The *non*-logic of quantum computation

OR:

How I learned to live without propositions-as-types

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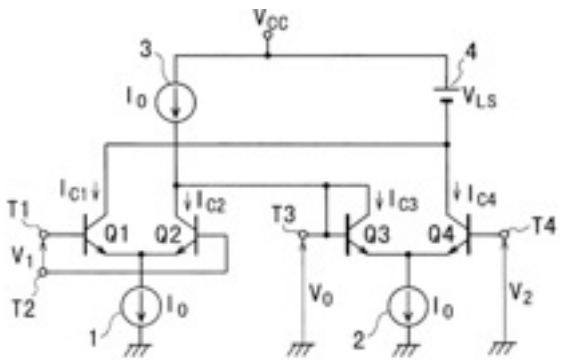
Background and Motivations

"I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more."

--John von Neumann, letter to G. Birkhoff, 1935

Motivations



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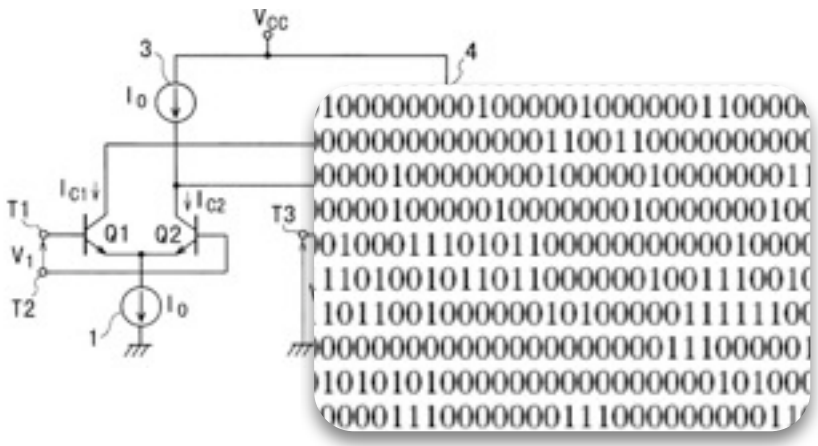

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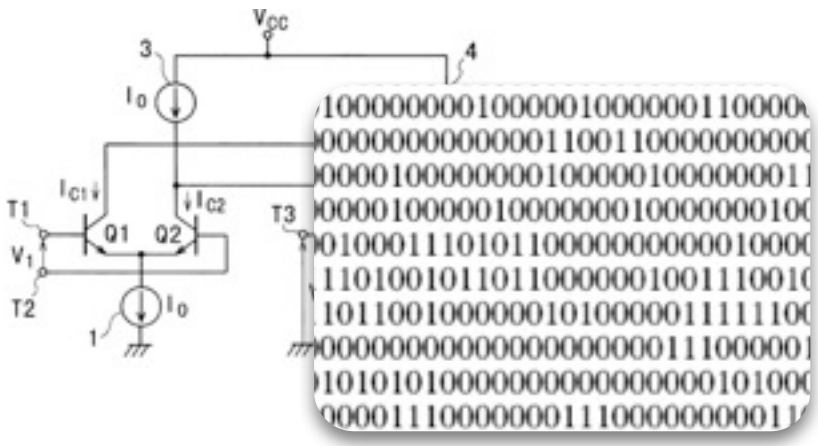


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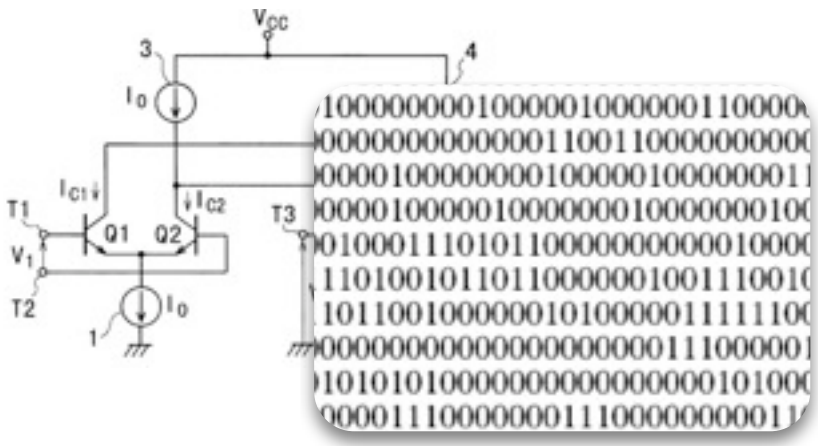

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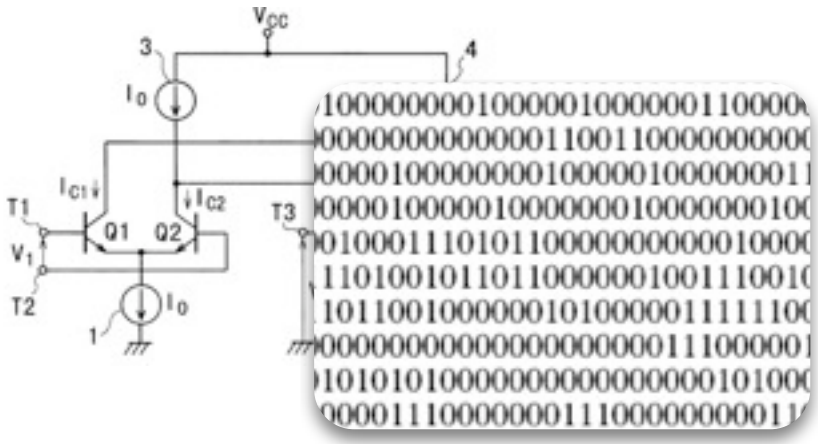

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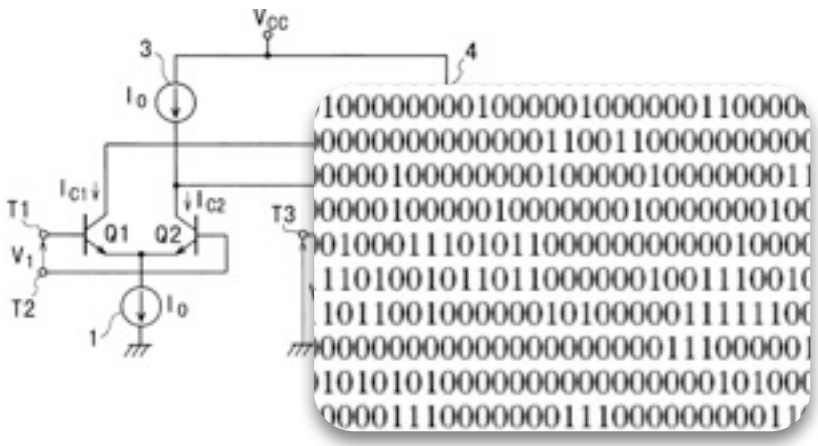

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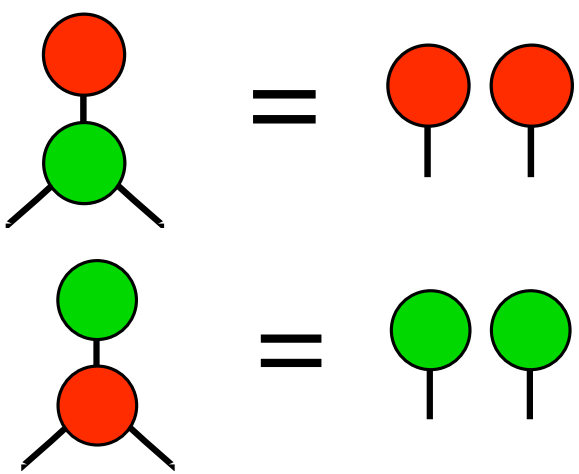
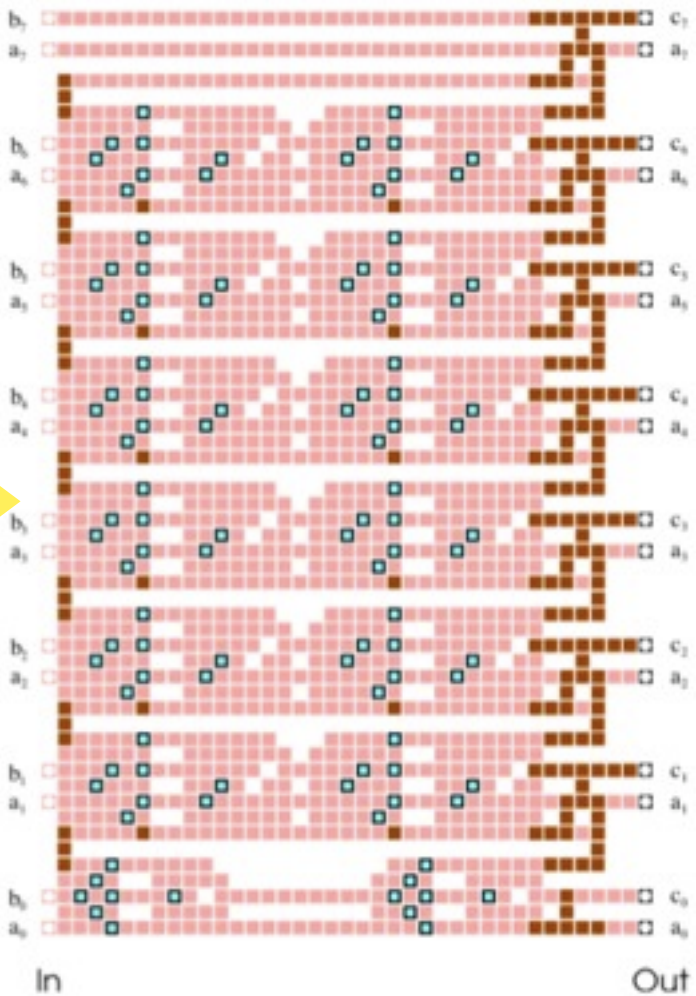


The need for abstraction:

This is an
8-bit adder



D ~ 2^1764

Motivation & Context

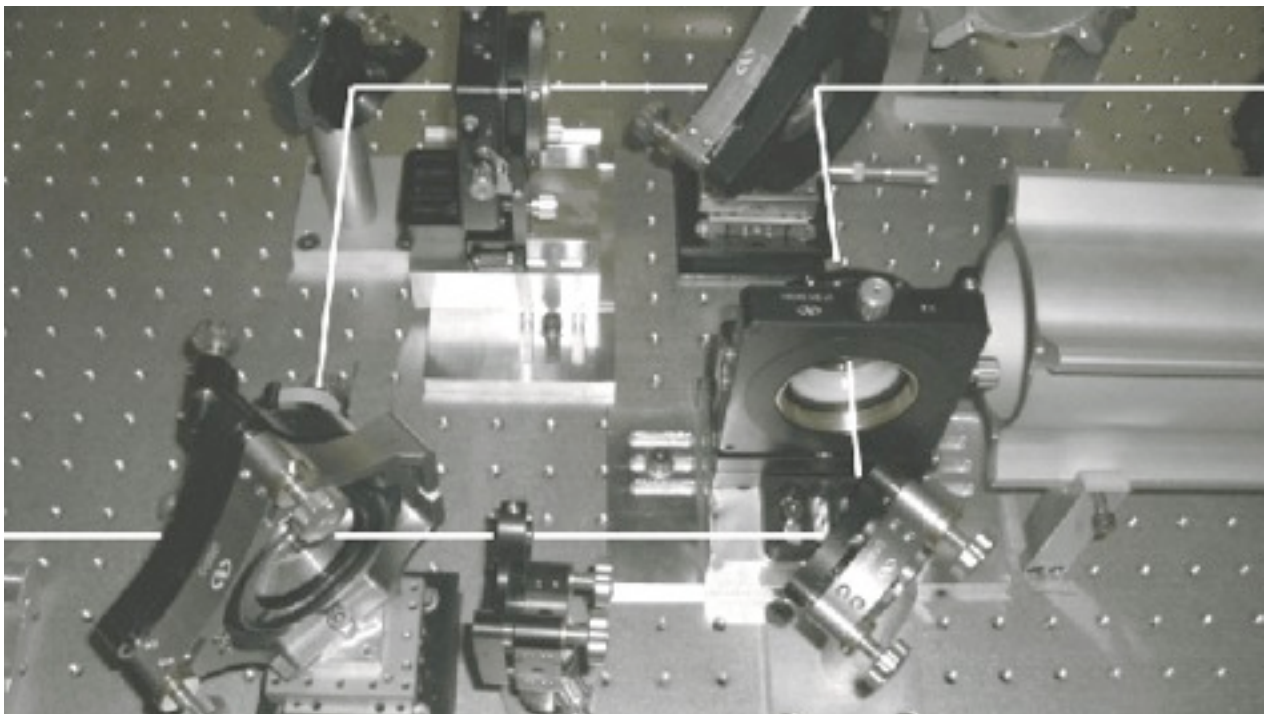
New **structural** axiomatisation of QM:

- Hilbert space formalism not a perfect fit for QM
- Can also study alternative quantum-like theories
- emphasis on **composition**

We use **Monoidal Categories**:

- Very general framework, including Hilbert spaces
- Well studied in literature
- **Beautiful diagrammatic presentation**

Quantum Mechanics



Overview of the physical theory

Quantum Mechanics (Abridged)

1. Quantum states are represented by unit vectors in a complex Hilbert space.

$$|0\rangle\,, |1\rangle\,, \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \in \mathbb{C}^2 \;=:\; \mathcal{Q}$$

2. The state space formed by combining two or more systems is the tensor product of their individual state spaces

$$|010\rangle := |0\rangle \otimes |1\rangle \otimes |0\rangle \;\in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \;=\; \mathcal{Q}^3$$

3. For each discrete time step, an undisturbed quantum system evolves according to a unitary operator acting on its state space

$$X, Z, H : \mathcal{Q} \to \mathcal{Q} \qquad \wedge X, \wedge Z : Q^2 \to Q^2$$

...**but** the quantum state is not directly accessible... more on this later!

No-Cloning and No-Deleting

Theorem: There are no quantum operations D such that

$$D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$$
$$D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no quantum operations E such that

$$E : |\psi\rangle \mapsto |0\rangle$$
$$E : |\phi\rangle \mapsto |0\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]

Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:

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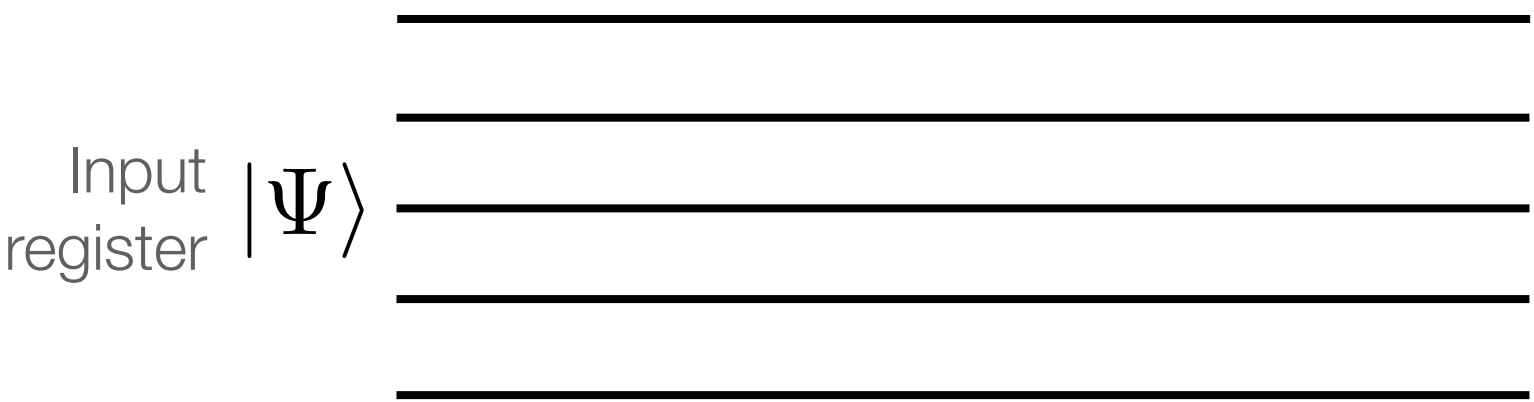

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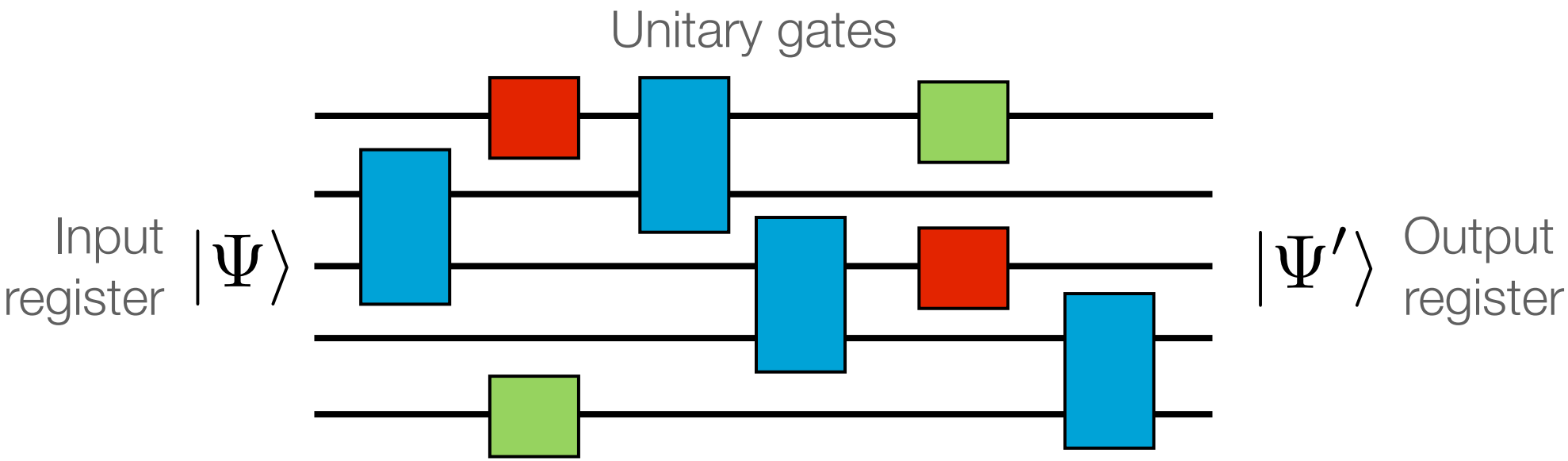

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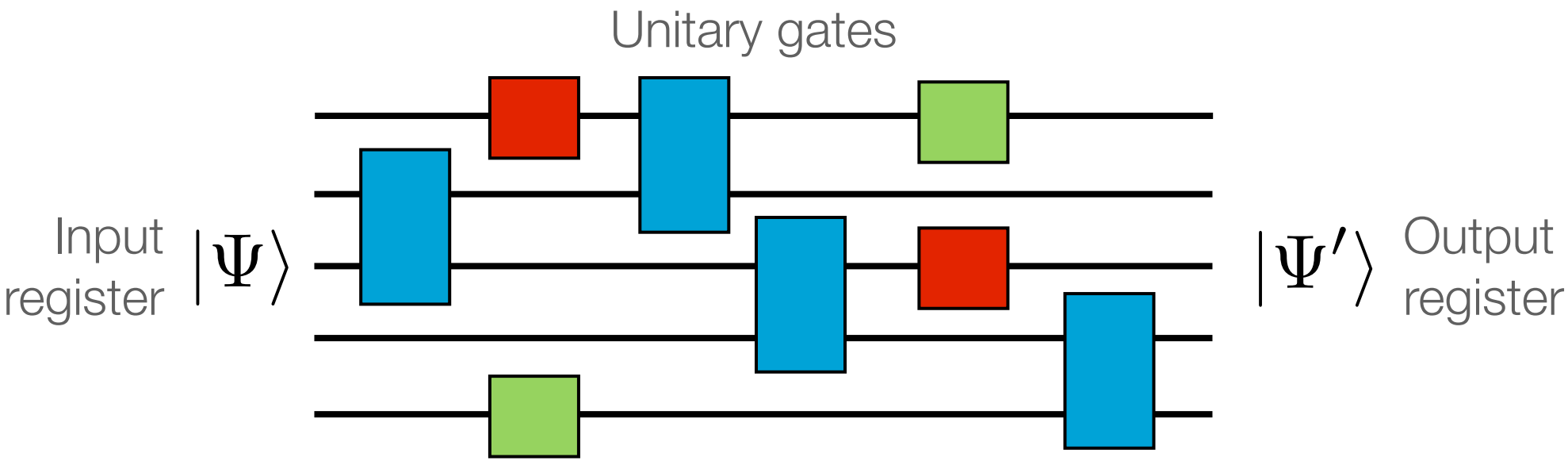
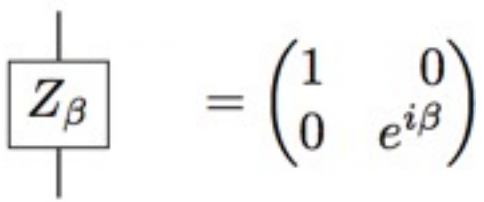


$$U : \mathbb{C}^{2^n} \longrightarrow \mathbb{C}^{2^n}$$

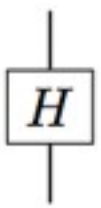
Universality

We say that a model of quantum computation is **universal** if it can represent all unitary maps.

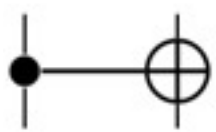
The circuit model requires a small set of gates to be universal:



$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix}$$



$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



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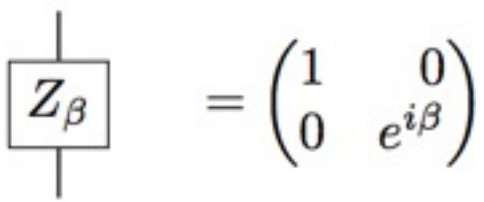

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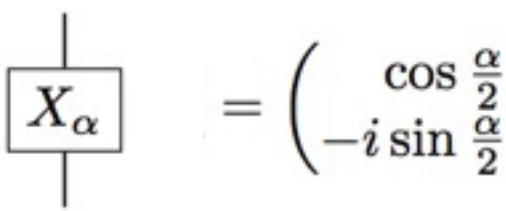
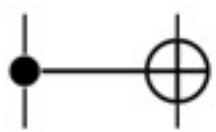


$$= \begin{pmatrix} \cos\frac{\alpha}{2} & -i\sin\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Quantum Observables

Each observable quantity O is represented as self-adjoint operator $\hat{O} = \sum_i \lambda_i \left| e_i \right\rangle \left\langle e_i \right|$:

- The possible values of O are the eigenvalues λ_i of $\hat{O}$
- When we observe O for a system in state $|\psi\rangle$, there is probability $\langle e_i \mid \psi \rangle^2$ of observing λ_i.
- If λ_i is the outcome of the measurement, the system is then in state $|e_i\rangle$.

In general, measuring O_1 then O_2 will give a different answer than measuring O_2 first!

Not every quantum observable is well defined at the same time.

