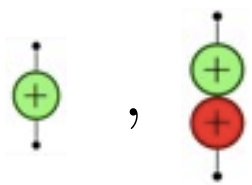
Several classes of 2-qubit cliffords:









$A \in \{1,$  $\}$ $B \in \{1,$  $\}$

This gives 24 * 24 * (1 + 1 + 9 + 9) = 11520 diagrams

Check mechanically: they are all different. Nice!

C_2

IDEA: compose an elements of C_2 with a generator of the group, show that you get another one of the diagrams already shown

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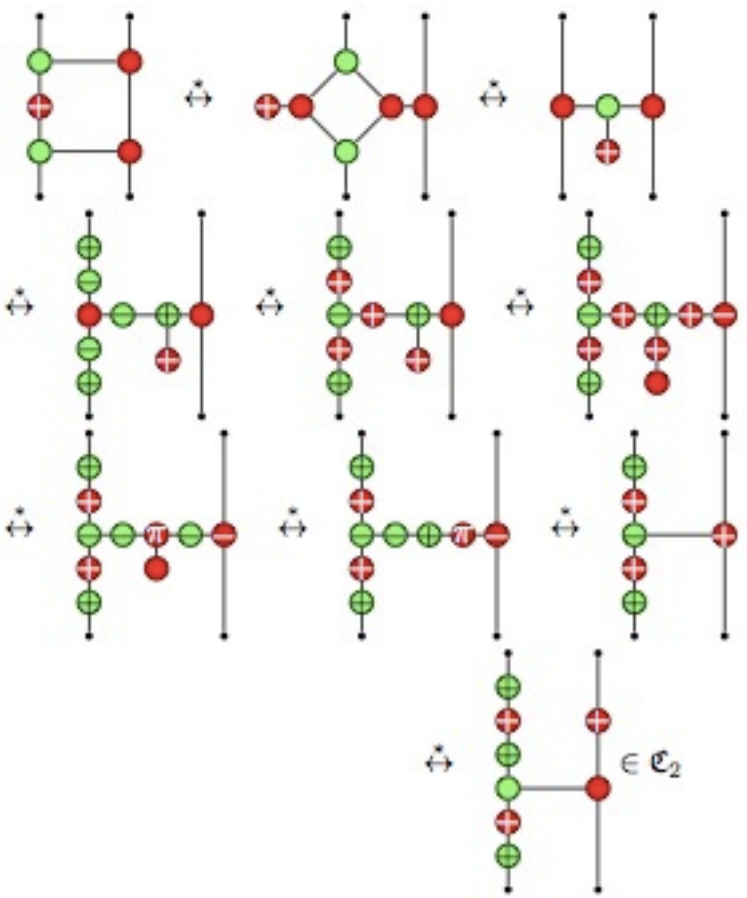
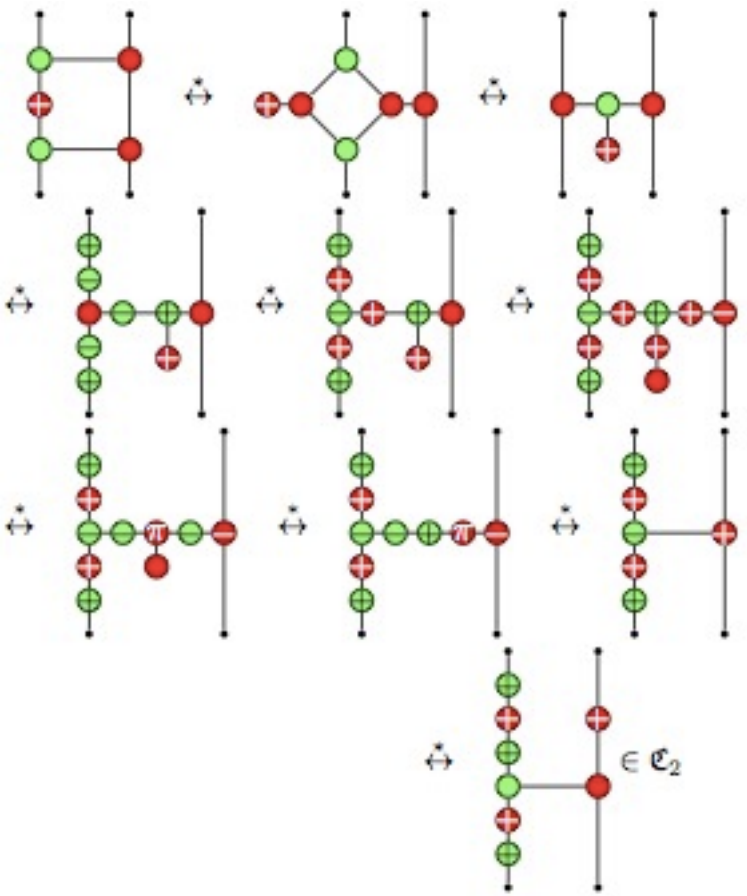
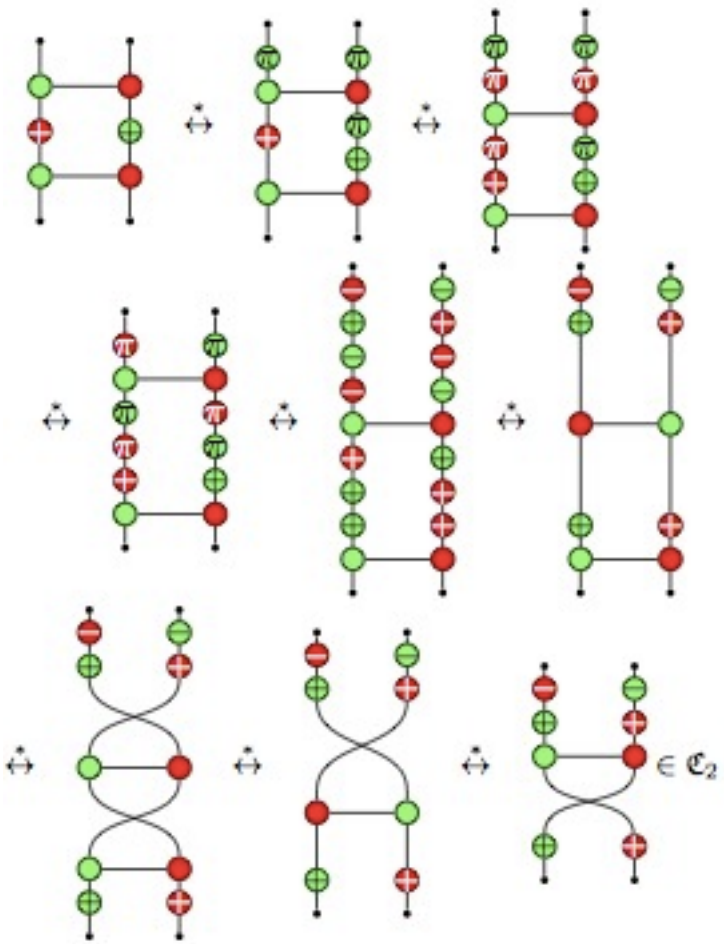