Pauli Matrices

The following 1-qubit unitaries are the Pauli spin matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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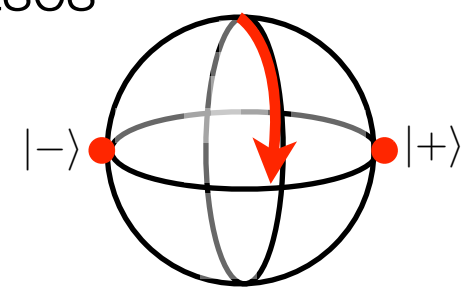
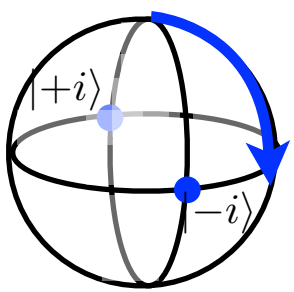
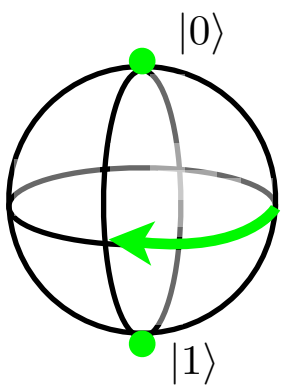
2. Their eigenvectors form a set of 3 Mutually Unbiassed Bases







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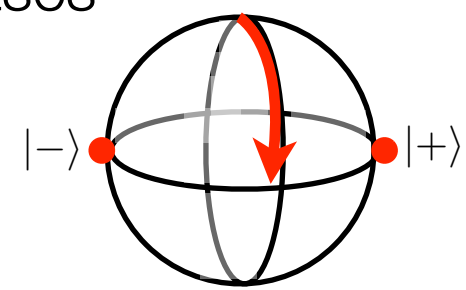

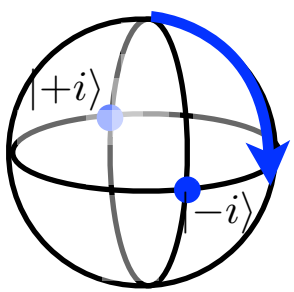

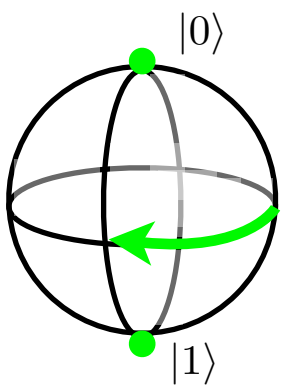

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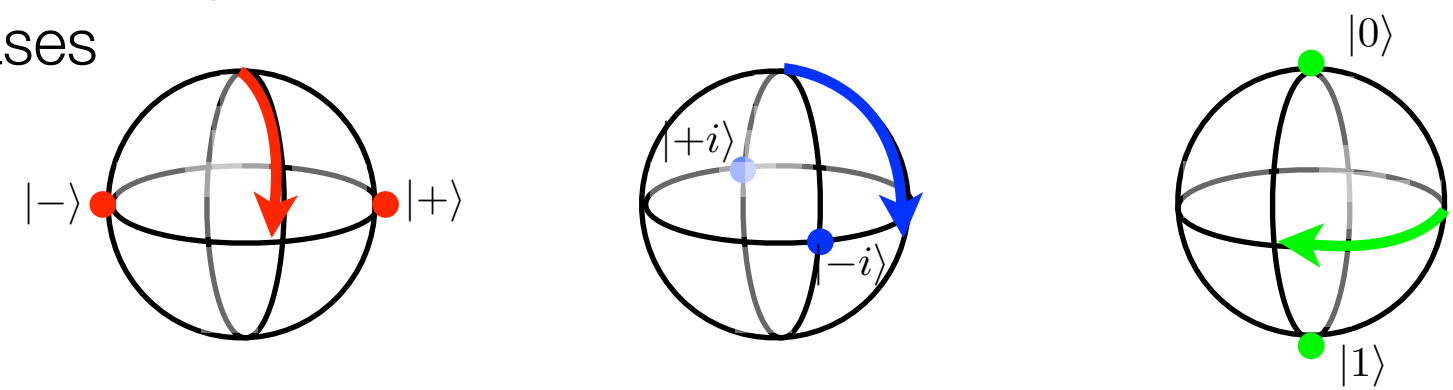

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X and Z Spins

We can measure the spin of qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

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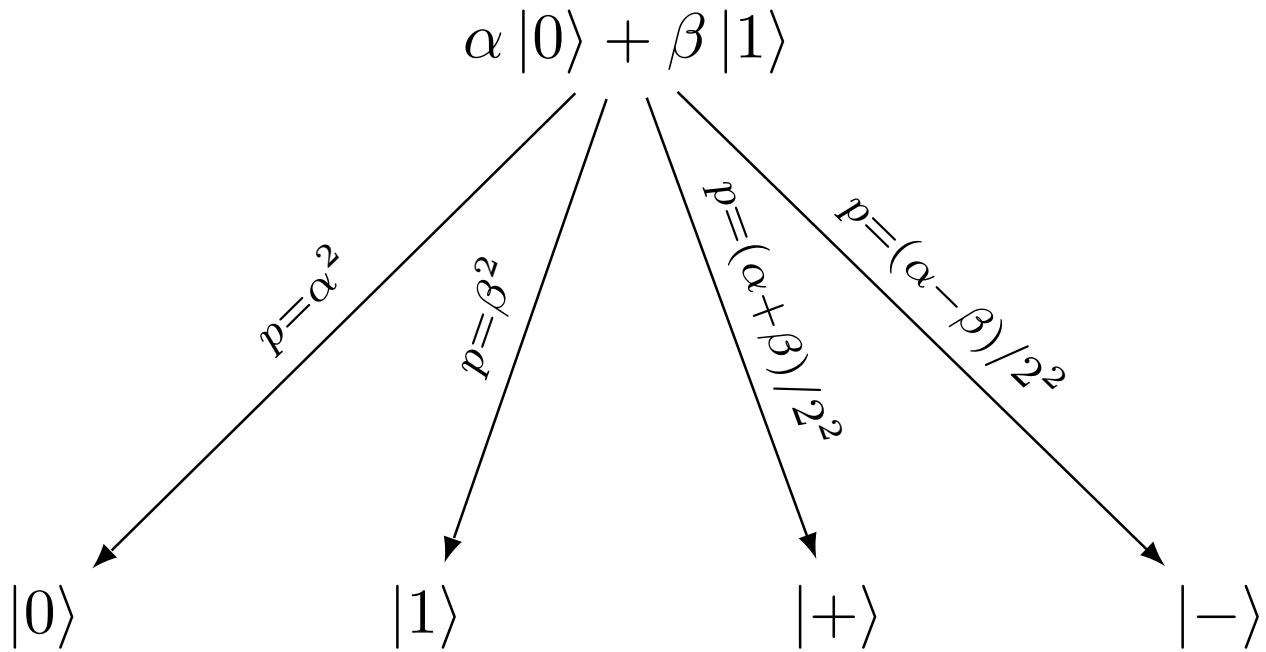

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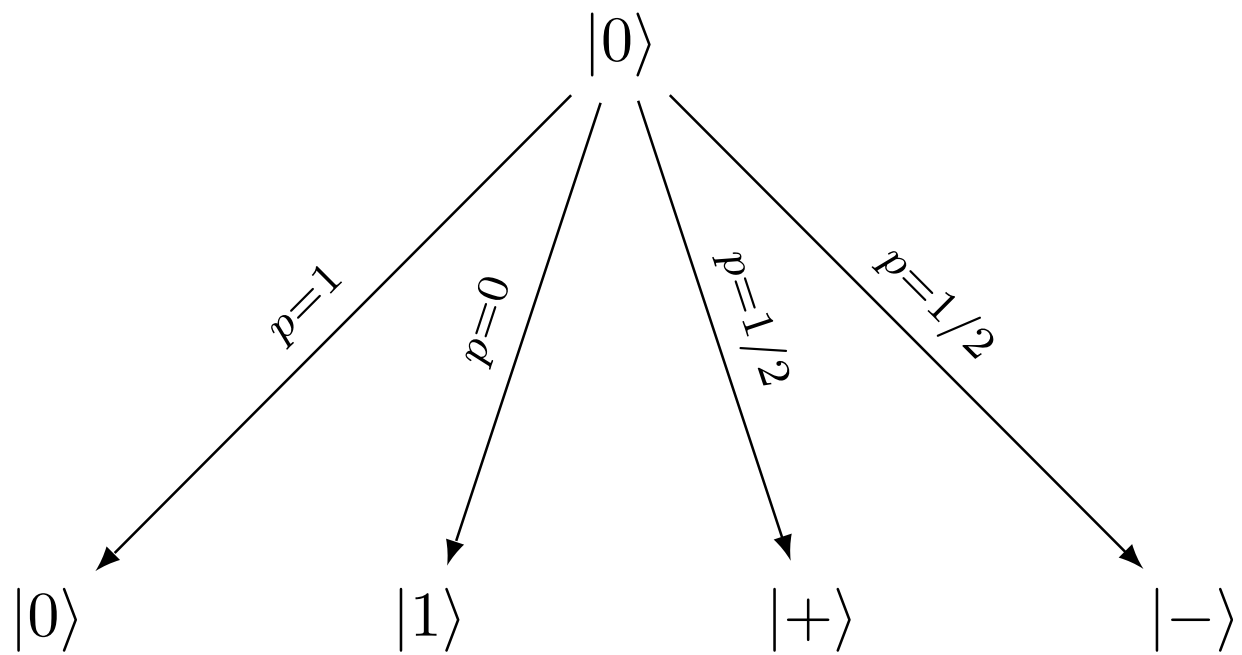

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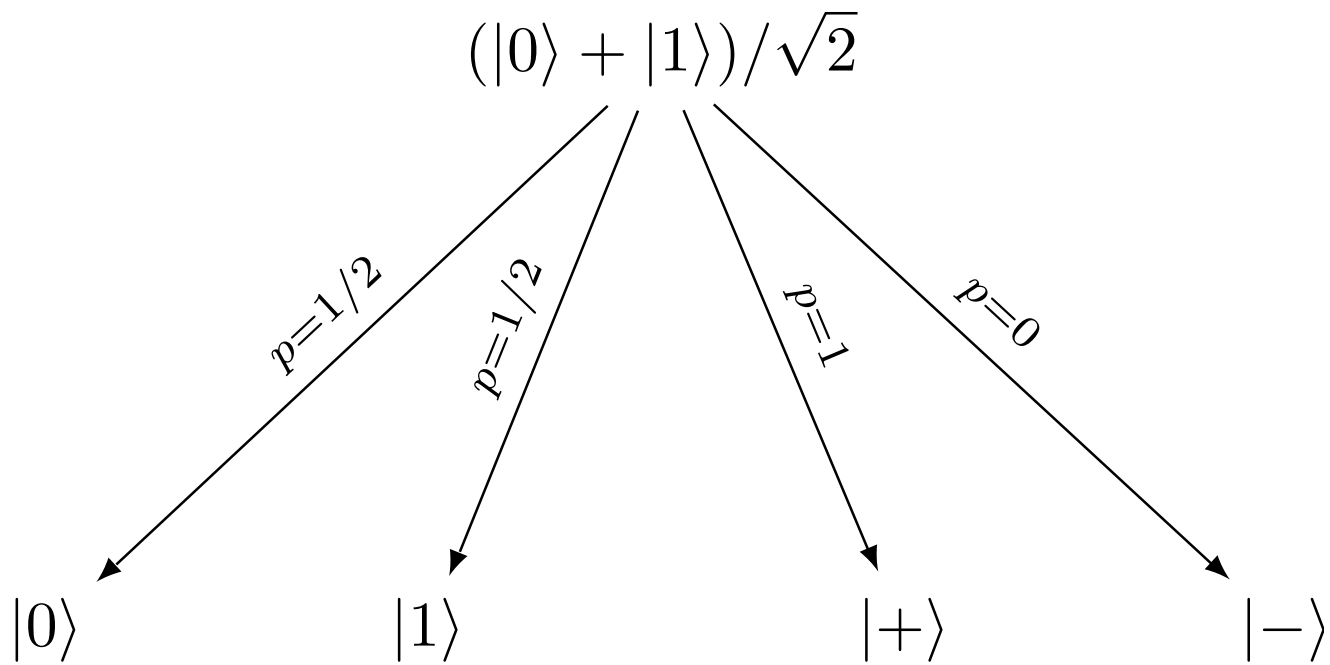

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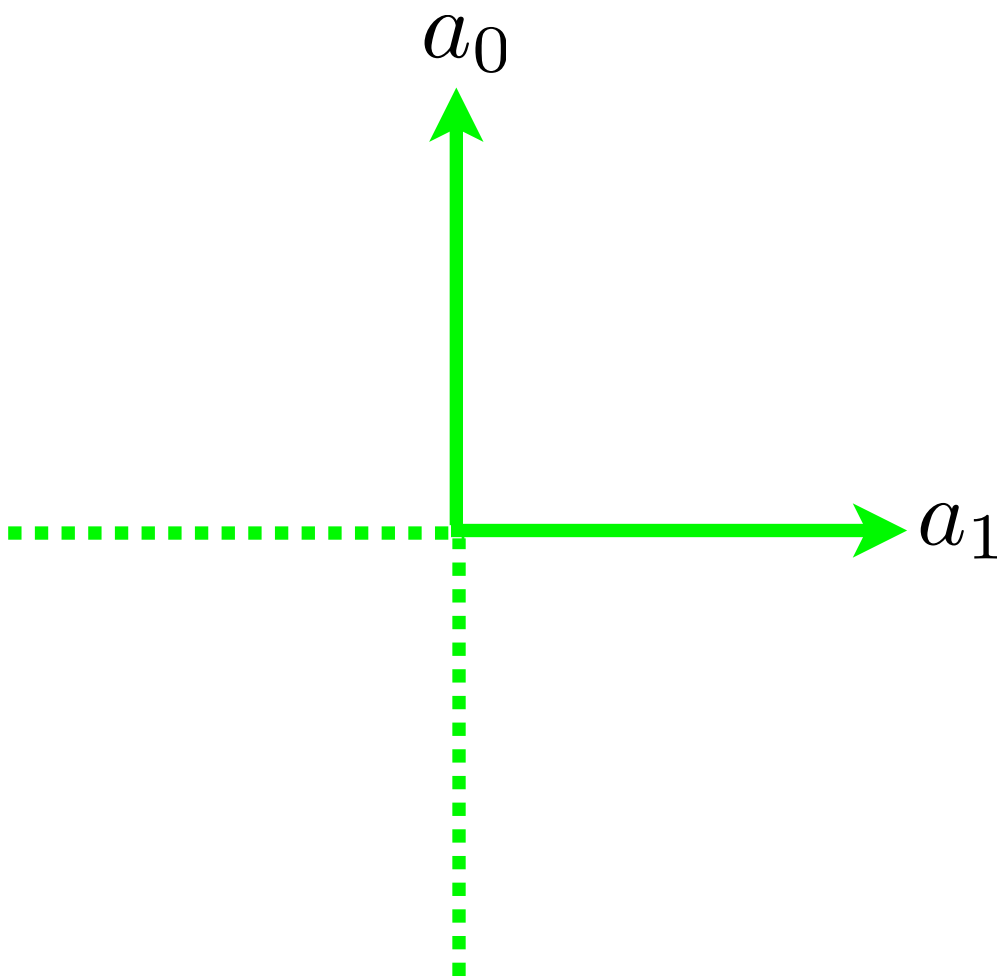

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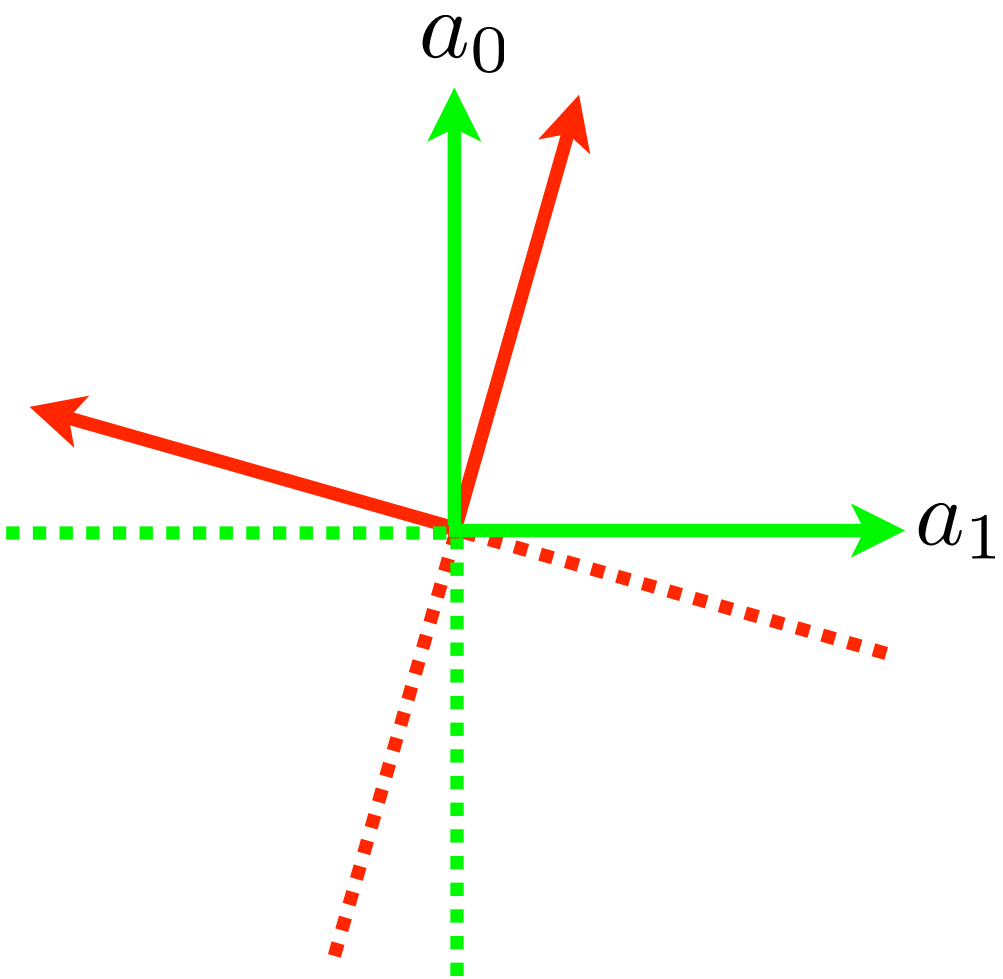

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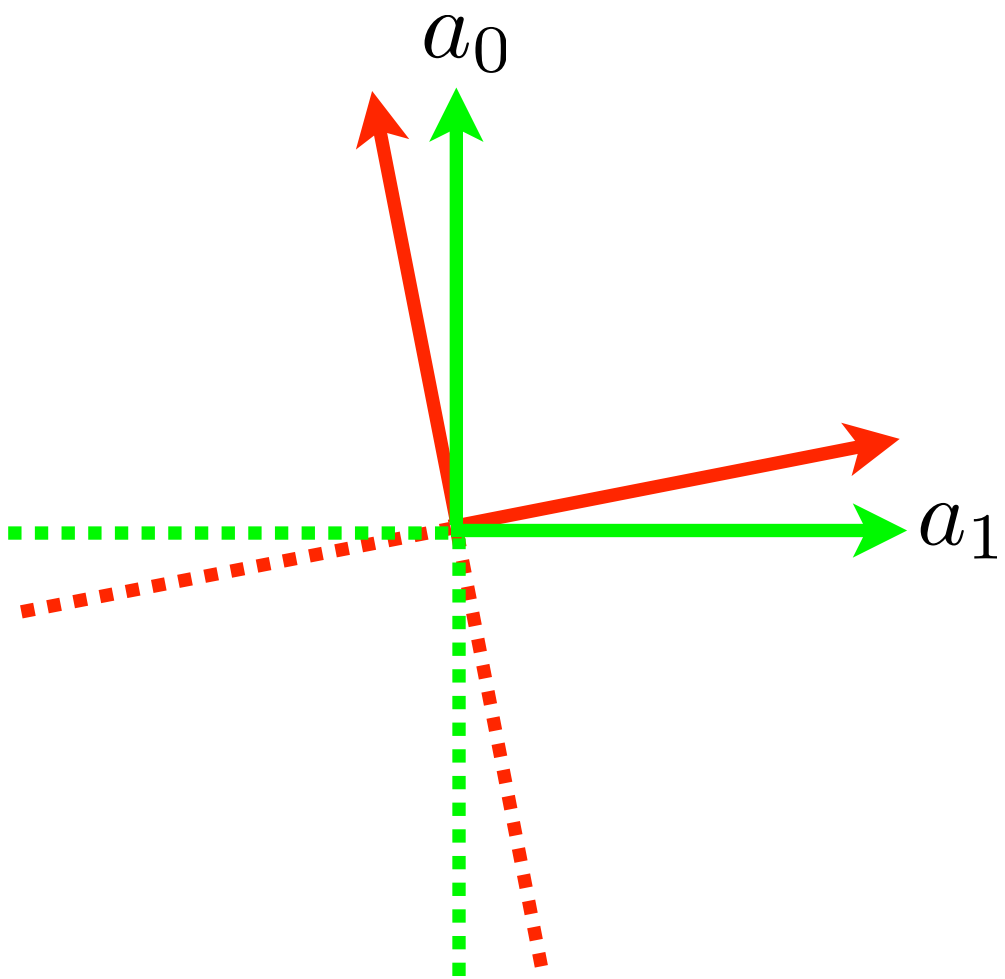

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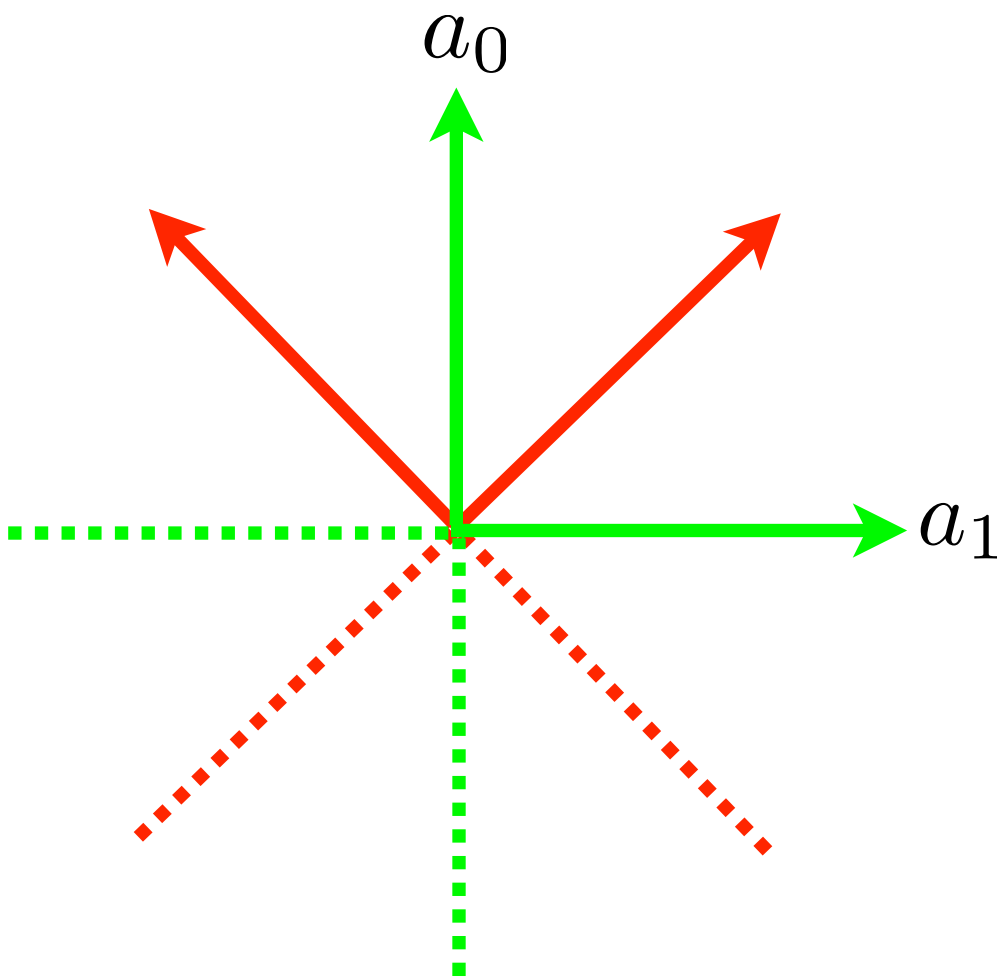

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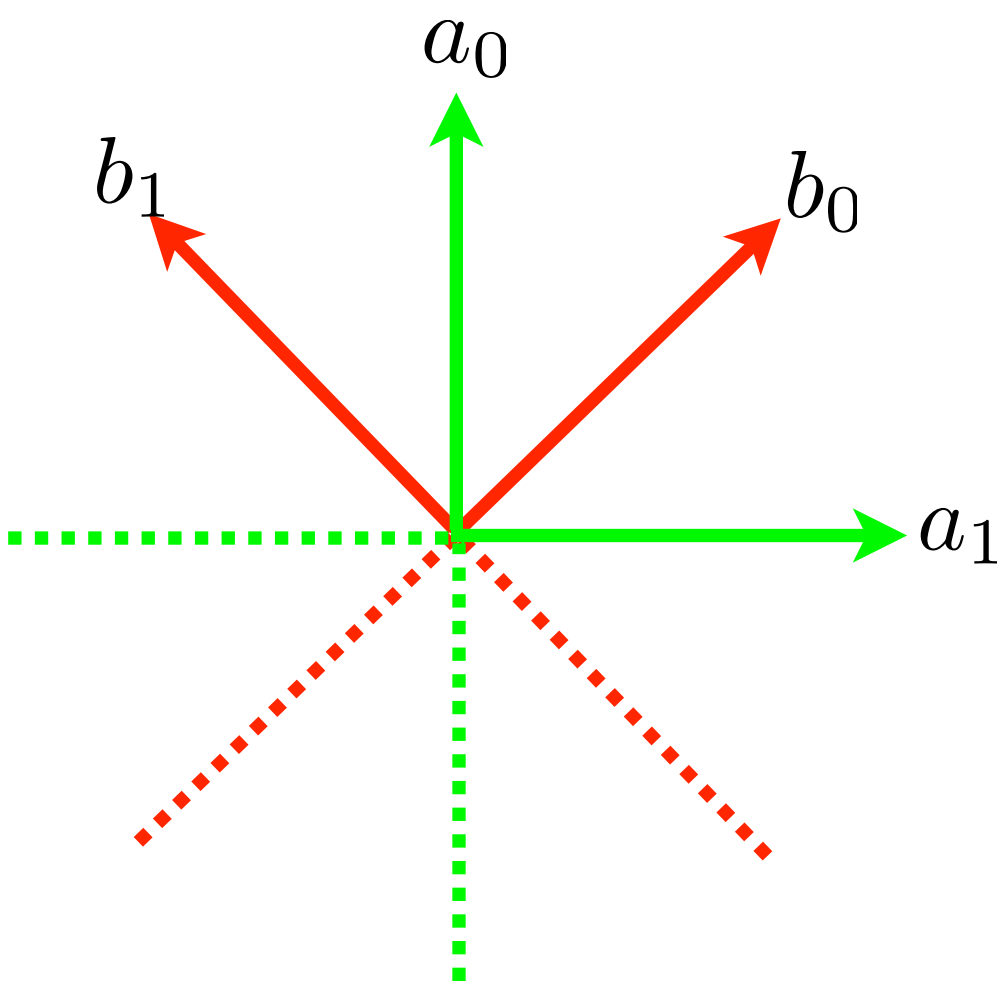

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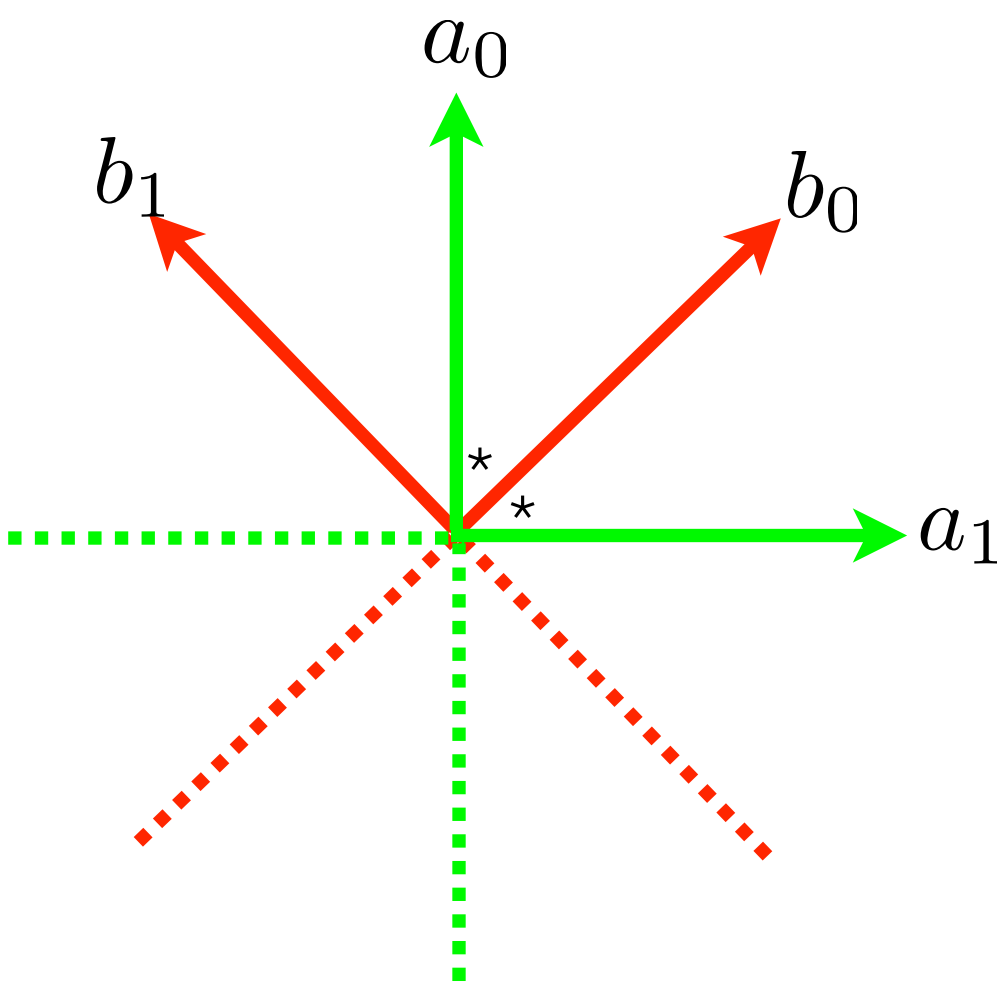

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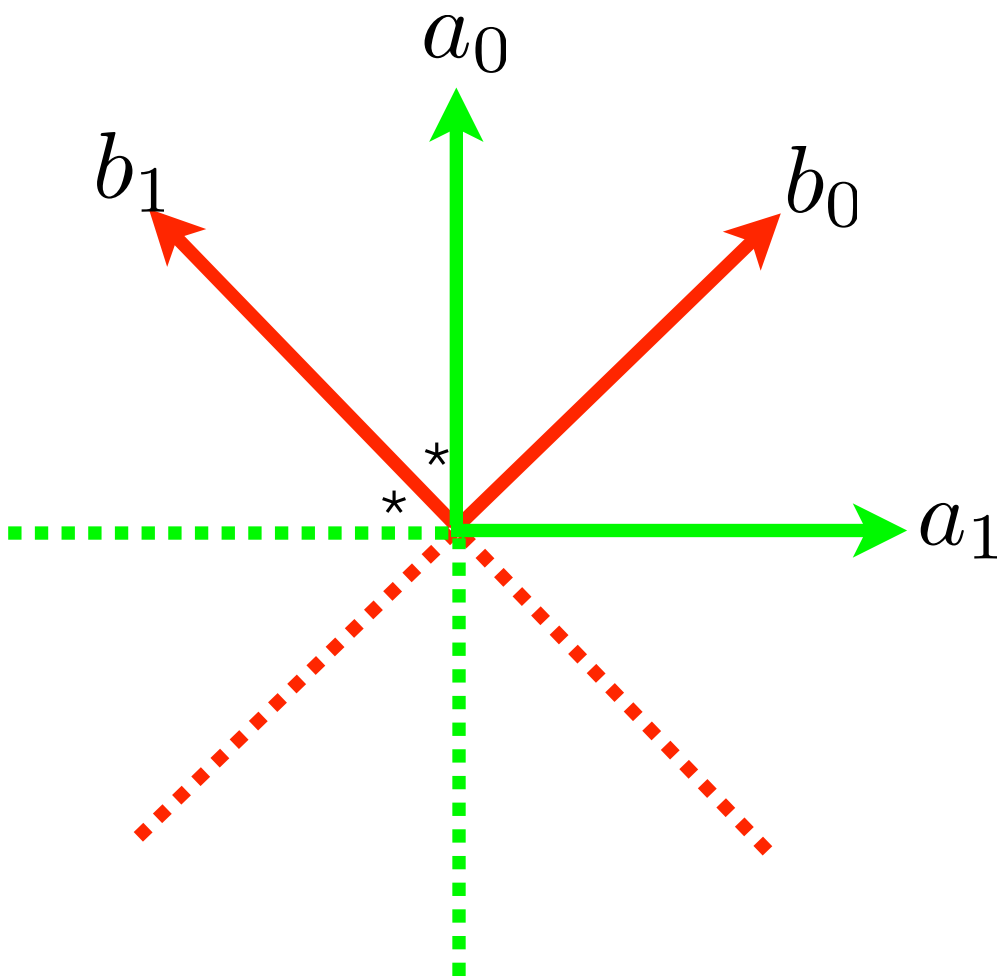

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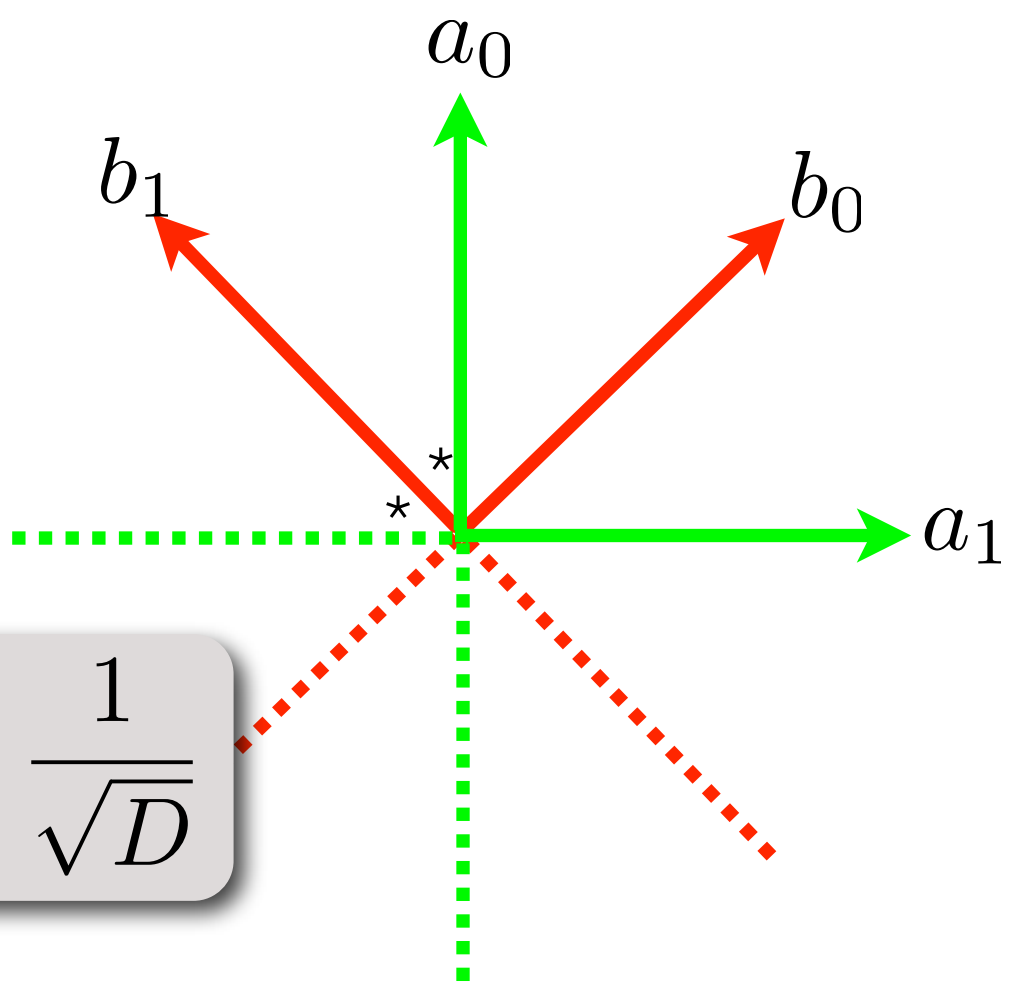

$$|\langle a_i \mid b_j \rangle| = \frac{1}{\sqrt{D}}$$

Entangled States

A state $|\psi\rangle \in A \otimes B$ is called *separable* if $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$; otherwise it is called *entangled;* i.e. it must be written as:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle$$

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Example 1: $|00\rangle = |0\rangle \otimes |0\rangle$ is separable

Example 2: $|00\rangle + |01\rangle + |10\rangle + |11\rangle$ is separable

$$= (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

$$= |++\rangle$$

Entangled States

A state $|\psi\rangle \in A \otimes B$ is called *separable* if $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$; otherwise it is called *entangled;* i.e. it must be written as:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle$$

Example 3: $|\mathrm{Bell}_1\rangle = |00\rangle + |11\rangle$ is entangled

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Example 3: $|\mathrm{Bell}_1\rangle = |00\rangle + |11\rangle$ is entangled

Example 4: $|H\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle$ is entangled

Entanglement and Measurement

- Entangled states consist of two or more systems which can be shared between distant parties
- If one party measures their system the other system can be affected.

$$|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B$$ Initial shared state

Entanglement and Measurement

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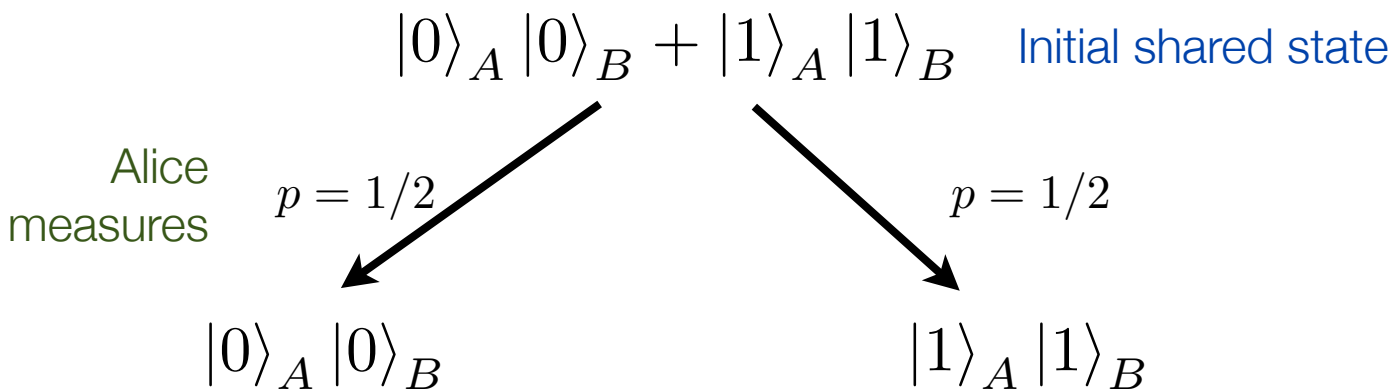

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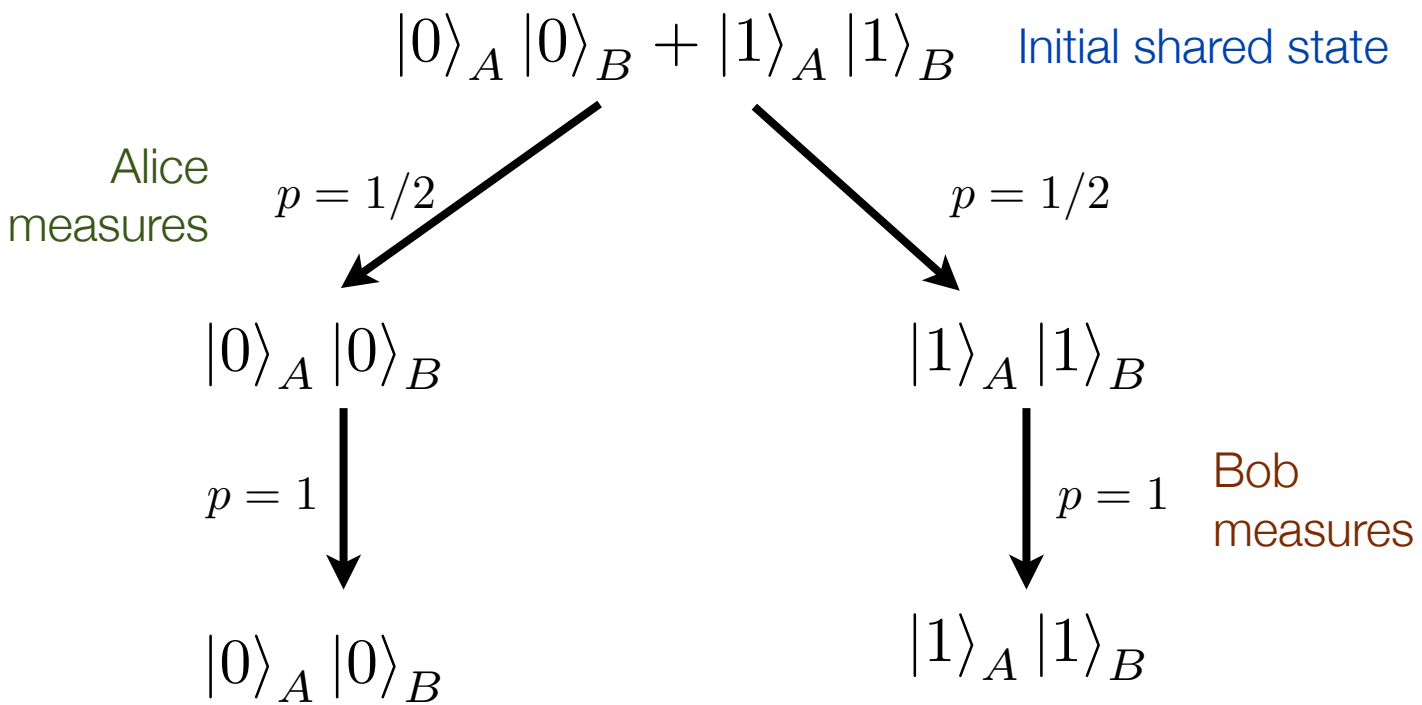


Map-State Duality

Recall that there is an isomorphism :

$$A \multimap B \cong A \otimes B$$

In particular:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \longleftrightarrow \quad |00\rangle + |11\rangle =: |\mathrm{Bell}_1\rangle$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \longleftrightarrow \quad |01\rangle + |10\rangle =: |\mathrm{Bell}_2\rangle$$

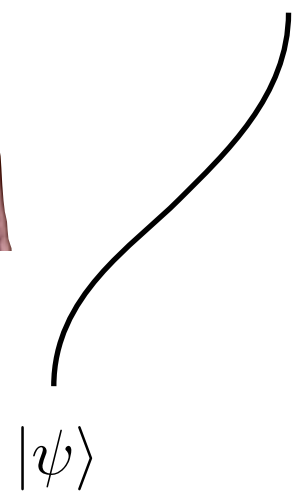
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \longleftrightarrow \quad |00\rangle - |11\rangle =: |\mathrm{Bell}_3\rangle$$

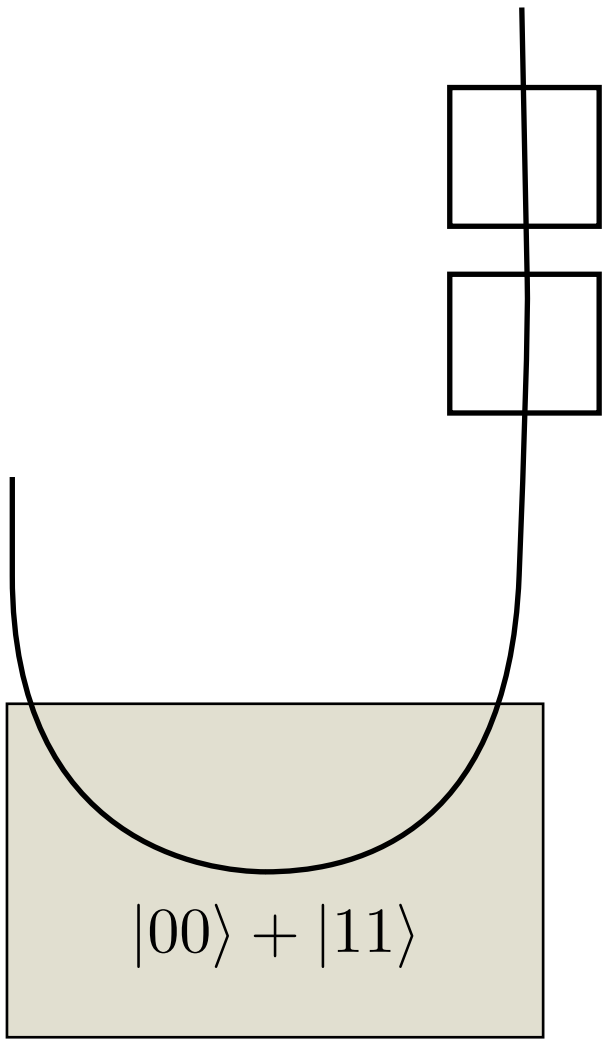
$$XZ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \longleftrightarrow \quad |01\rangle - |10\rangle =: |\mathrm{Bell}_4\rangle$$

Since they form a basis we can measure with them.

Quantum Teleportation

~~Alice~~
Audrey





Bob

Quantum Teleportation

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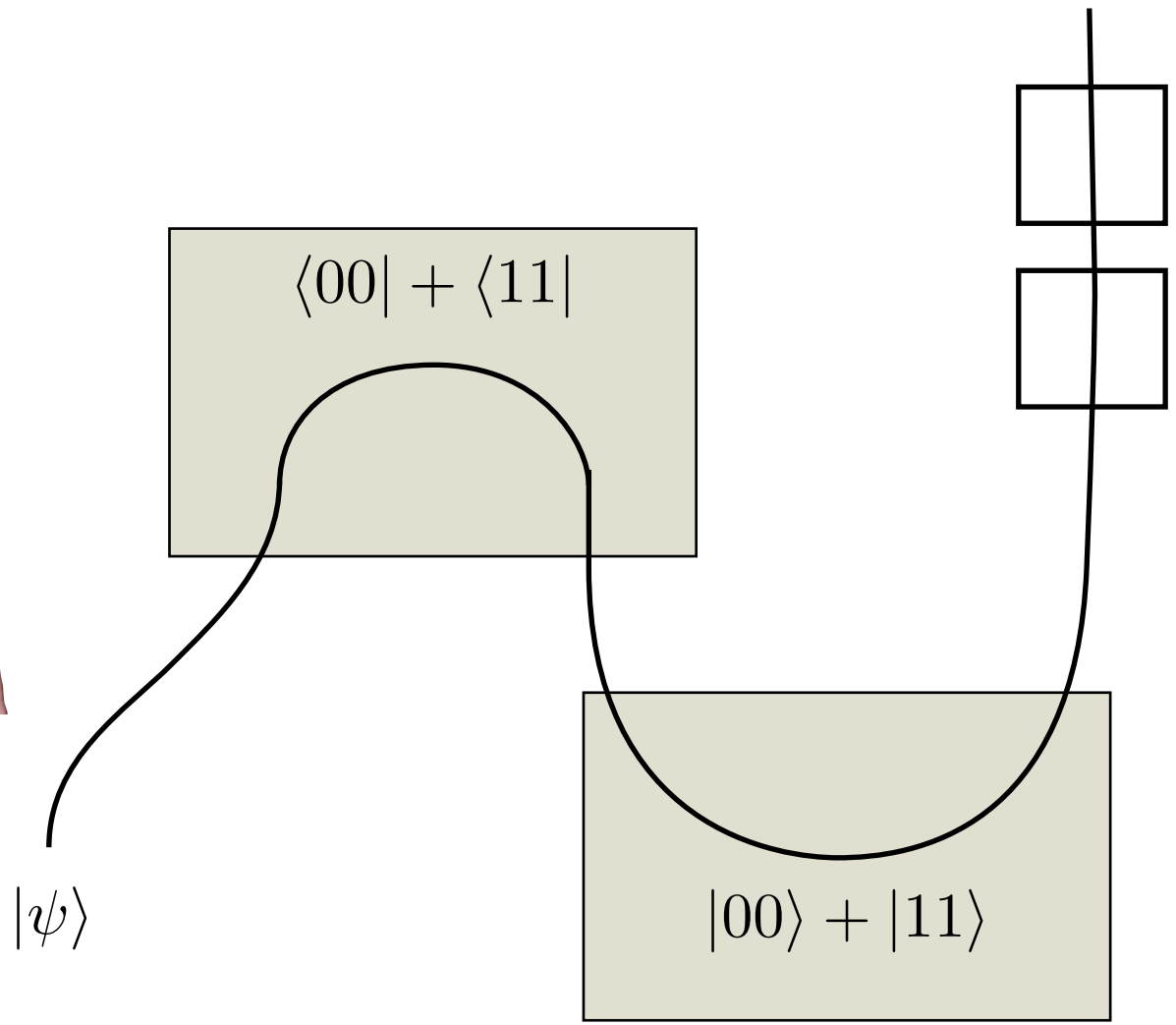

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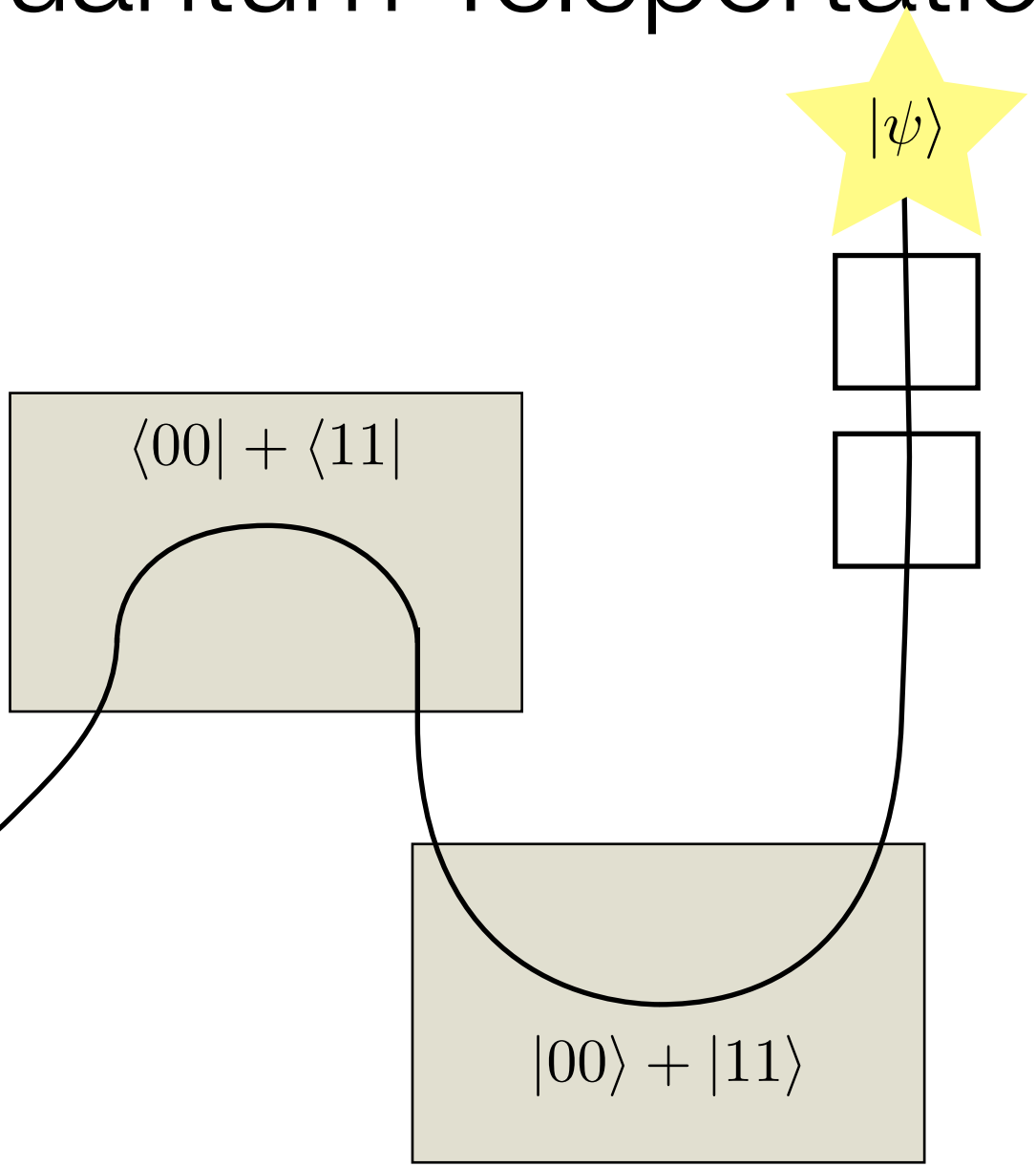