C_2

IDEA: compose an elements of C_2 with a generator of the group, show that you get another one of the diagrams already shown





Theorem: the ZX-calculus decides the word problem for C_2

Papers

Monoidal Categories and Diagrams

- G. M. Kelly and M. L. Laplaza. Coherence for Compact Closed Categories. *Journal of Pure and Applied Algebra*, 19 pp193-213, 1980.
- A. Joyal and R. Street. Geometry of Tensor Categories. *Advances in Mathematics* 88, pp55-113, 1991.
- A. Joyal, R. Street and D. Verity. Traced Monoidal Categories. *Math. Proc. Camb. Phil. Soc.* 119 (3), 1993.
- P. Selinger. A survey of graphical languages for monoidal categories. In *New Structures for Physics*, Springer Lecture Notes in Physics 813, pp 289-355, 2011.

†-Compact Categories and QM

- S. Abramsky and B. Coecke. A categorical semantics of quantum protocols. *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science: LICS 2004*, IEEE Computer Society, 2004.
- S. Abramsky. No-Cloning In Categorical Quantum Mechanics. In *Semantic Techniques in Quantum Computation, ed. S. Gay and I. Mackie*, pages 1--28, Cambridge University Press 2010. arXiv:0910.2401v1.
- S. Abramsky and B. Coecke. Categorical quantum mechanics. In *Handbook of Quantum Logic and Quantum Structures*, Elsevier, 2008.
- H. Barnum, R. Duncan, and A. Wilce. Convexity, Categorical Semantics and the Foundations of Physics. *Proceedings of 7th Workshop on Quantum Physics and Logic (QPL 2010)* 2010.

Papers

Observable Structures

- A. Carboni and R.F.C. Walters. Cartesian bicategories i. *Journal of Pure and Applied Algebra*, 49(1-2), 1987.
- A. Carboni. Matrices, relations, and group representations. *Journal of Algebra*, 136(2), 1991.
- B. Coecke, E. O. Paquette, and D, Pavlovic. Classical and quantum structuralism. 2009.
- B. Coecke, D. Pavlovic, and J. Vicary. A new description of orthogonal bases. *Math. Structures in Comp. Sci.* 13pp, to appear, arxiv.org/abs/0810.0812
- D. Pavlovic. Geometry of abstraction in quantum computation. 2010. arXiv:1006.1010
- B. Coecke, E. O. Paquette and S. Perdrix. Bases in diagrammatic quantum protocols. *Proceedings of Mathematical Foundations of Programming Semantics XXIV*, Electronic Notes in Theoretical Computer Science, Elsevier, 2008.

Complementary Observables

- B. Coecke and R. Duncan. Interacting quantum observables. Interacting Quantum Observables: Categorical Algebra and Diagrammatics. *New J. Physics* 13 (043016) 2011. arXiv:0906.4725

Non-locality and the phase group

- B. Coecke, W. Edwards, and R. W. Spekkens Phase groups and the origin of non-locality for qubits Electronic Notes in Theoretical Computer Science 270, 15 (2011) arXiv:1003.5005
- W. Edwards. Non-locality in Categorical Quantum Mechanics. DPhil Thesis, Oxford University, 2009.
- B. Coecke and W. Edwards. Spekkens's toy theory as a category of processes. 2011. arxiv:1108.1978.
- B. Coecke, R. Duncan, A. Kissinger and Q. Wang. Strong Complementarity and non-locality in categorical quantum mechanics. Submitted. 2012.

Papers

MBQC - classic

- R. Raussendorf and H. J. Briegel. A One-Way Quantum Computer *PRL* 86 2001.
- R. Raussendorf, D. E. Browne and H. J. Briegel. Measurement-based quantum computation with cluster states, *Phys. Rev. A*. 68, 2003.
- V. Danos, E. Kashefi and P. Panangaden. The Measurement Calculus. *Journal of ACM*, 54 (2) , 2007.
- V. Danos and E. Kashefi. Determinism in the one-way model. *Phys. Rev. A* 74, 2006.
- D. E. Browne, E. Kashefi, M. Mhalla, and S .Perdrix. Generalized Flow and Determinism in Measurement-based Quantum Computation. *New J. Phys*. 9 (250), 2007.

MBQC - categorical

- R. Duncan and S. Perdrix. Graph states and the necessity of Euler decomposition. In K. Ambos-Spies, B. Löwe, and W. Merkle, editors, *Computability in Europe: Mathematical Theory and Computational Practice (CiE'09)*, volume 5635 of Lecture Notes in Computer Science, pages 167–177. Springer, 2009.
- R. Duncan, and S. Perdrix. Rewriting measurement-based quantum computations with generalised flow. In *Automata, Languages and Programming, 37th International Colloquium, ICALP 2010, Proceedings Part II*. Springer Lecture Notes in Computer Science 6199, 2010.
- R. Duncan. A graphical approach to measurement-based quantum computation. To appear.