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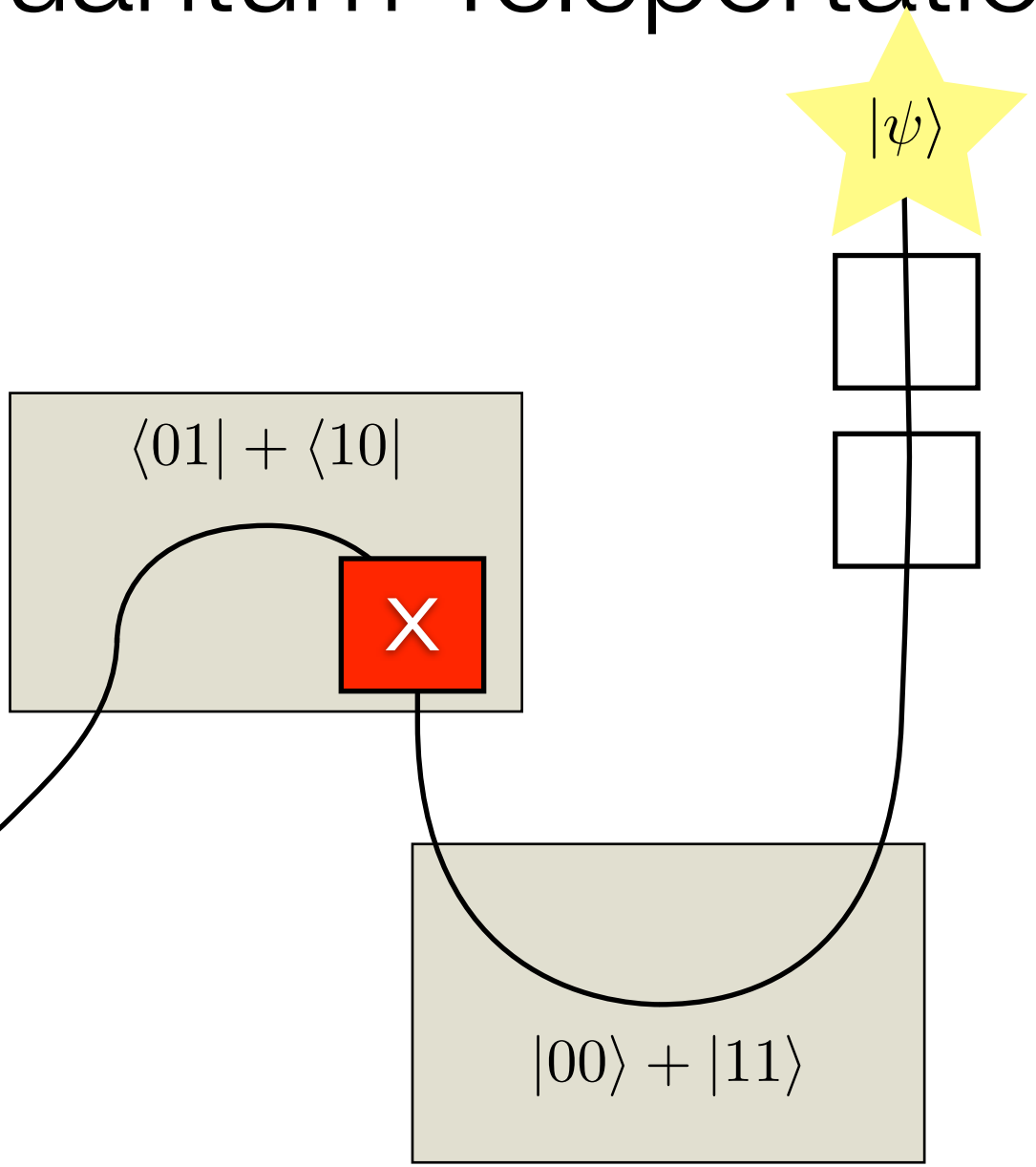

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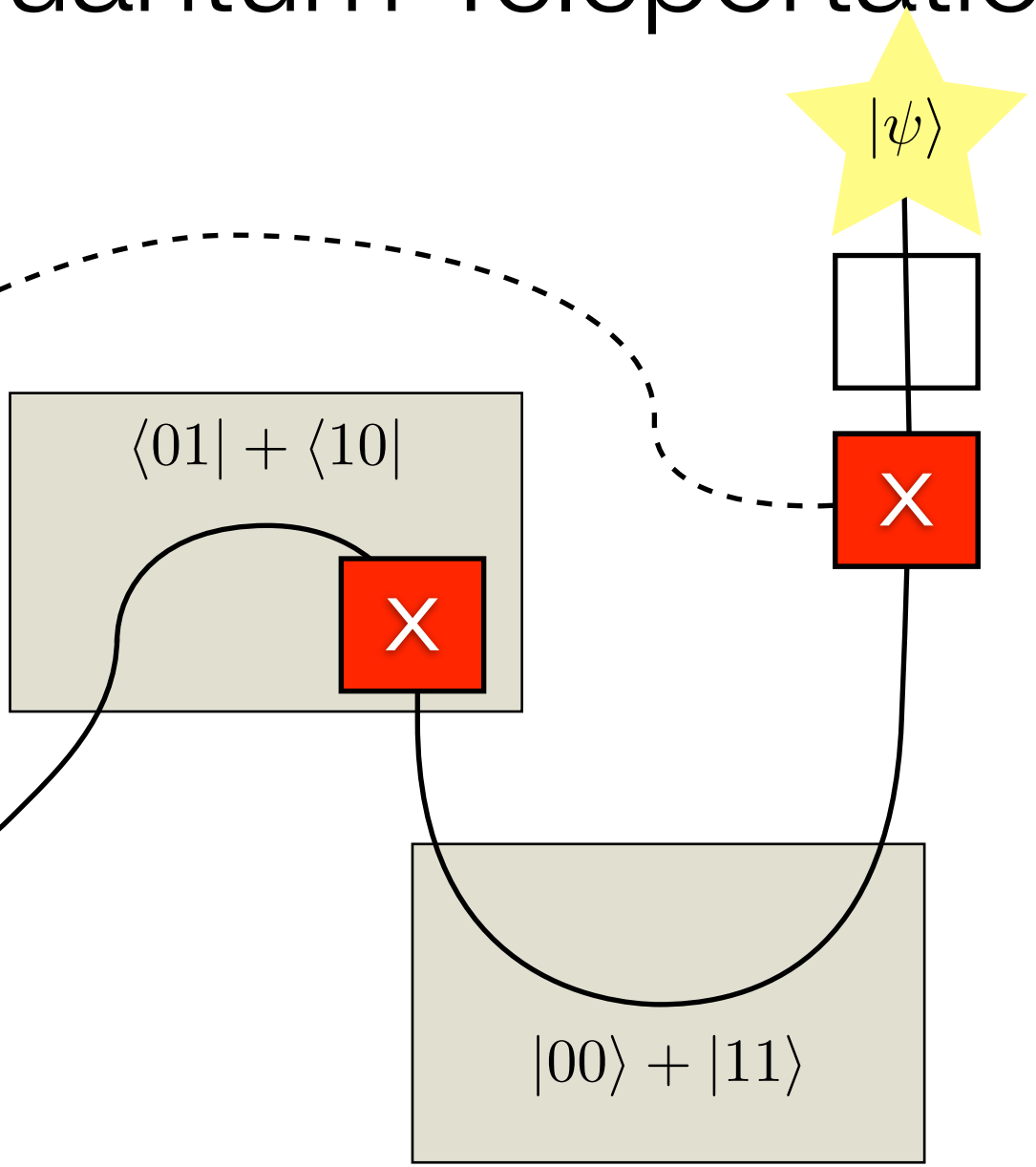

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Quantum Teleportation

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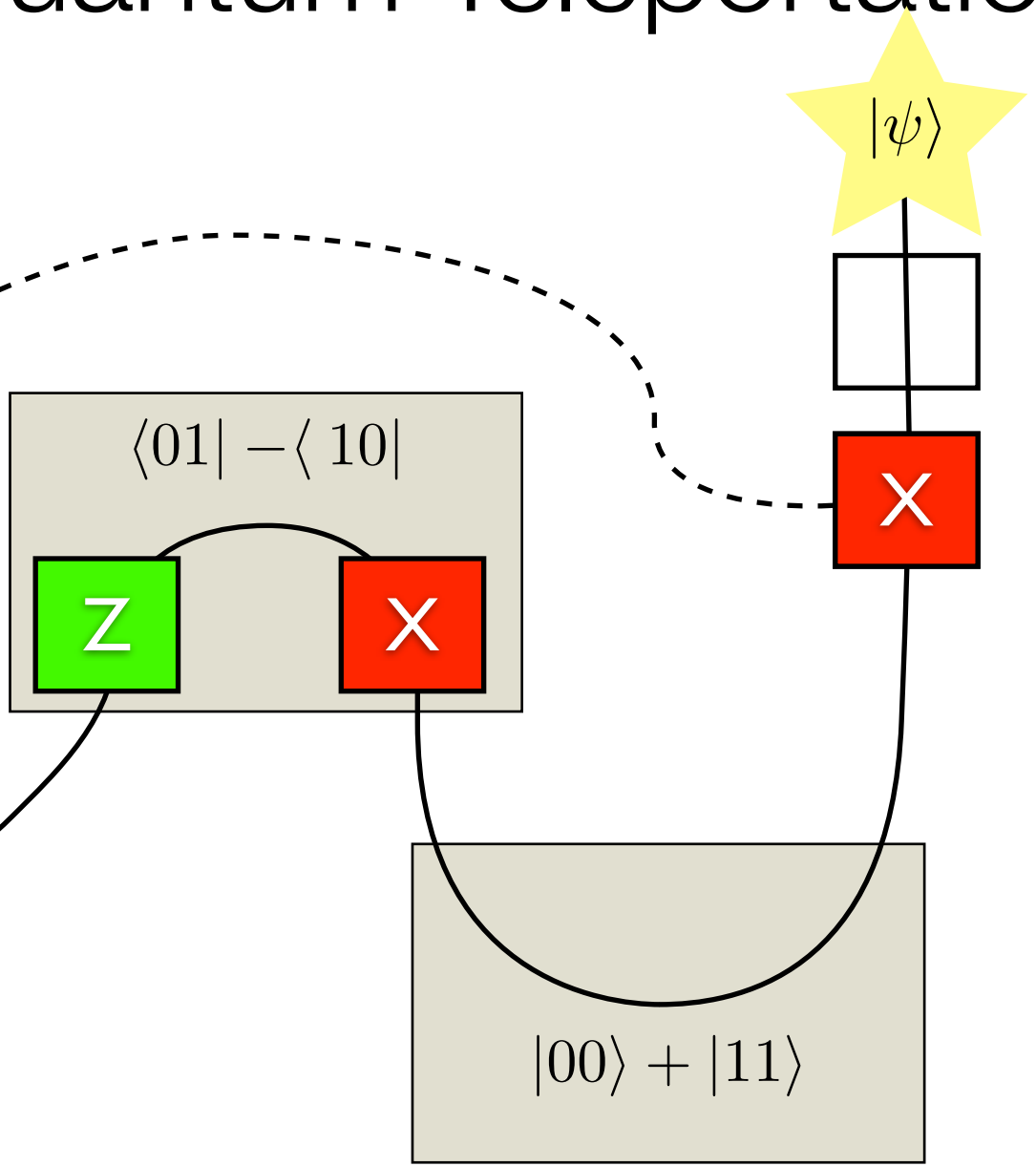

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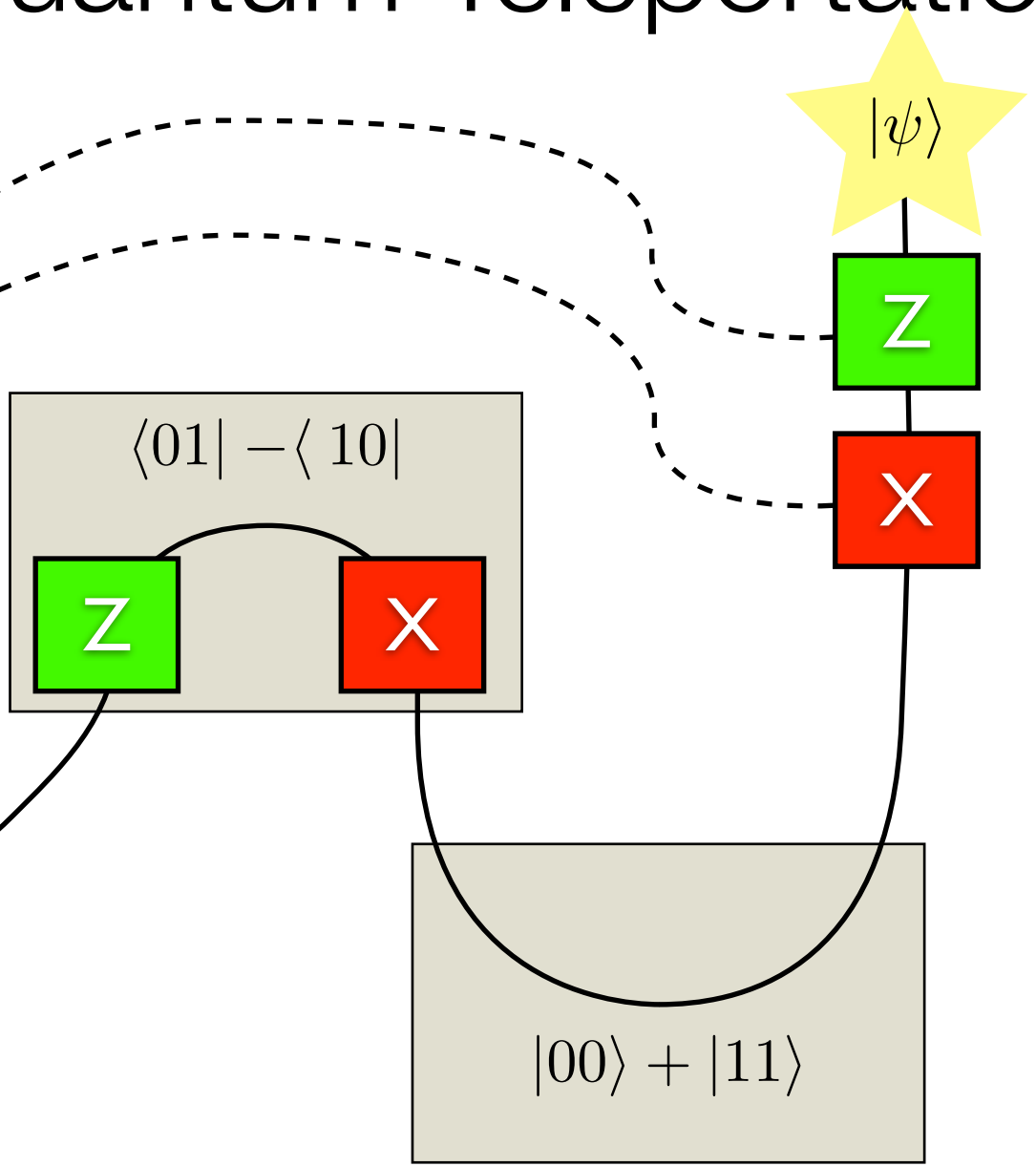


Bob

Channels via entanglement

Bennett at al:

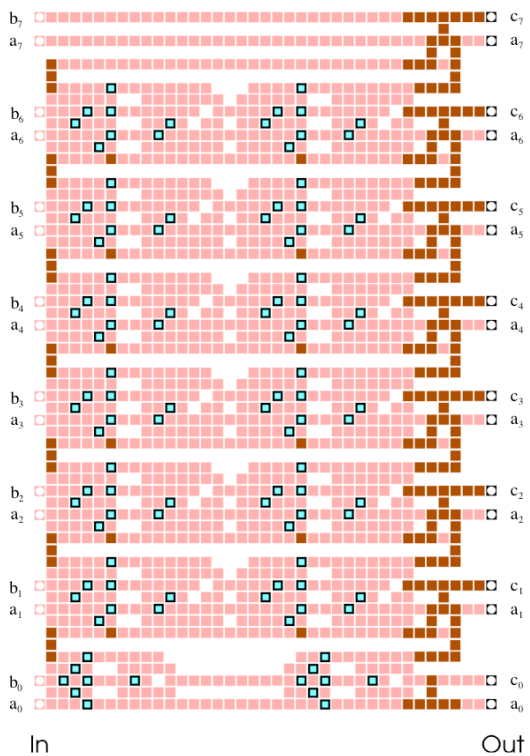
> *"Note that qubits are a directed channel resource, sent in a particular direction from the sender to the receiver; by contrast [entangled pairs] are an undirected resource shared between the sender and receiver."*

Teleporting an unknown quantum state via dual classical and EPR channels, PRL, 1993

This suggests that the type of an entangled pair should be the *linear* type $Q \otimes Q$ rather than the usual $Q \to Q$.

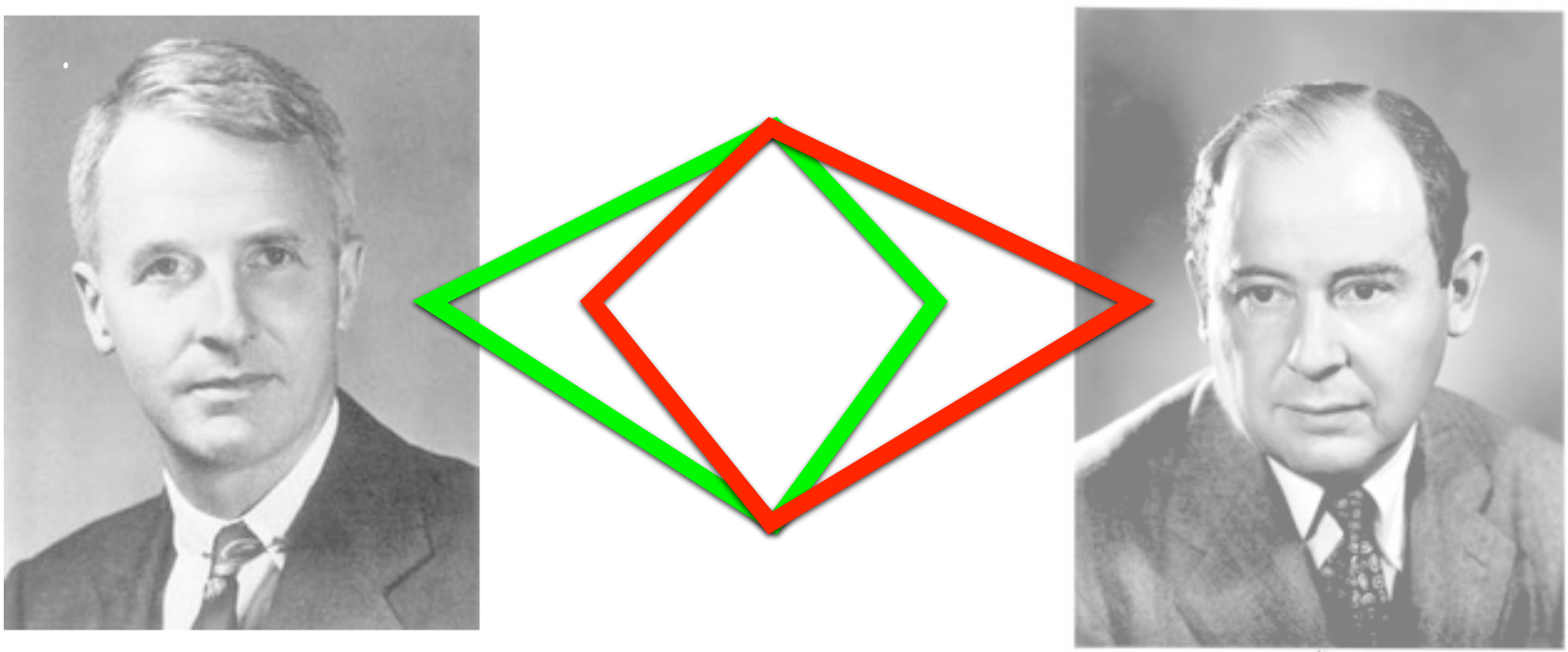
More Entanglement

Entanglement can be used for a lot more than just transmitting information:



MBQC is a universal model of computation which is based on the flow of information through large entangled states.

Quantum Logic



The Birkhoff-von Neumann approach and its

A small historical detour

Quantum logic was an attempt to do two things at once:

- Develop a logic that took the limitations of knowledge imposed by quantum mechanics seriously;
- Re-found quantum theory on a more abstract logical basis.

It is a “Tarskian” approach based purely on *what the propositions mean*, and not at all concerned with the proofs.

Propositions and projectors

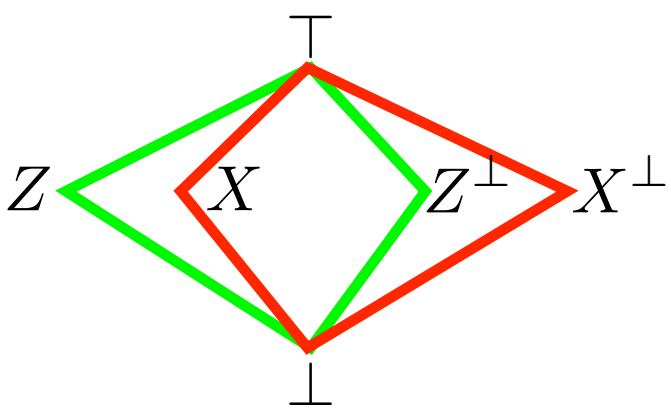
A proposition is a question with a yes/no answer:

A = *"Is the spin up?"*

but the answer will be given by a quantum measurement:

$$\psi \models A \quad \Leftrightarrow \quad p_A \left|\psi\right\rangle = \left|\psi\right\rangle$$

hence each proposition corresponds to a pair of orthogonal subspaces.

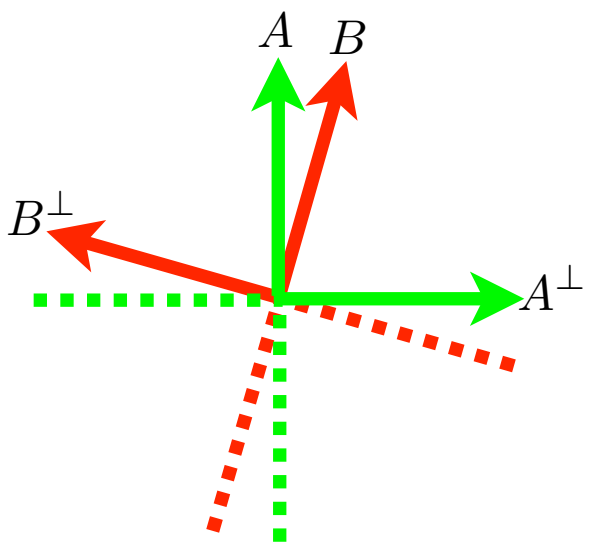


The "lattice of propositions" is simply the collection of closed subspaces ordered under inclusion.

Distributivity Fails

In general we have $p_A p_B \neq p_B p_A$ which implies the failure of distributivity.

Consider:



we have

$$\bot = (A \wedge B) \vee (A^\perp \wedge B) \neq (A \vee A^\perp) \wedge B = B$$

hence such a lattice is not distributive.

(It does satisfy a weaker law called *orthomodularity* which I won't discuss.)

No deduction theorem

Theorem: Suppose we can define a connective $\rightarrow$ such that

$$A \wedge X \leq B \qquad \Leftrightarrow \qquad X \leq A \rightarrow B$$

then the lattice is distributive.

Corollary: Quantum logic does not admit modus ponens.

Note that the sub-lattice defined by any set of commuting projectors is just a boolean lattice.

No “good” tensor product

Given finite dimensional Hilbert spaces H_1 and H_2, we can construct their subspace lattices $\mathcal{L}(H_1)$ and $\mathcal{L}(H_2)$. In fact this is a functor:

$$\mathcal{L} : \mathbf{FDHilb} \to \mathbf{OML}$$

But what about the tensor product?

$$\mathcal{L}(H_1) \otimes \mathcal{L}(H_2) = \mathcal{L}(H_1 \otimes H_2) \; ?$$

To date no one has been able to find a tensor product on $\mathbf{OML}$ to make this functor monoidal.

Quantum logic today

The failure of both sequential and parallel modes of composition in quantum logic means that the projection lattice approach cannot support any notion of ***process***.

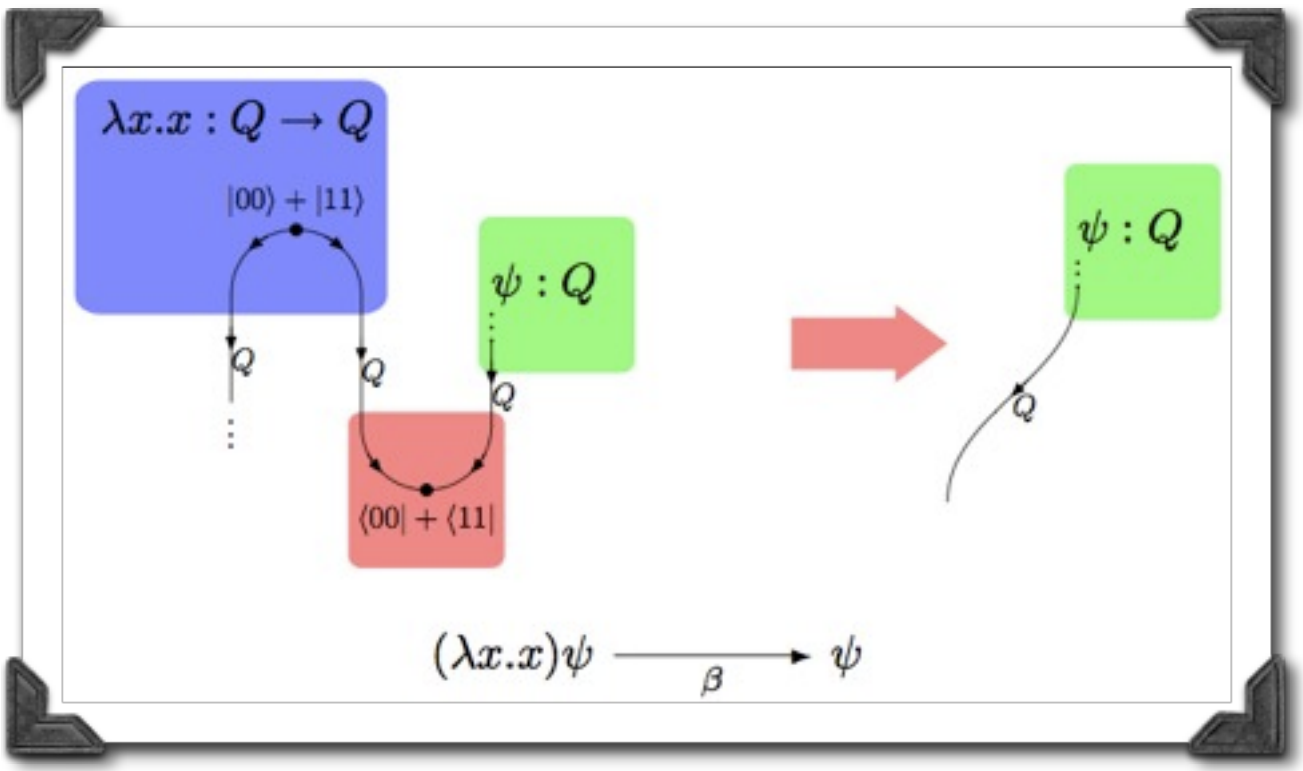
Hence for quantum computation, a new approach must be found.

Quantum Logic Today

Some modern developments based on quantum logic:

- The topos approach: essentially aims to get back realism by working in a suitable topos; Isham, Döring, Butterfield; Heunen, Landsman, Spitters.
- Jacobs and Heunen have shown that the lattice of subobjects in a dagger-kernel category is orthomodular; hence we can carry out quantum logic *internally* in a suitable category.
- Coecke, Heunen and Kissinger constructed orthomodular lattices internally using tensor products of certain Frobenius algebras.

Propositions as types for QM



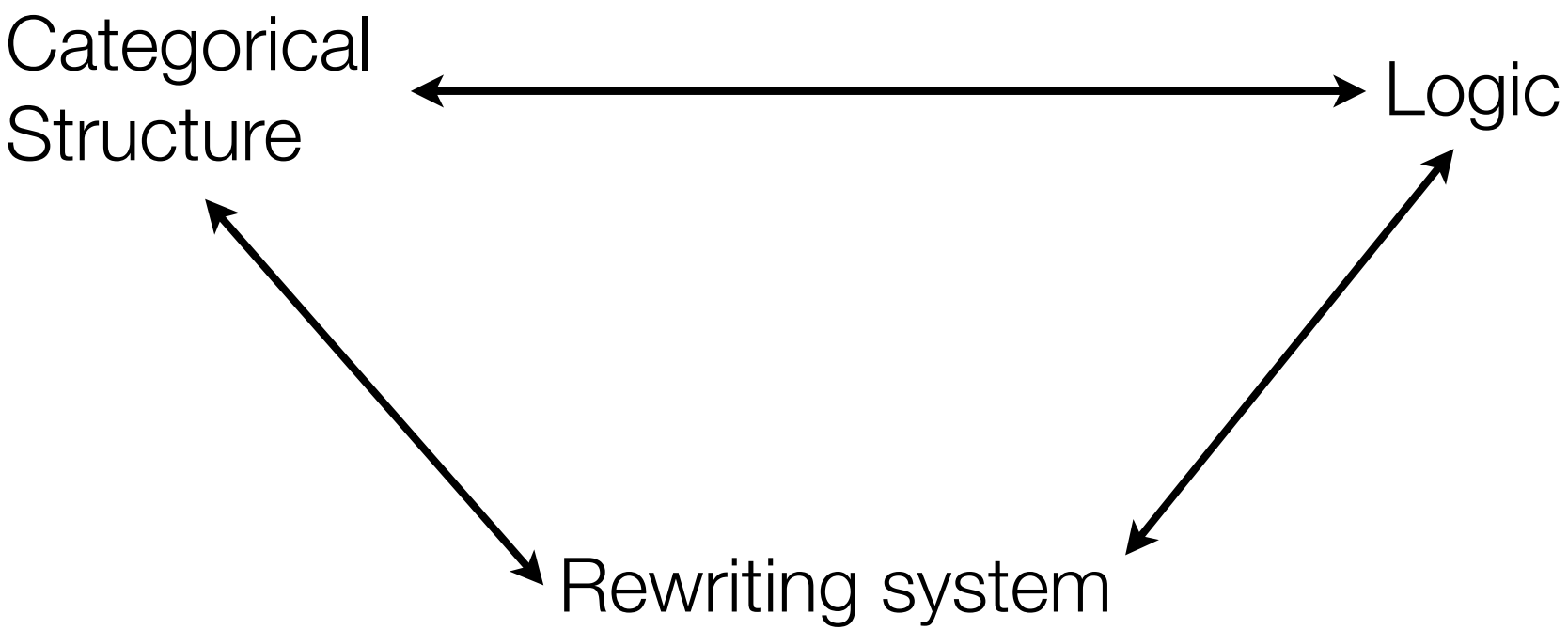
A logic based on processes not properties

Proofs and processes

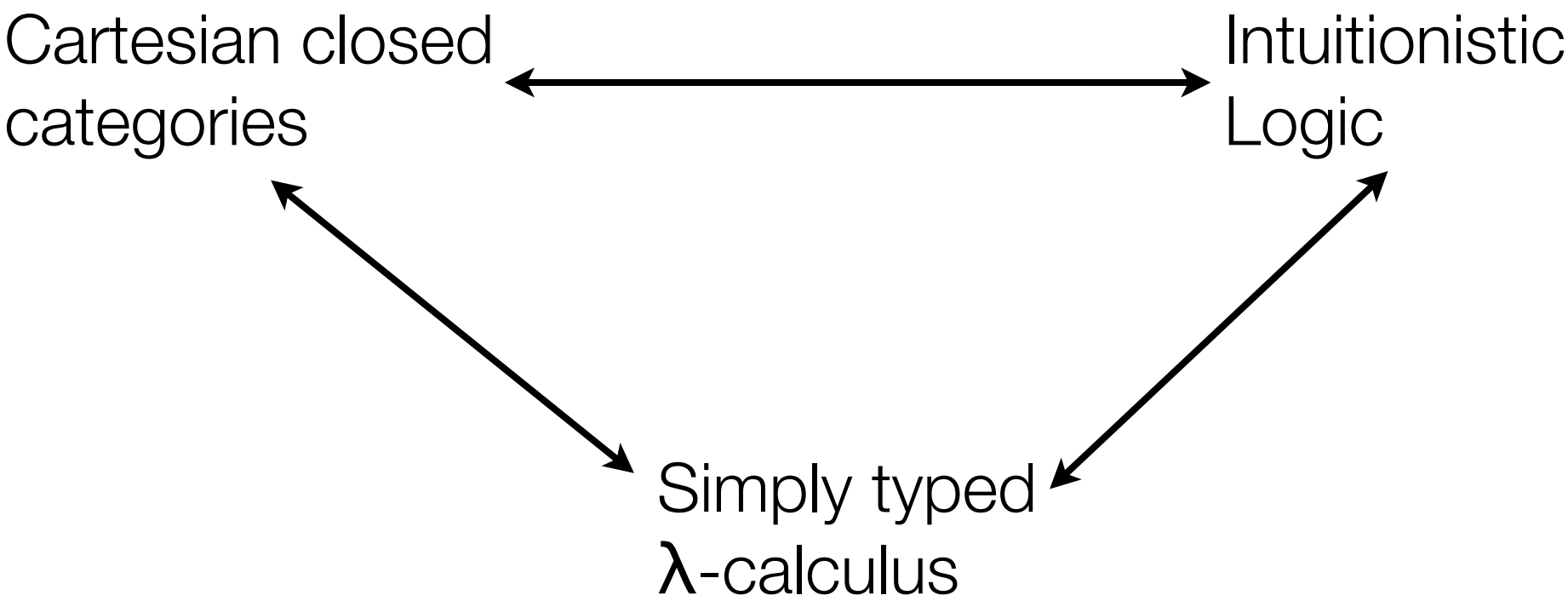
A long tradition in computer science is to treat the proof as the more important object.

- Propositions are *types.*
- Many different proofs of the same theorem, *processes* producing output of that *type.*
- Different possibilities for *equivalence of proofs*: denotational/static vs operational/dynamic.

General Scheme



The Curry-Howard-Lambek correspondence



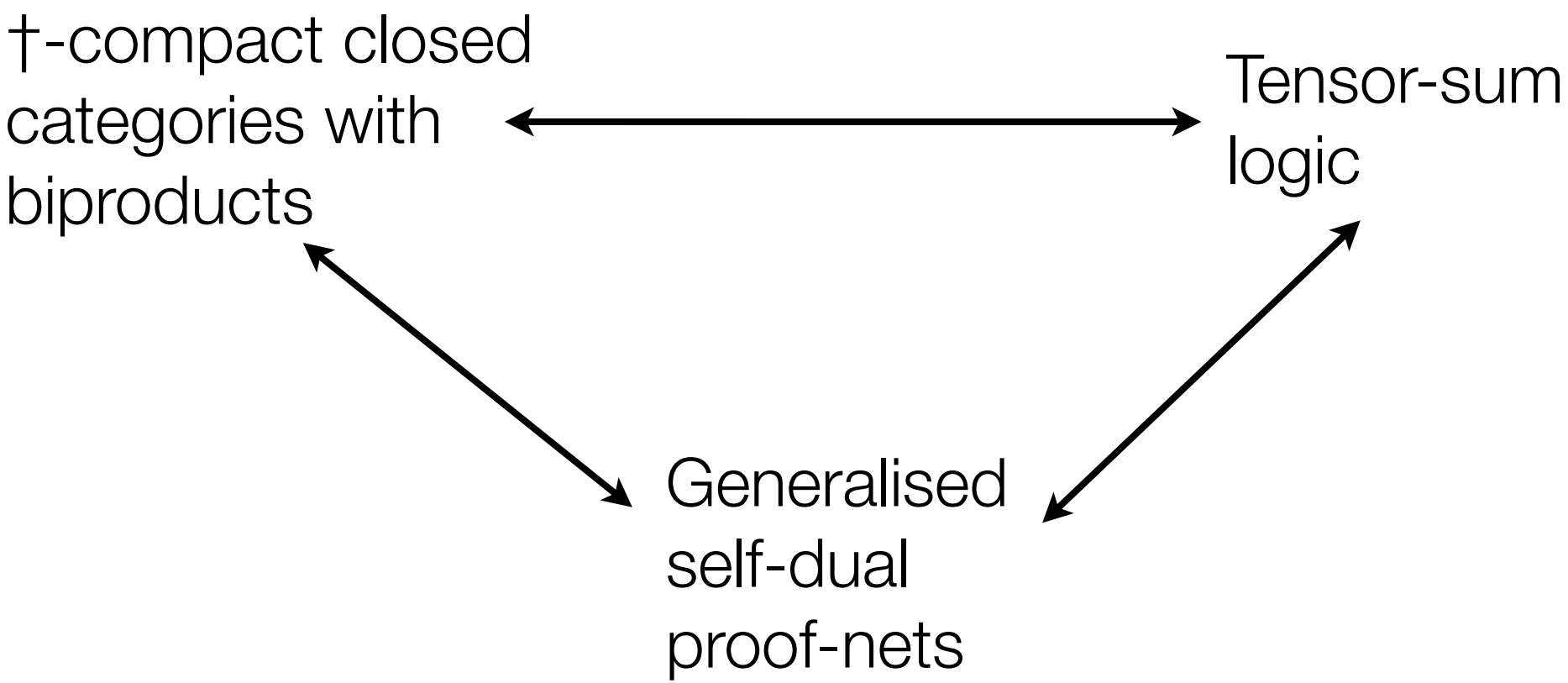
What is the quantum version?

- **We want a logic of "quantum processes"**

Some hints as to what this should be:

- entangled systems can't be described by a Cartesian product
- map-state duality suggests we should have a "function-type"
- no-cloning and no-deleting imply that the underlying setting should be *linear*
- ....however we still need some way to represent non-determinism

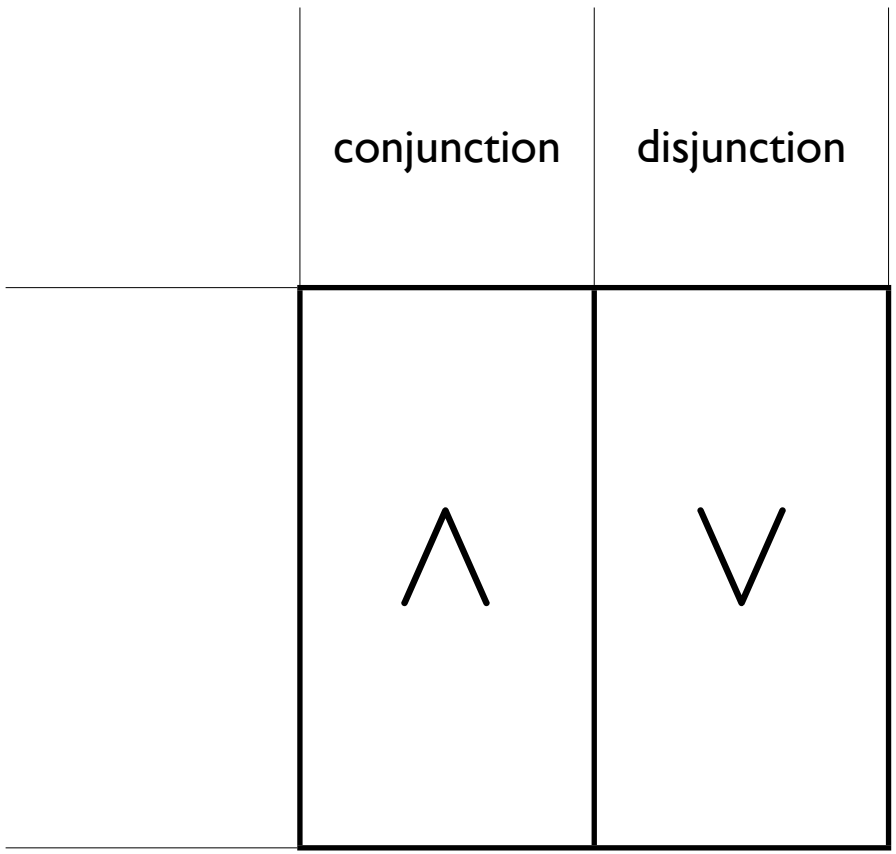
My (old) approach:



The connectives

Classical logic

$$\neg\neg A = A$$
$$\neg(A \wedge B) = \neg A \vee \neg B$$
$$\neg(A \vee B) = \neg A \wedge \neg B$$



The connectives

Linear logic
(MALL)

$$A^{\perp\perp} = A$$
$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$$
$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$
$$(A \& B)^{\perp} = A^{\perp} \oplus B^{\perp}$$
$$(A \oplus B)^{\perp} = A^{\perp} \& B^{\perp}$$

	conjunction	disjunction
multiplicative	$\otimes$	$\parr$
additive	$\&$	$\oplus$

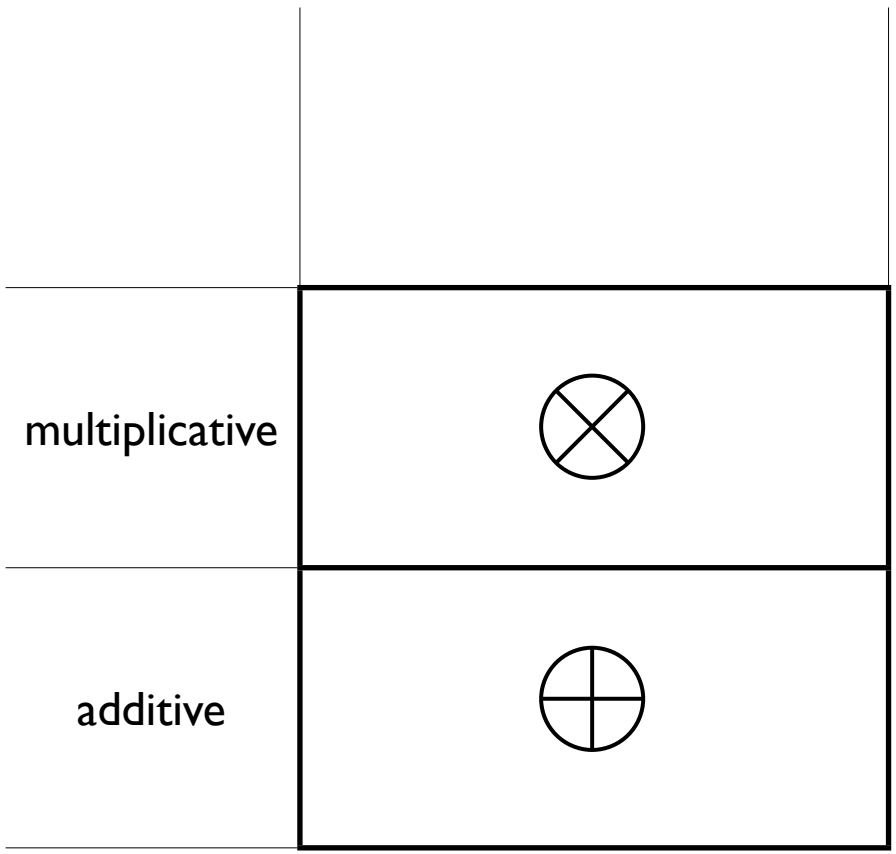
The connectives

Tensor-sum “logic”

$$A^{**} = A$$

$$(A \otimes B)^* = A^* \otimes B^*$$

$$(A \oplus B)^* = A^* \oplus B^*$$



Tensor-Sum Logic

Tensor-sum logic is a Gentzen system, deisgned to capture the structure of a certain free category on some generators $\mathcal{A}$.

- Essentially it is MALL with self-dual connectives
- Every proof has an interpretation as an arrow of $F\mathcal{A}$
- Every arrow of $F\mathcal{A}$ has a corresponding proof
- The system is cut-eliminating, and the cut-elimination procedure is sound wrt the interpretation.

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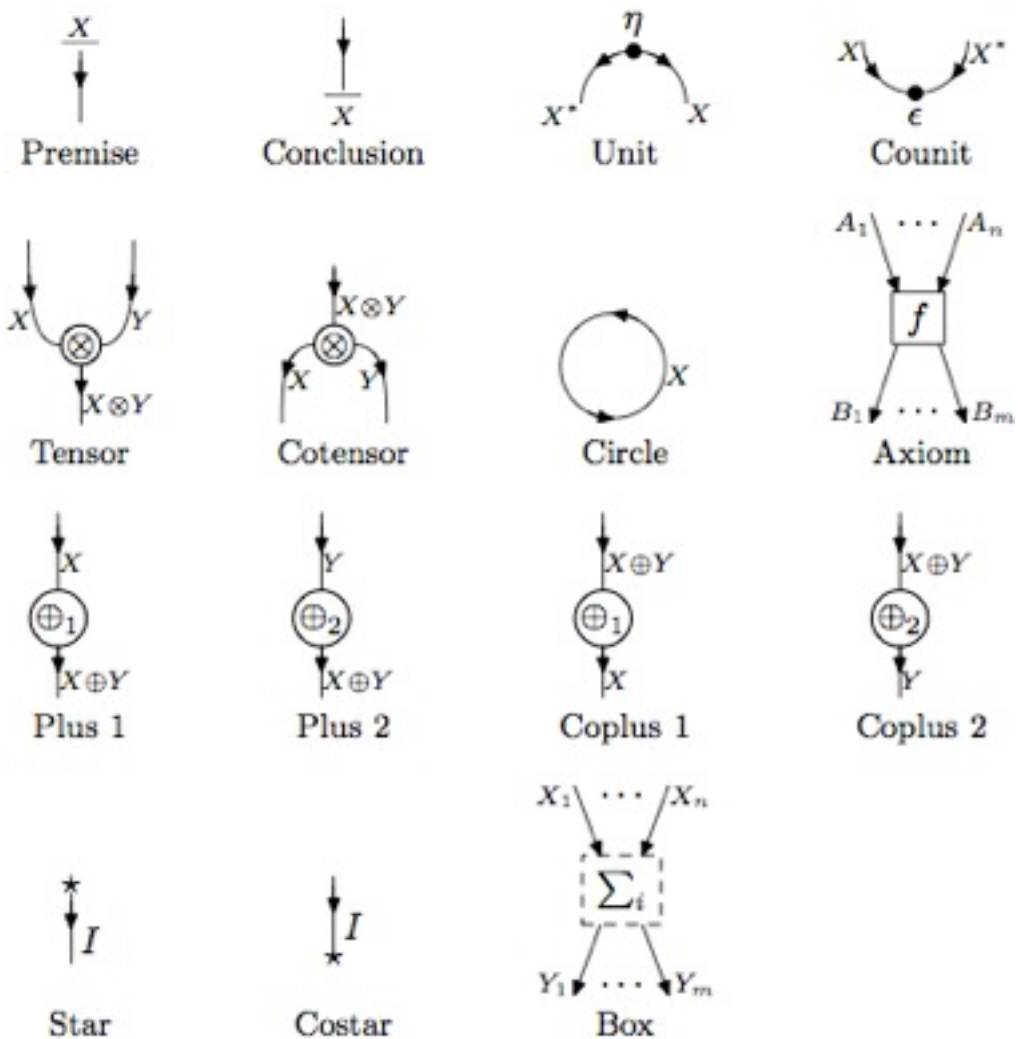
It has some *oddities* as a logical system:

- Every entailment $A \vdash B$ is derivable with a *zero proof*
- Self-duality allows the formation of *self-cuts*
  - the empty sequent is derivable in many *inequivalent ways*

Proof-nets for tensor and sum

We update the system of proof-nets with generalised axioms:



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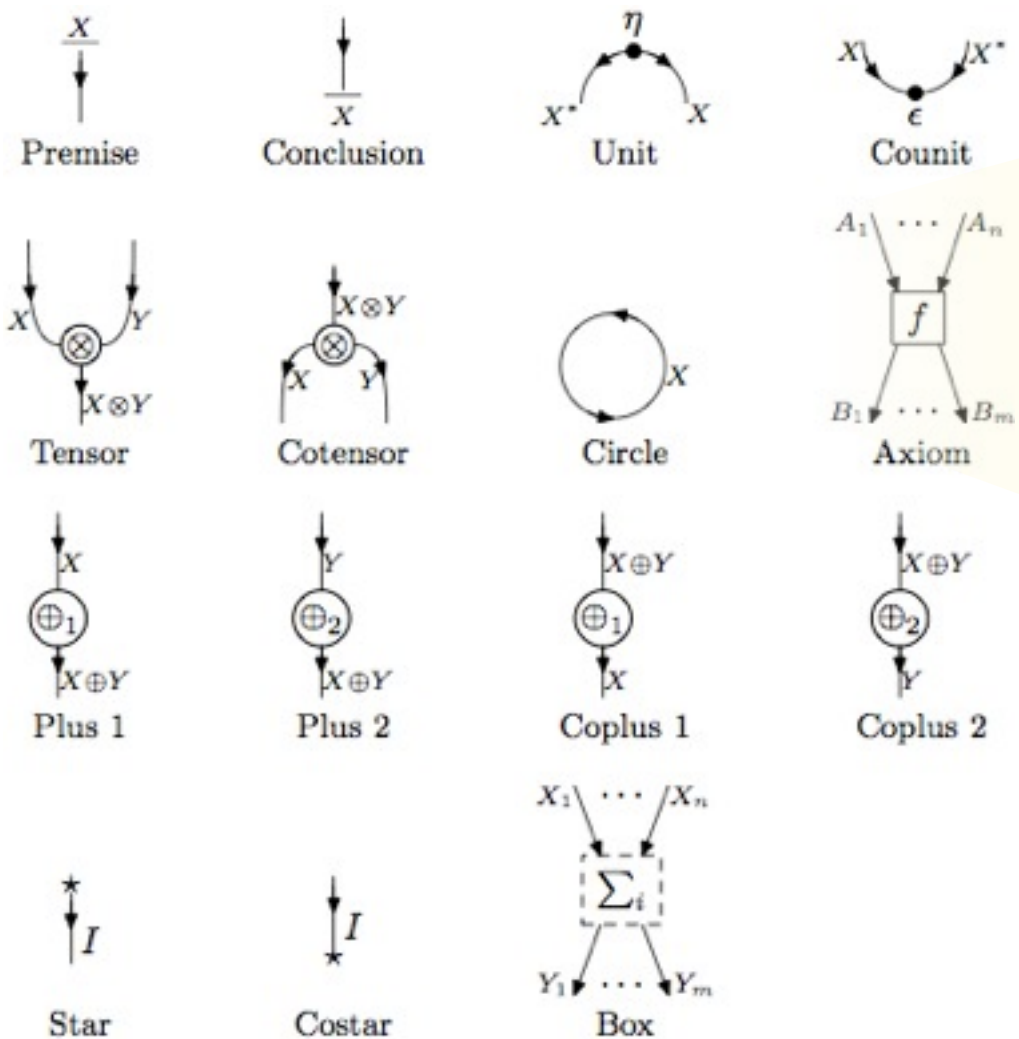
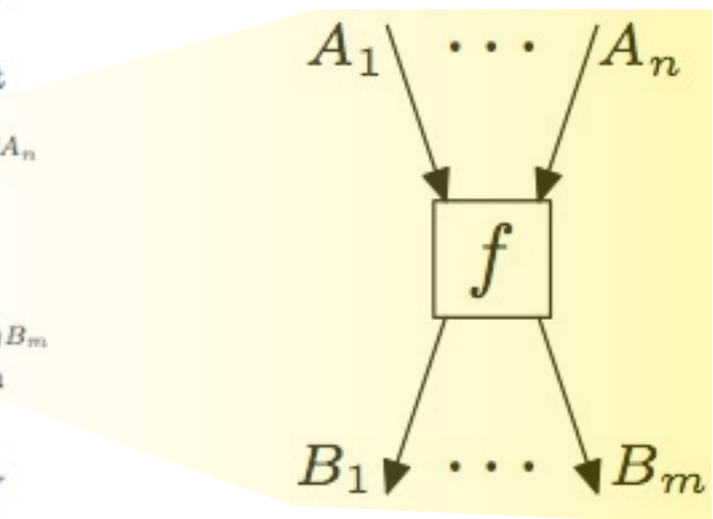




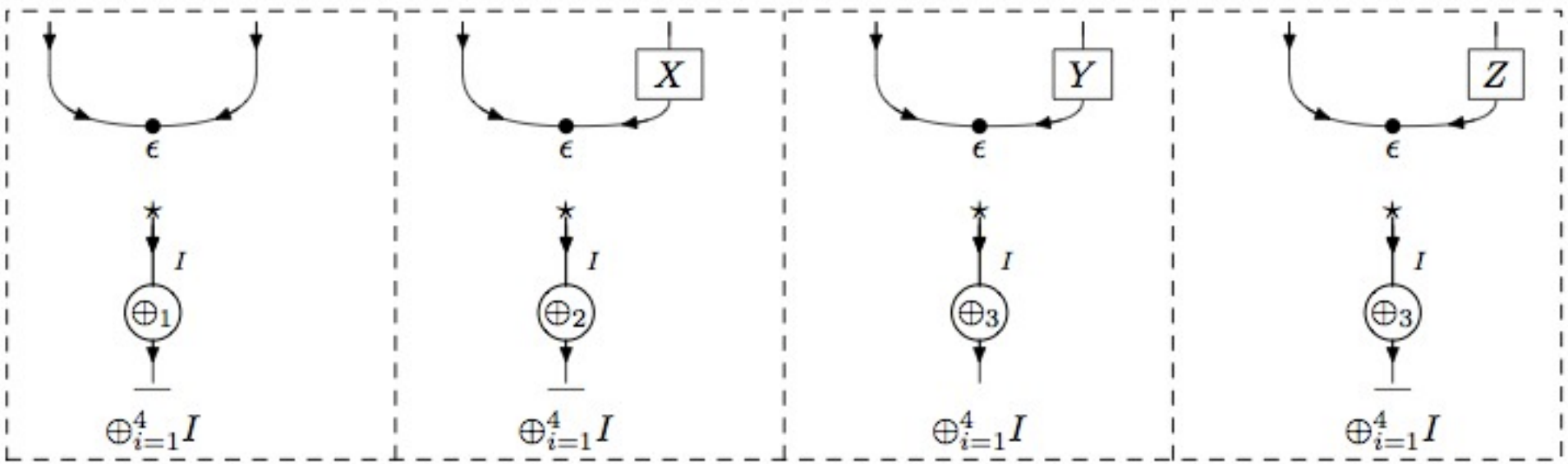
The rewrite rules are unchanged.

Example: teleportation

The shared Bell state and the input qubit:

Example: teleportation

The Bell basis measurement

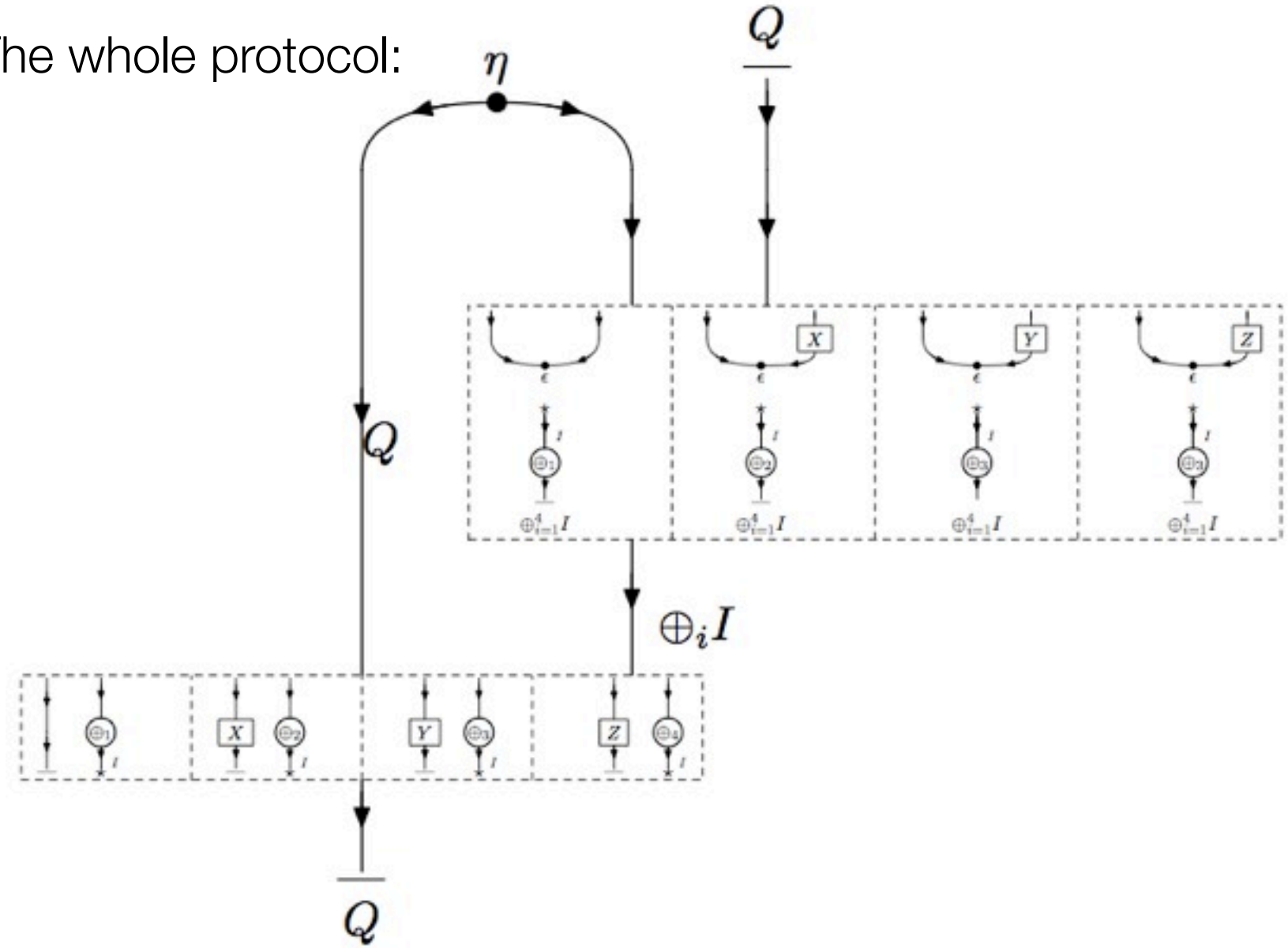


Example: teleportation

The classically controlled corrections:

Example: teleportation

The whole protocol:



Example: teleportation

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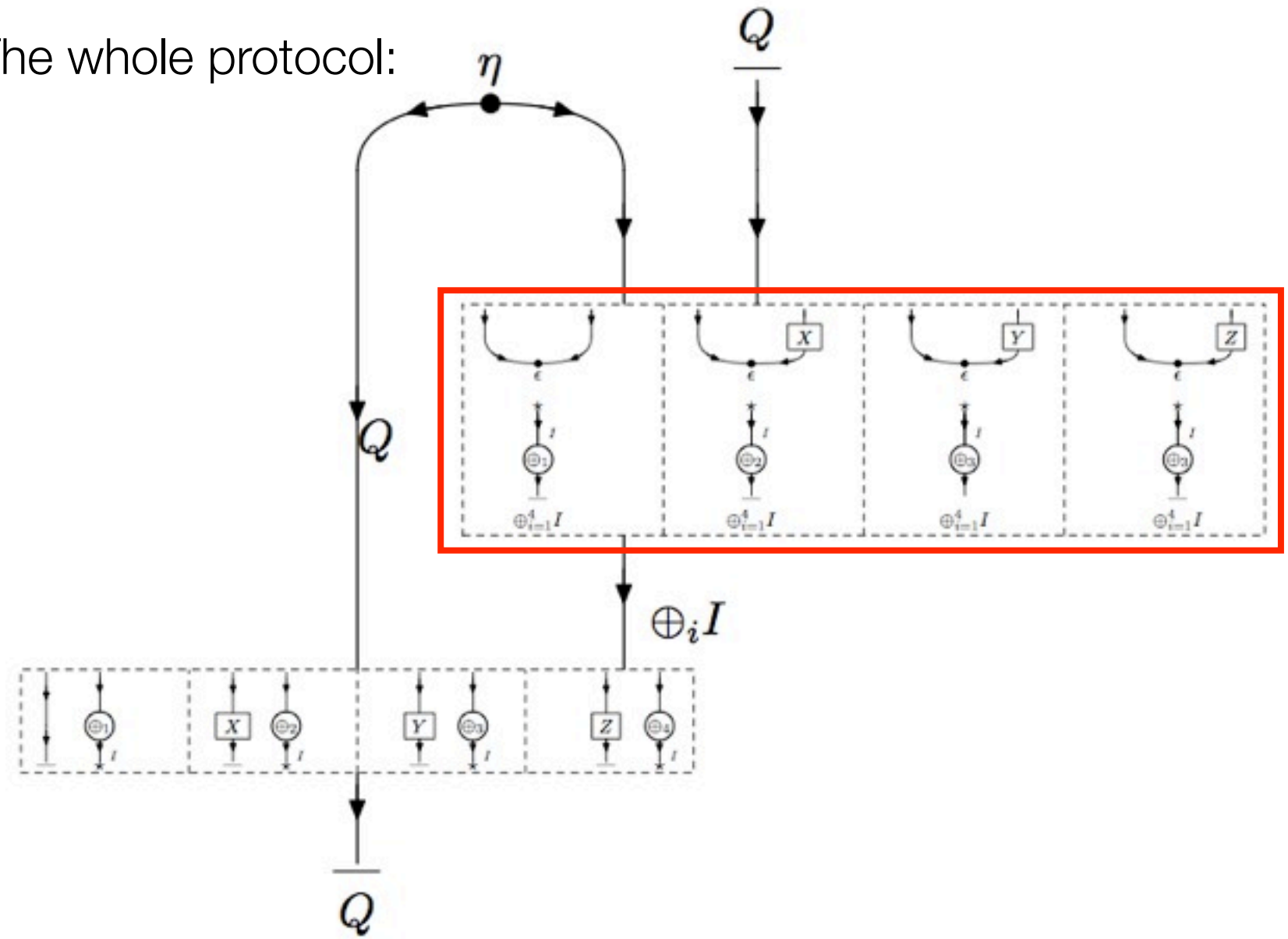

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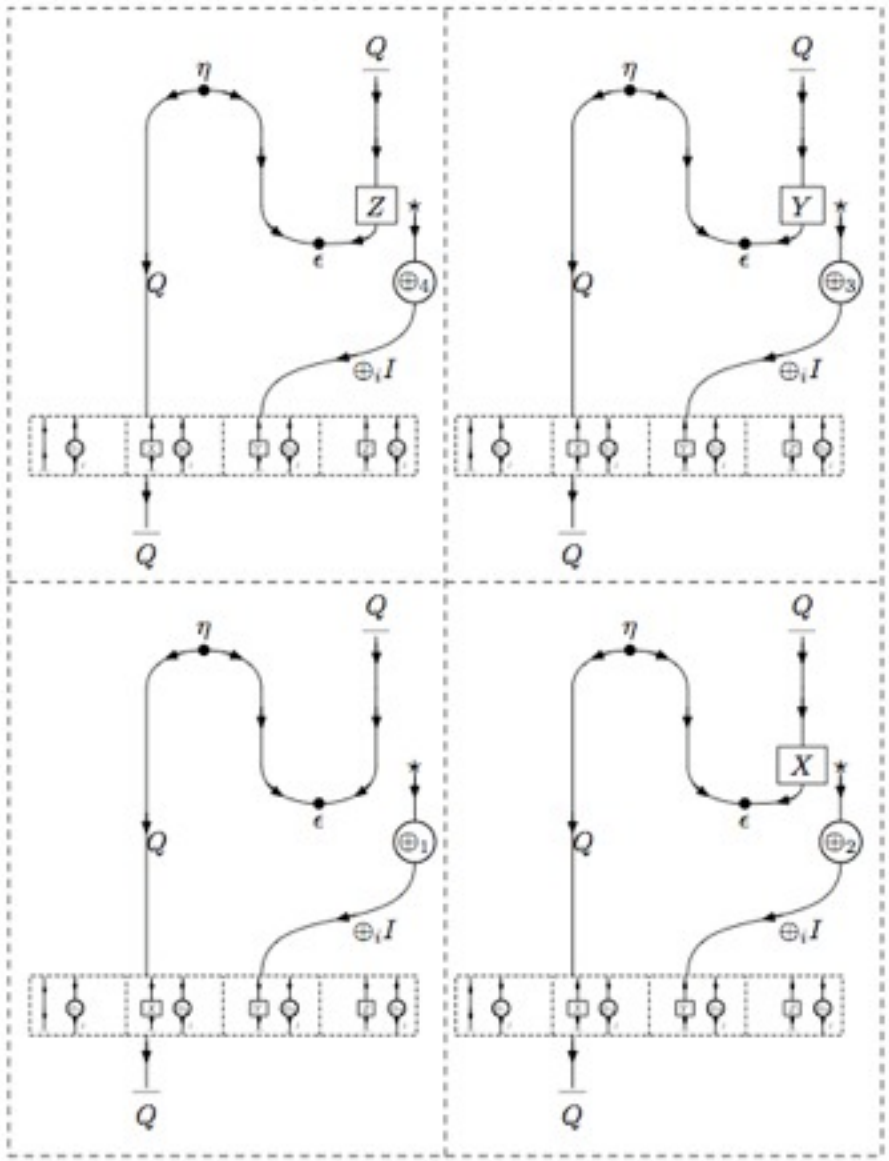

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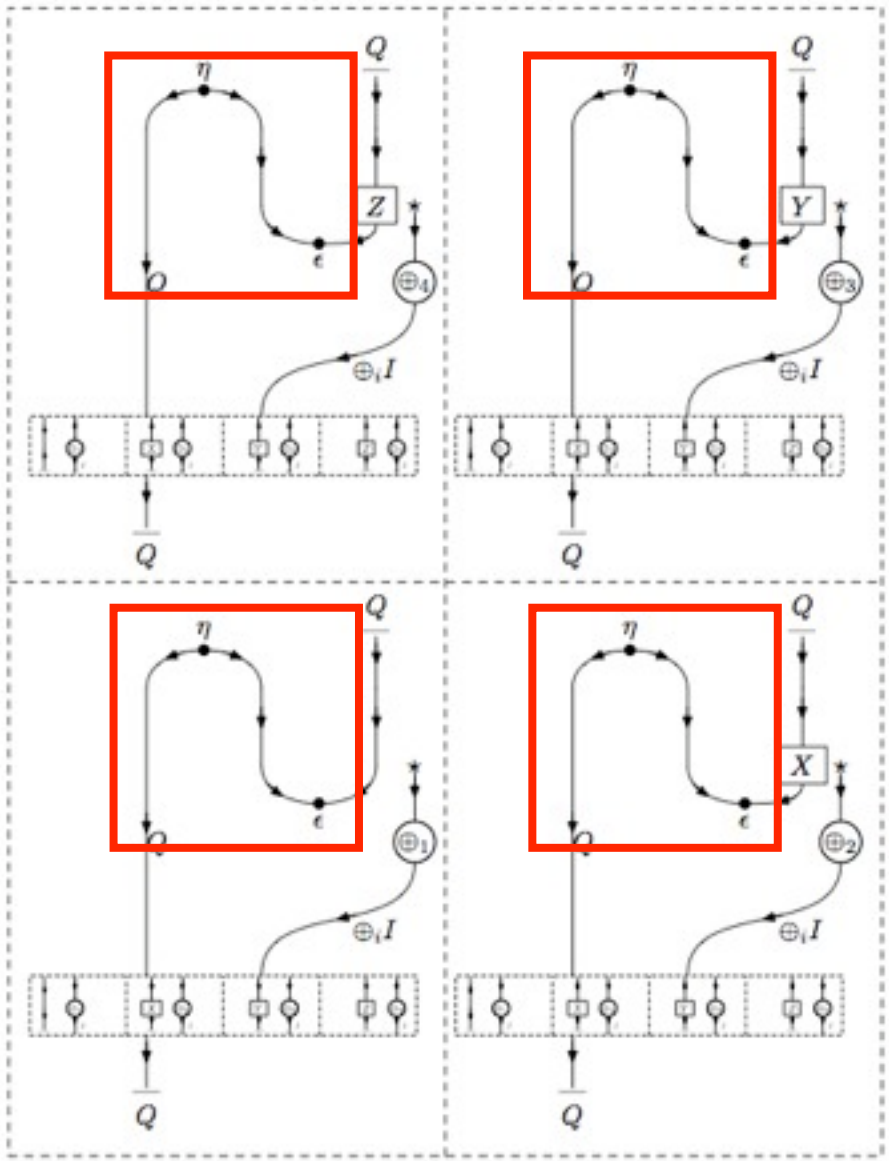

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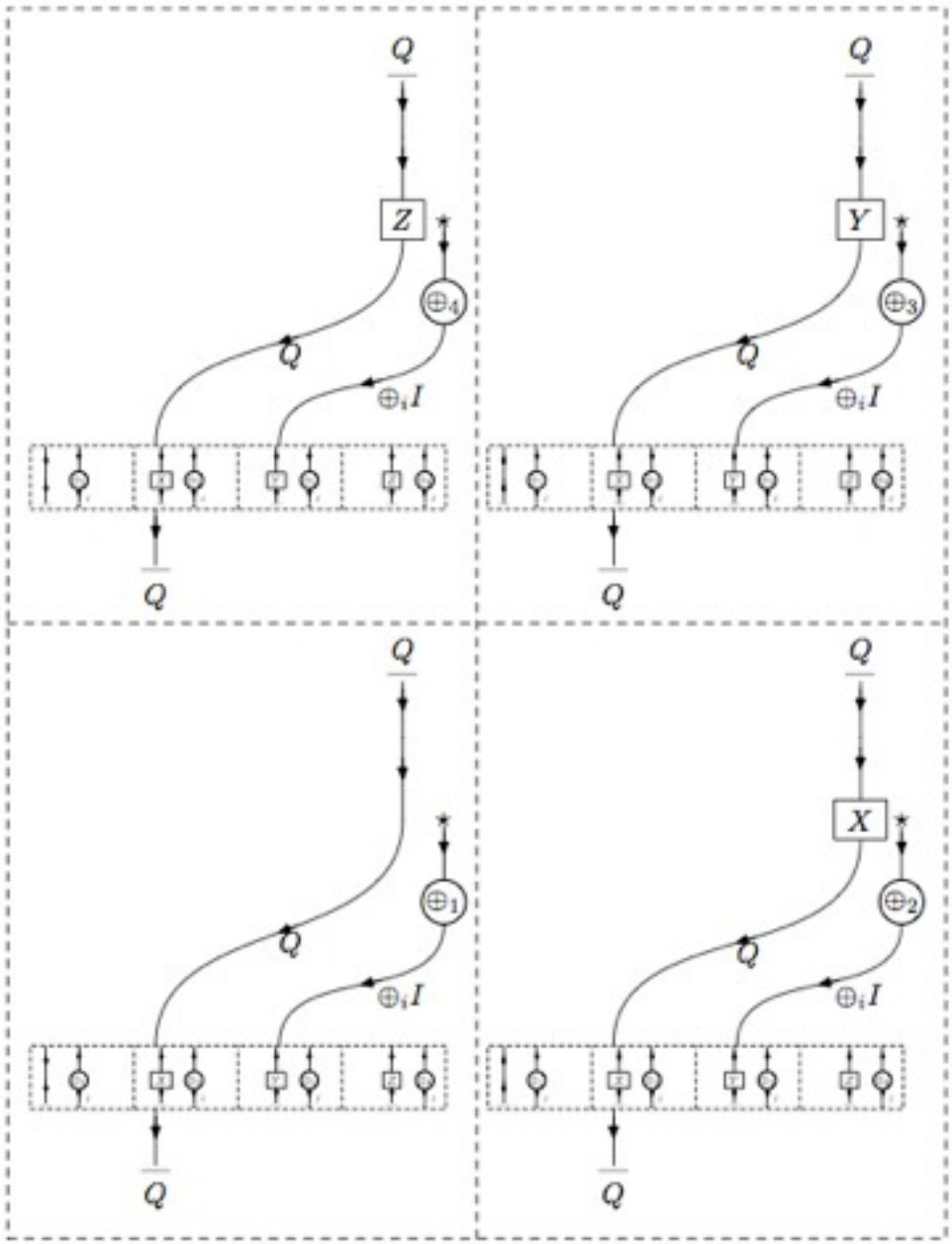

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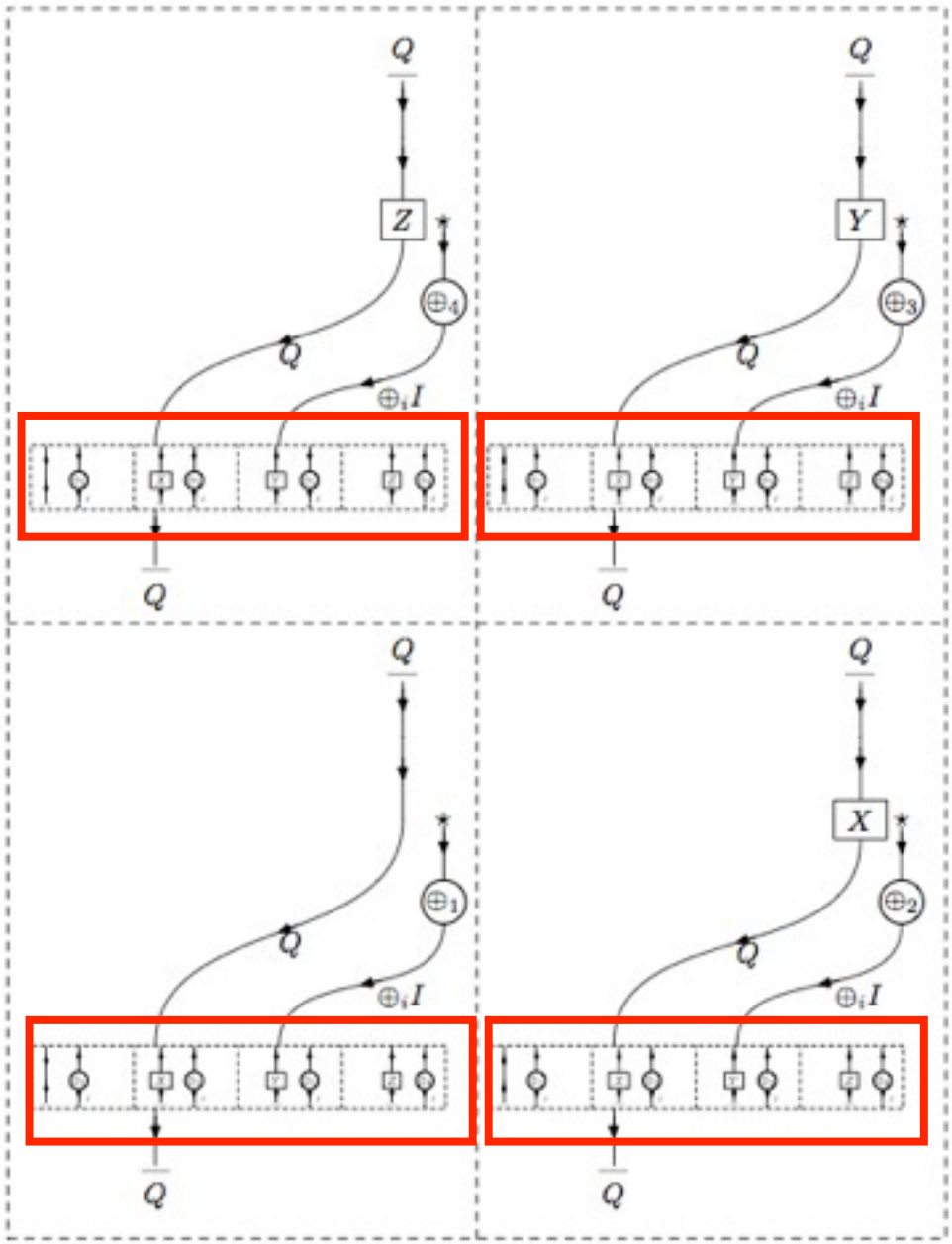

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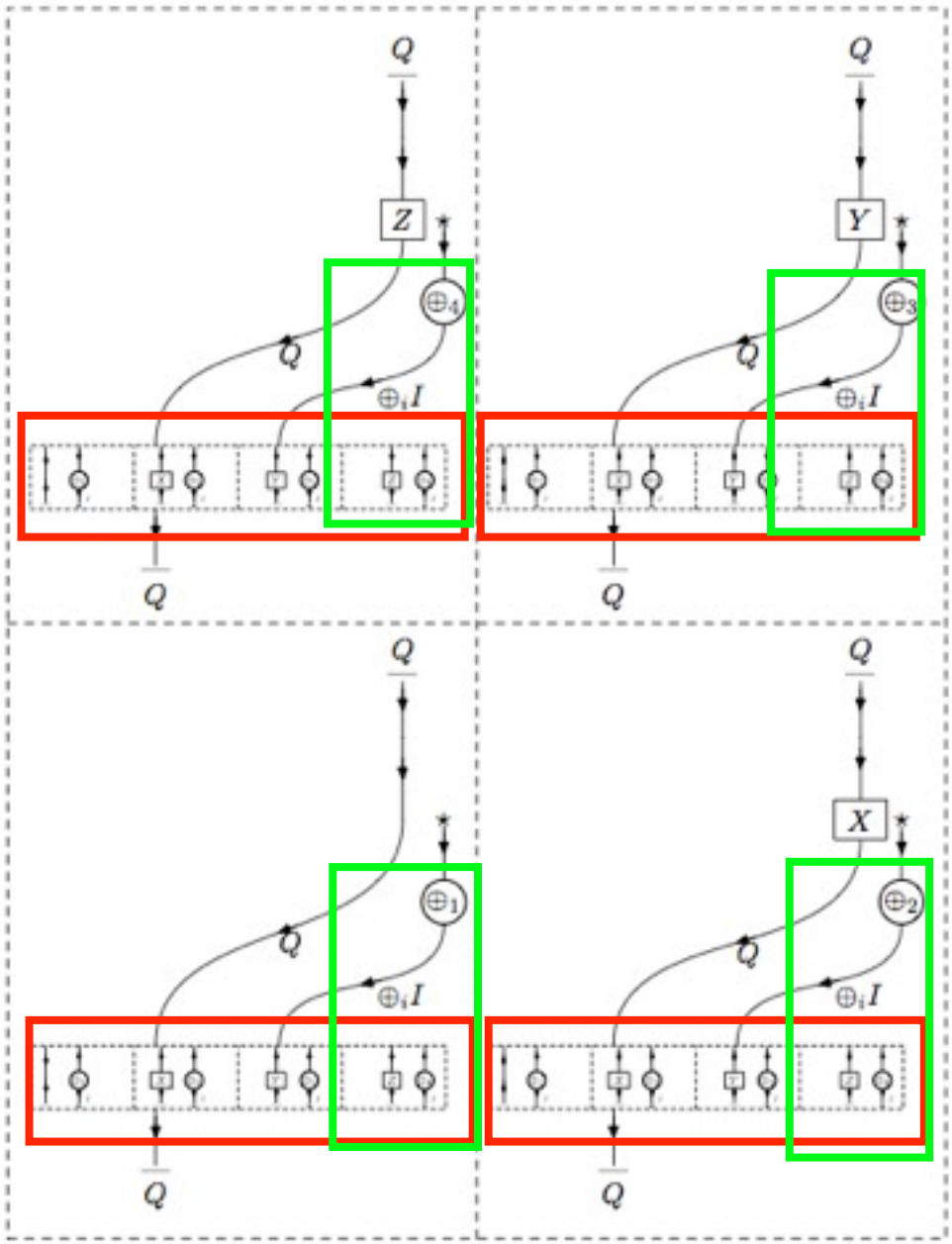

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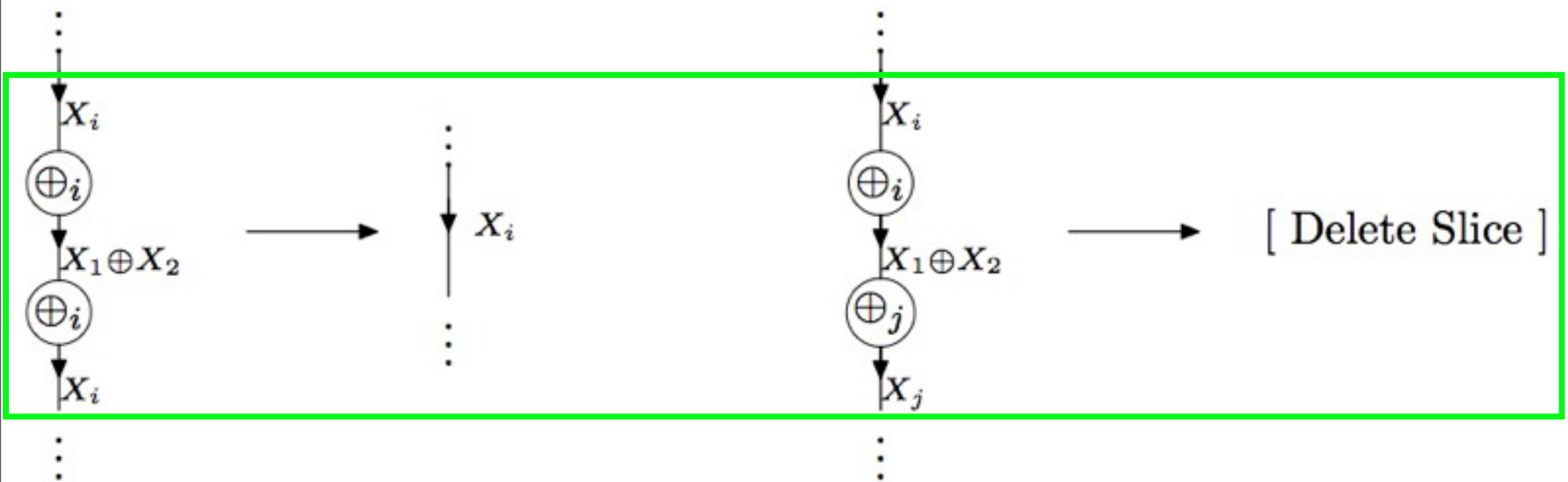

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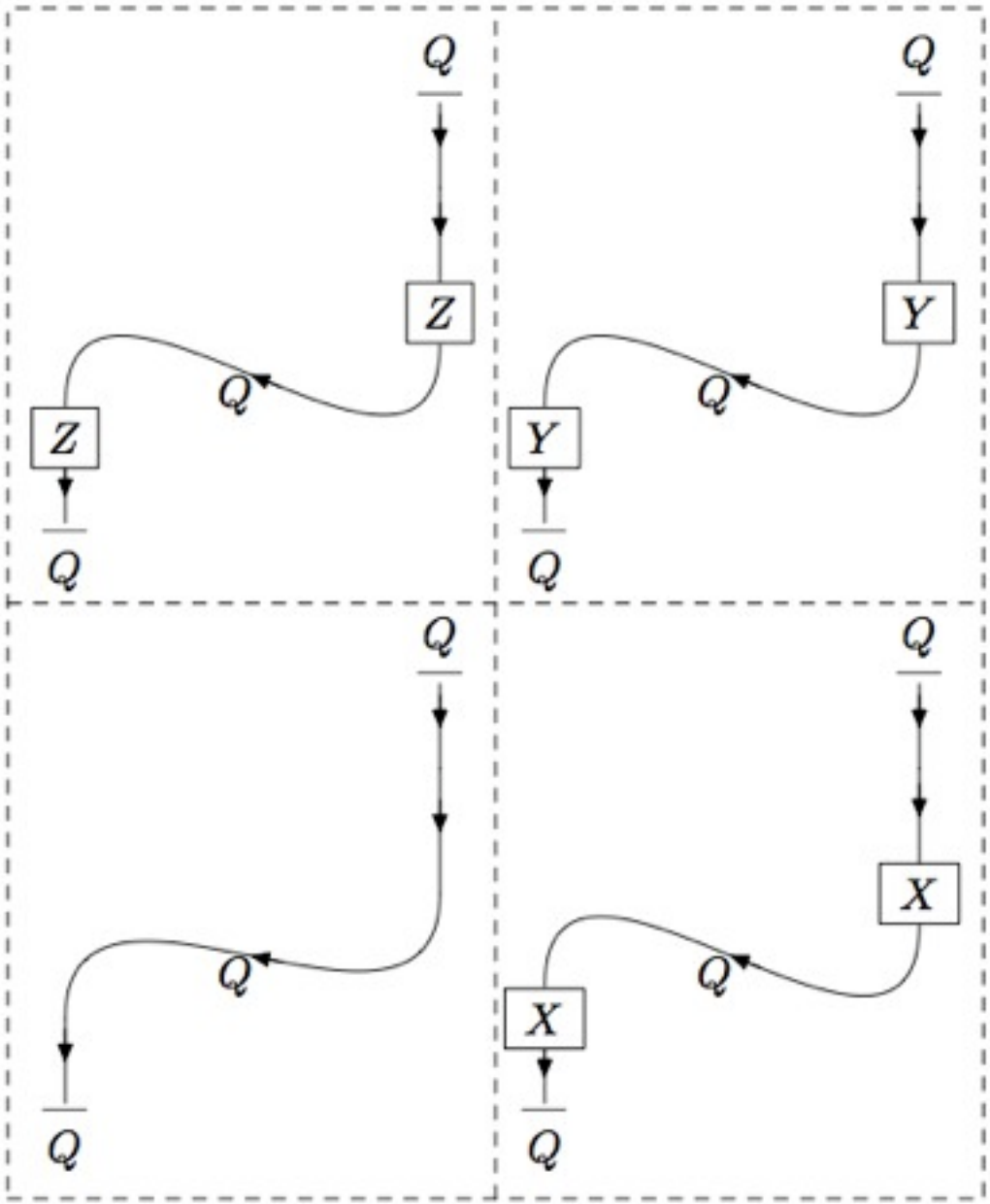

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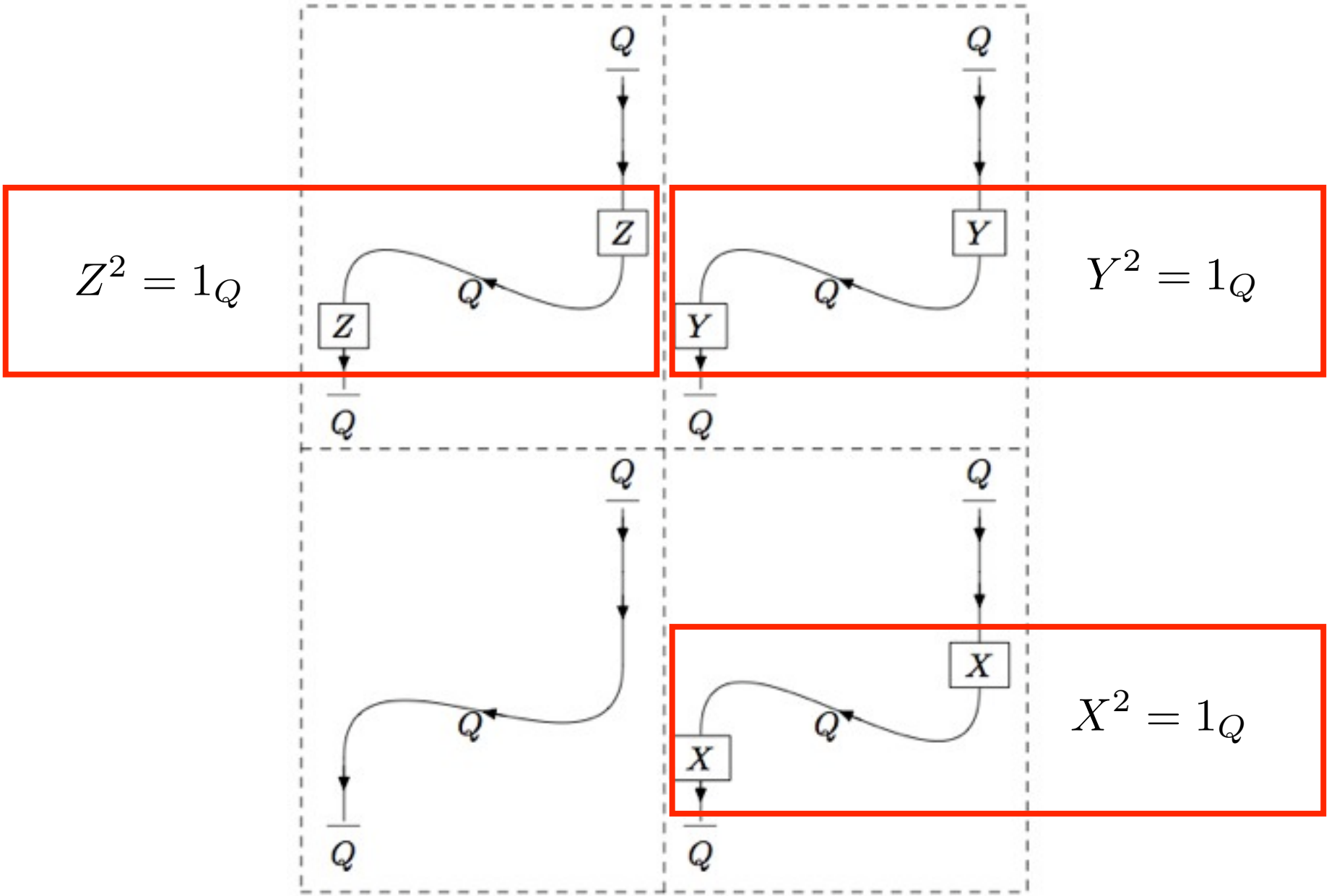

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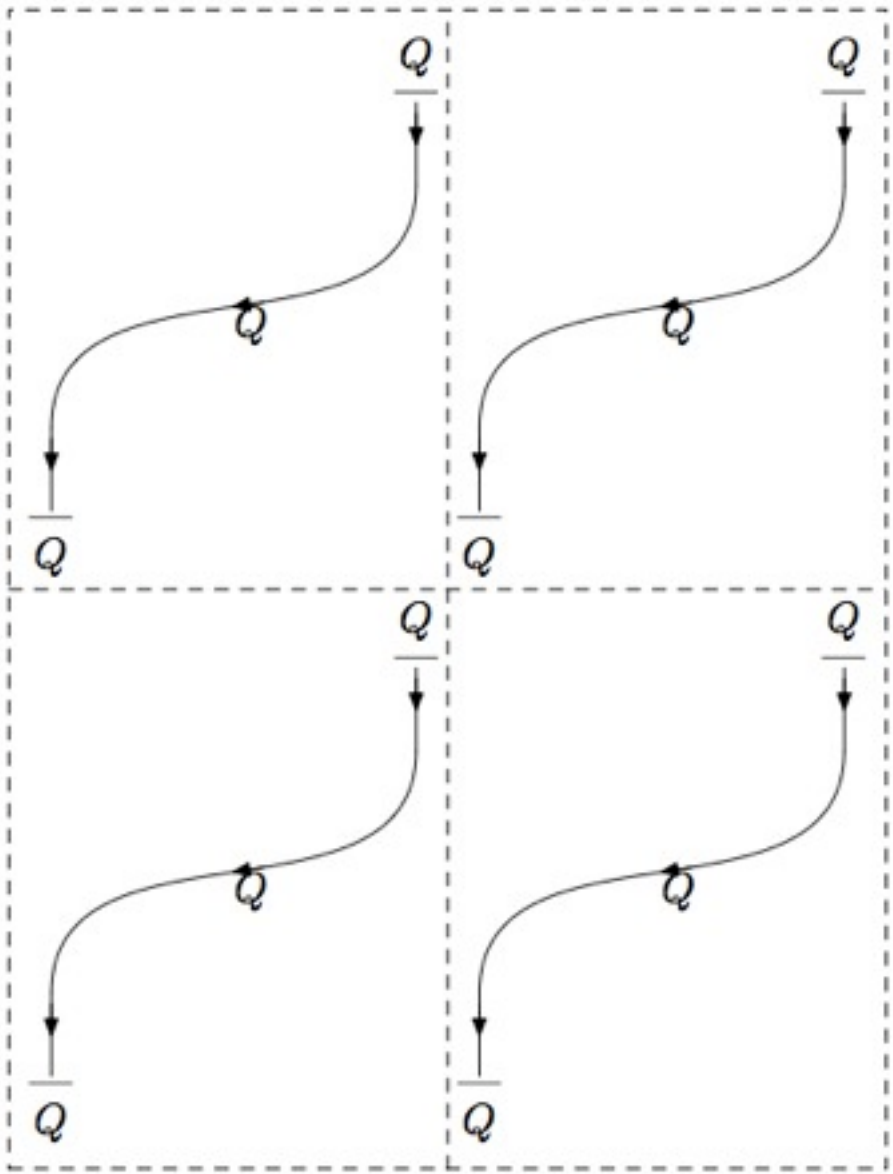


The biproduct

We used the biproduct to encode the branching nature of quantum processes.

- The diagonal map shows the possibility of different choices:

$$Q \xrightarrow{\Delta} Q \oplus Q$$

- But what about the codiagonal?

$$Q \oplus Q \xrightarrow{\nabla} Q$$

- Semantically this corresponds to superposition rather than probabilistic mixing --- the wrong interpretation
- To properly address the issue of probabilities in QM we use Selinger's CPM construction --- but this kills the biproducts.

The biproduct

The biproduct fixes a particular matrix representation for the arrows, that is *a basis.*

- Bases play represent measurements in quantum mechanics - the spectra of self-adjoint operators
- In order to fully handle quantum measurements we must deal with different, incompatible measurements at the same time.
- Bases cannot be represented by natural transformations - unlikely any “categorical logic” can help here

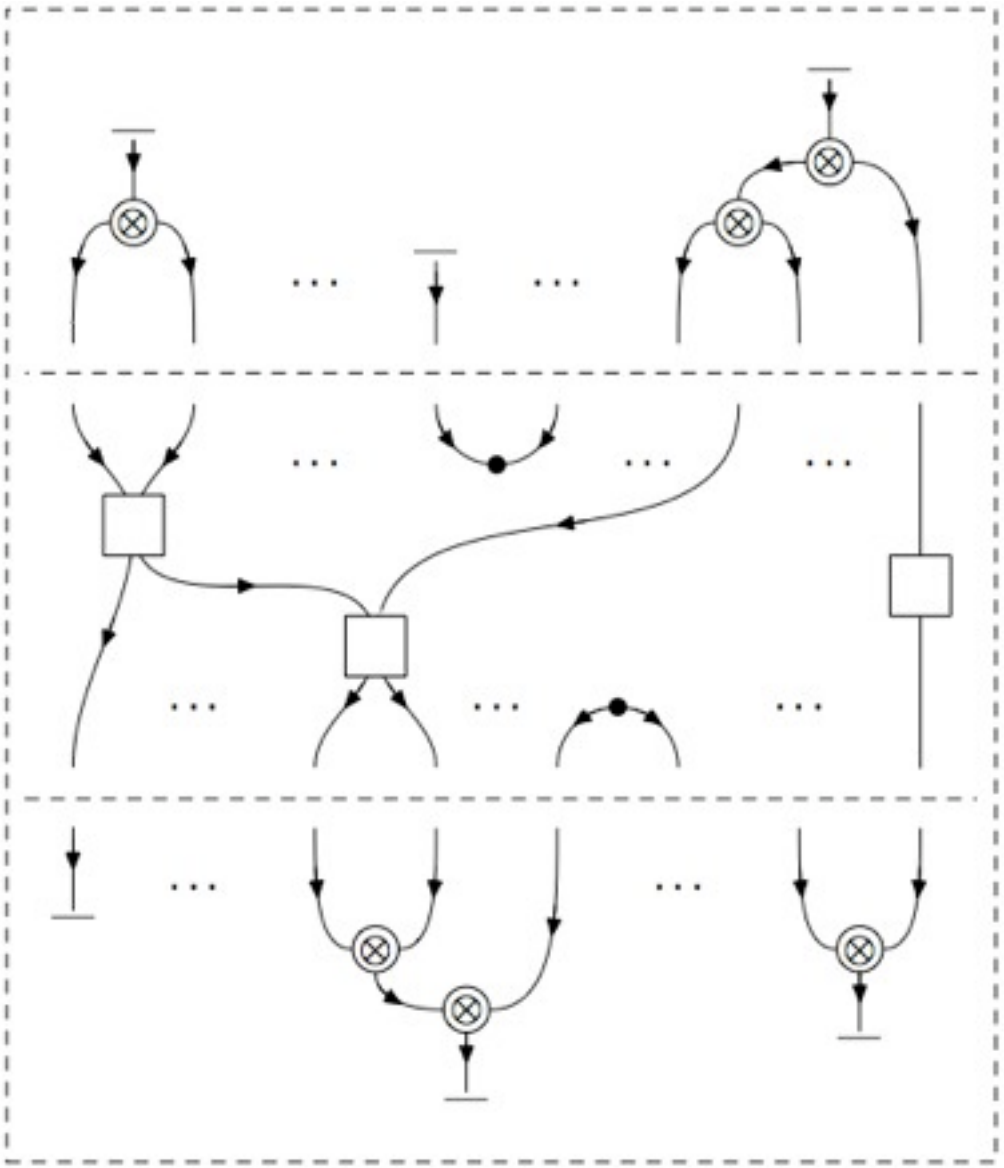
The biproduct does not offer a good formalisation of this. We will drop the biproduct and find a more useful way to formalise quantum computation.

Normal form theorem

Pure logic, determined by premise

A unique A-labelled circuit

Pure logic determined by conclusion



Types for entanglement?

Can we regain the the separation between $\otimes$ and $\parr$ to talk about entanglement?

- Entangled states do not form a subspace
- Do double gluing on fdHilb
  - $\otimes$ gives product states
  - $\parr$ gives **all** states
- Hence $\otimes$ is a subtype of $\parr$

How many types anyway?

Defn: A state S is said to be *SLOCC reachable* from state S' if there is a sequence of stochastic local operations and classical communications producing S from S'

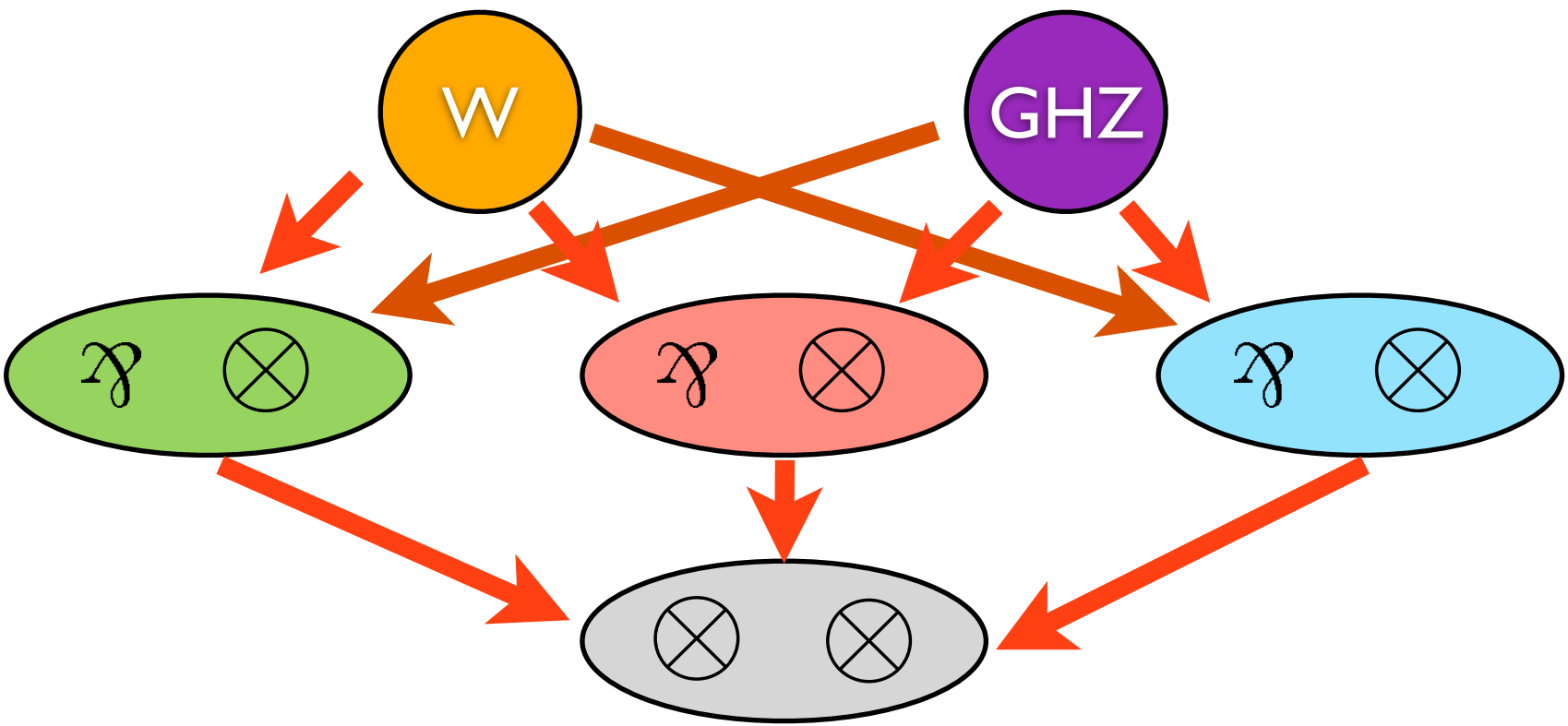
Defn: If S and S' are mutually SLOCC reachable then they are *SLOCC equivalent*.

How many types anyway?

Prop: For 2-qubit states there are 2 SLOCC classes:

How many types anyway?

Prop: For 3-qubit states there are 6 SLOCC classes:



How many types anyway?

Prop: For 4-qubit states there are **uncountably many** SLOCC classes

How many types anyway?

Prop: For 4-qubit states there are **uncountably many** SLOCC classes

Forget about types to describe entanglement

Just look at the terms

Overview

Hilbert Space QM

Categorical QM

Overview

Hilbert Space QM

Hilbert space

Categorical QM

Overview

Hilbert Space QM

Hilbert space

Categorical QM

†-symmetric
monoidal category

Overview

Hilbert Space QM

Hilbert space

Observables

Categorical QM

†-symmetric
monoidal category

Overview

Hilbert Space QM	Categorical QM
Hilbert space	†-symmetric monoidal category
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Complementarity		

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Qubits	ZX-calculus

Monoidal Categories & Diagrams

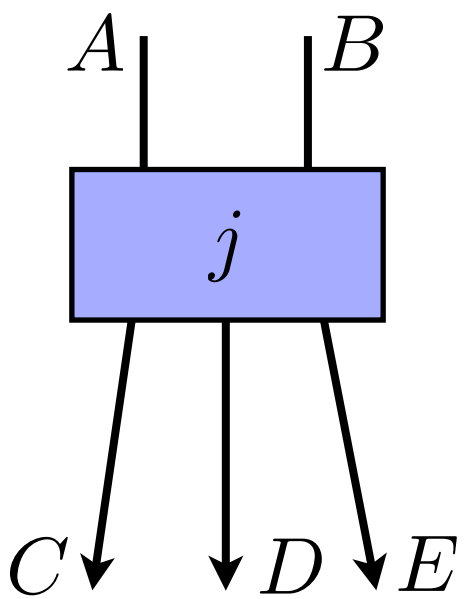
PAST / HEAVEN

FUTURE / HELL

Pessimistic Diagram Convention

Diagrams

a.k.a. Monoidal categories

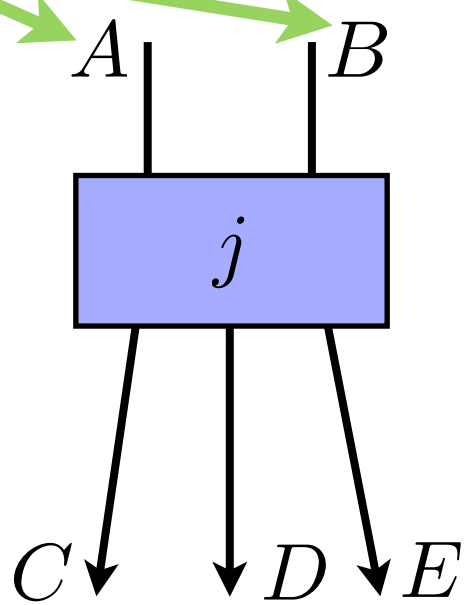


$$j : A \otimes B \to C \otimes D \otimes E$$

Diagrams

a.k.a. Monoidal categories

Input Systems



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Diagrams

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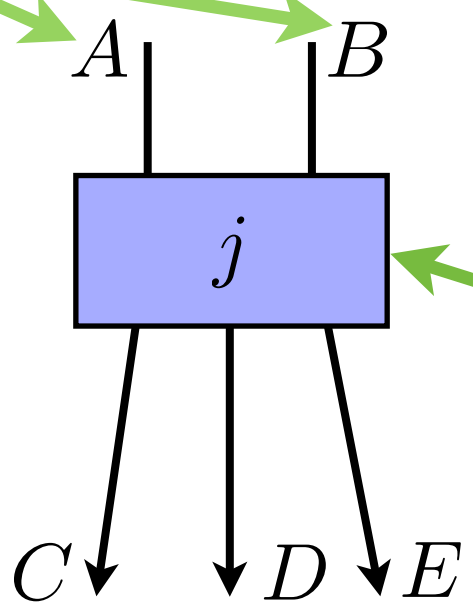

Interaction, or process, or state change, or ...

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Diagrams

a.k.a. Monoidal categories

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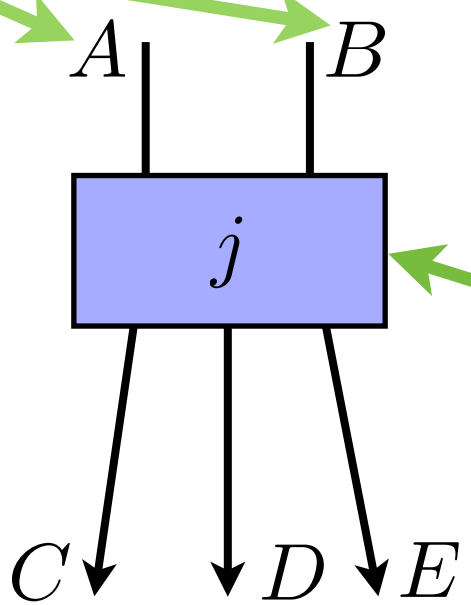

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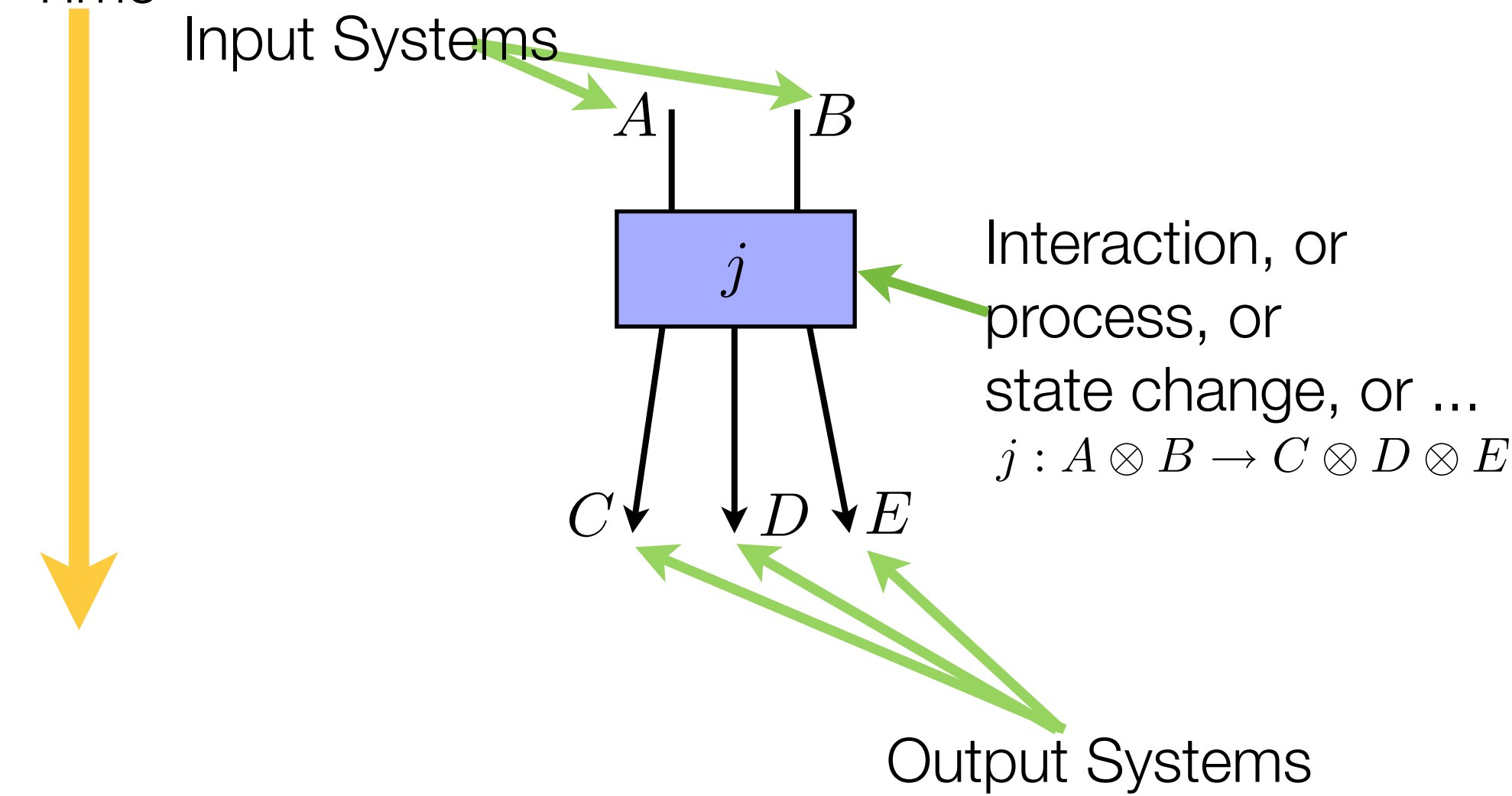

A hierarchy of categories

Categories

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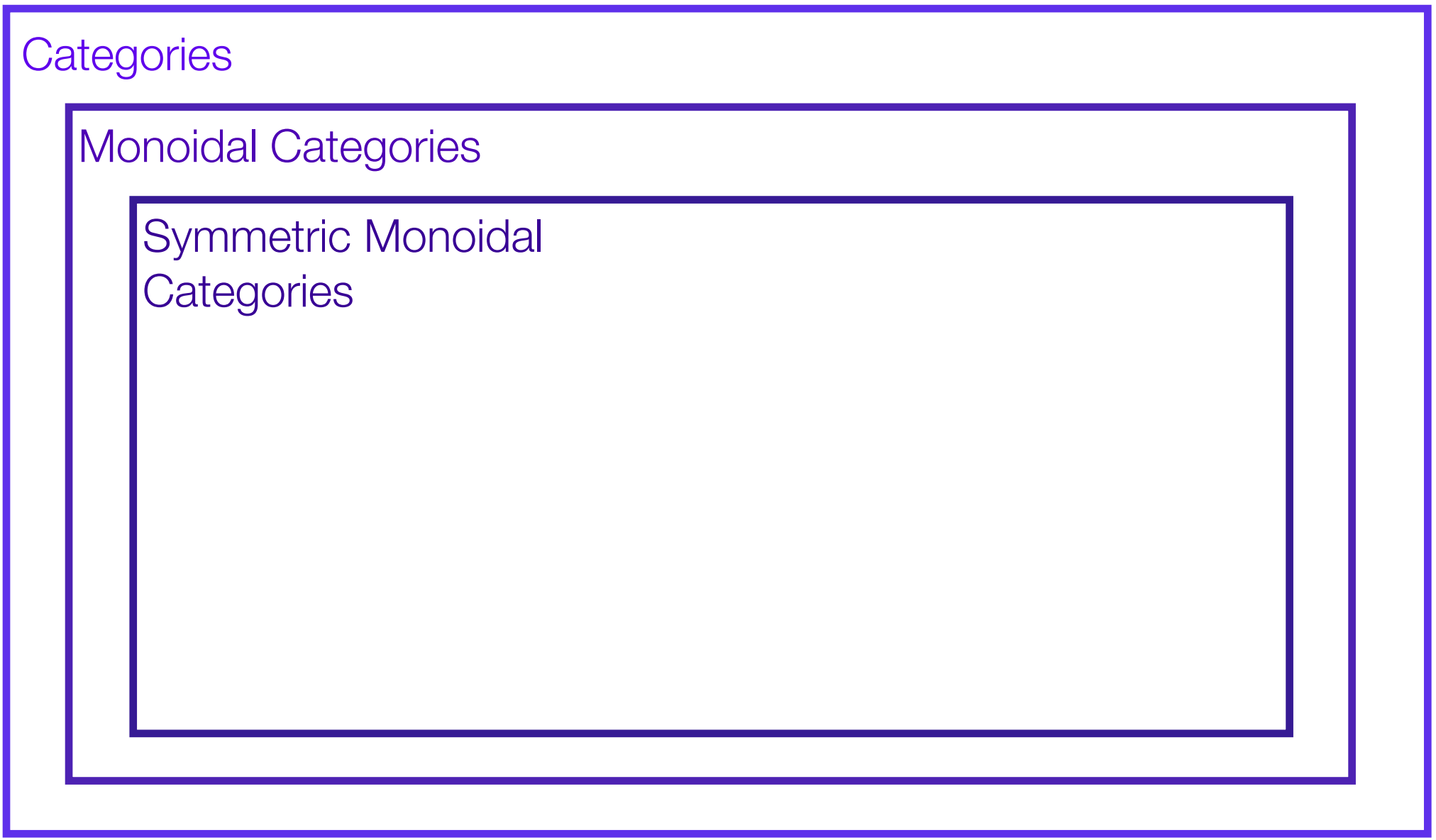

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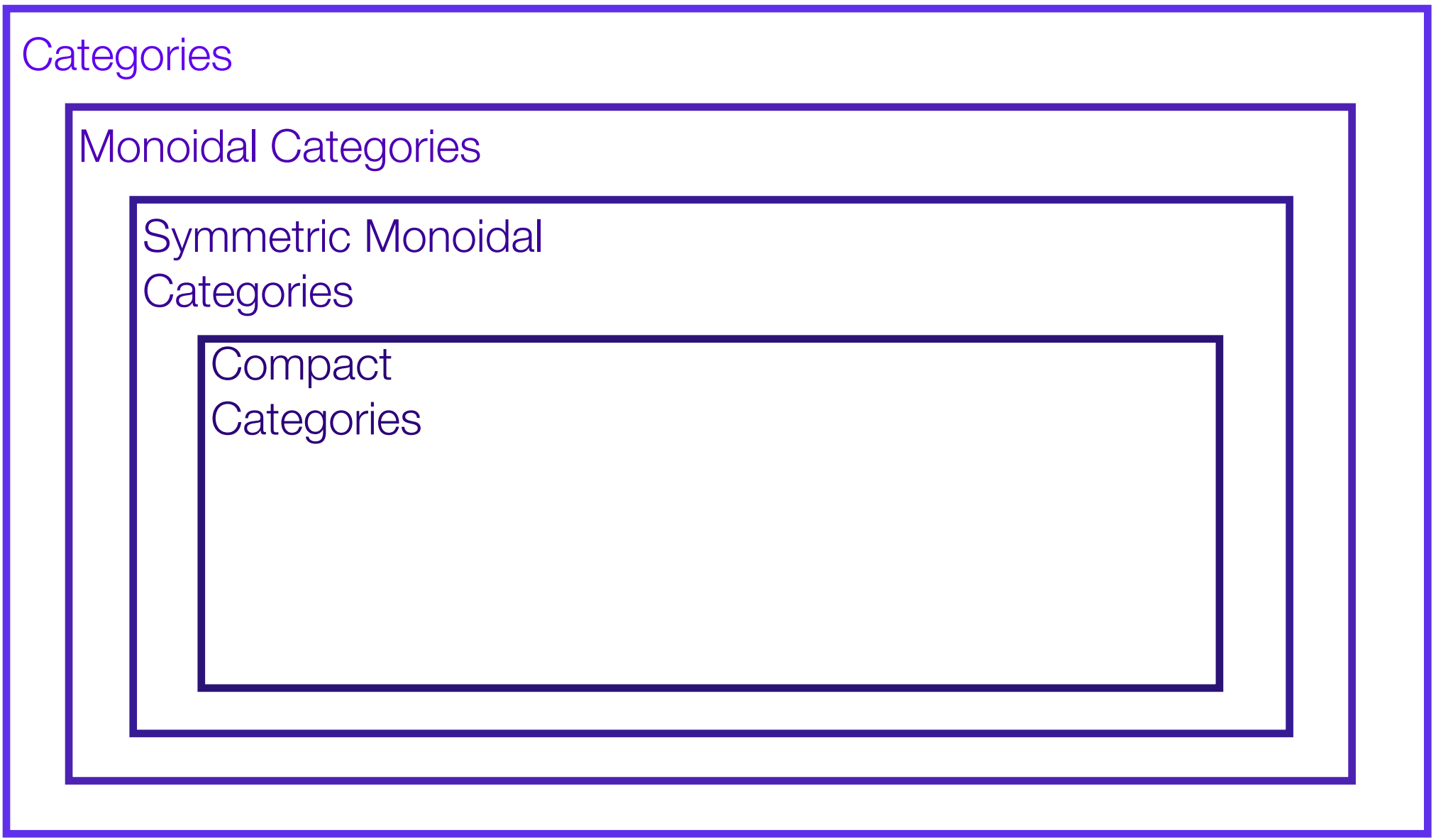

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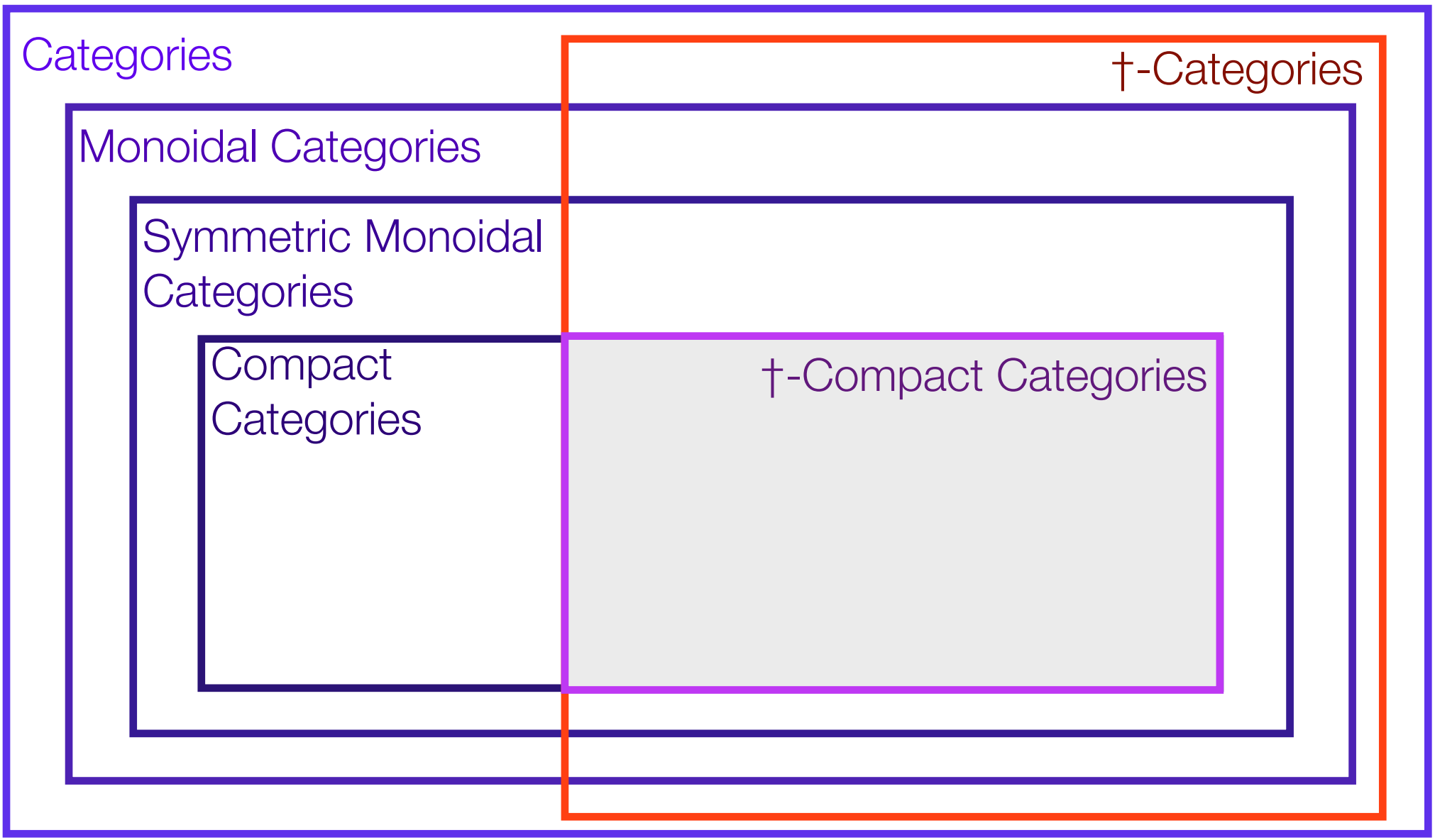

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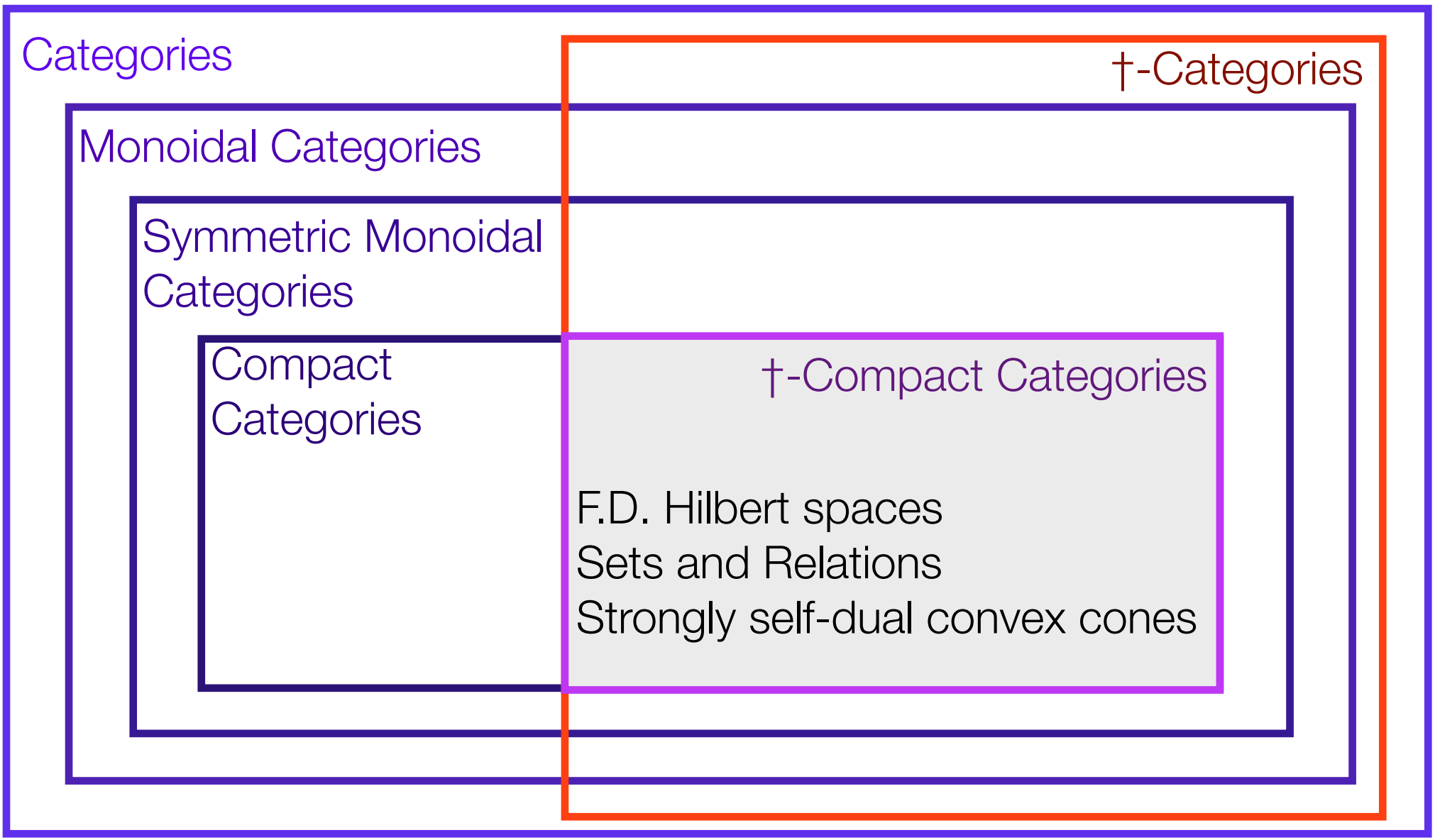
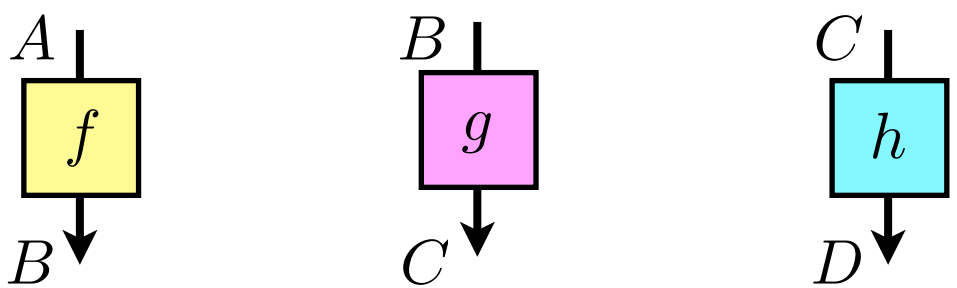


Categories

A *category* consists of *objects* A, B, C, etc, and *arrows* between them:

$$f : A \to B \qquad g : B \to C \qquad h : C \to D$$



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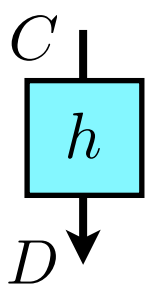
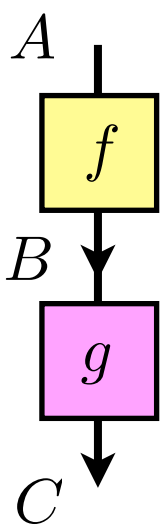


$g \circ f : A \to C$



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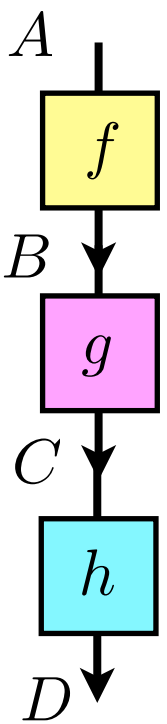

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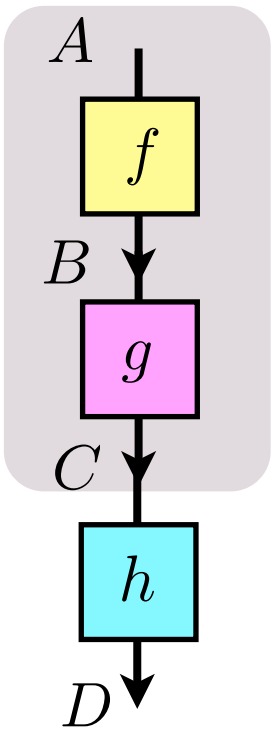

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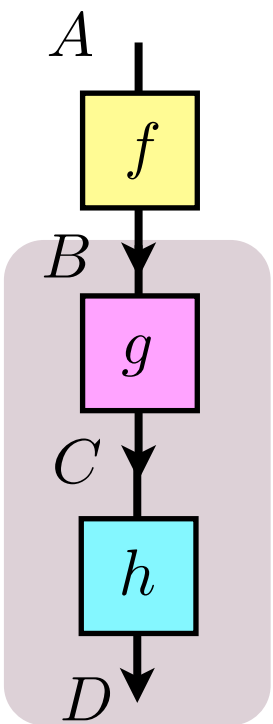

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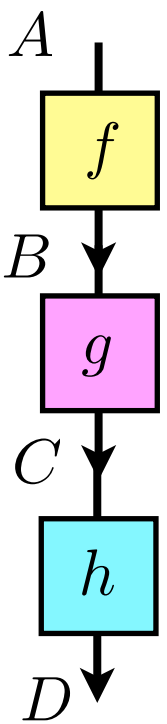
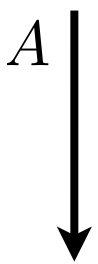


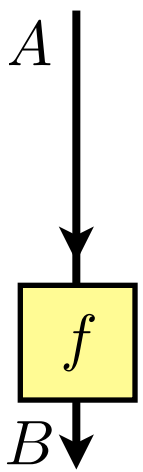
Categories

$$\mathrm{id}_A : A \to A$$



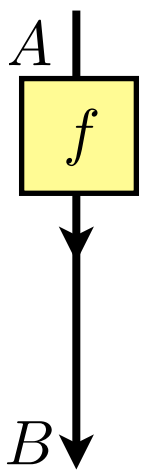
Categories

$$f \circ \mathrm{id}_A : A \to B$$



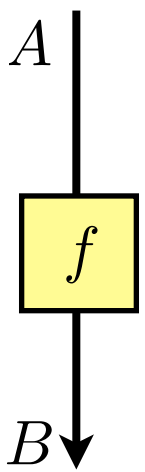
Categories

$$\mathrm{id}_B \circ f : A \to B$$



Categories

$$f : A \to B$$



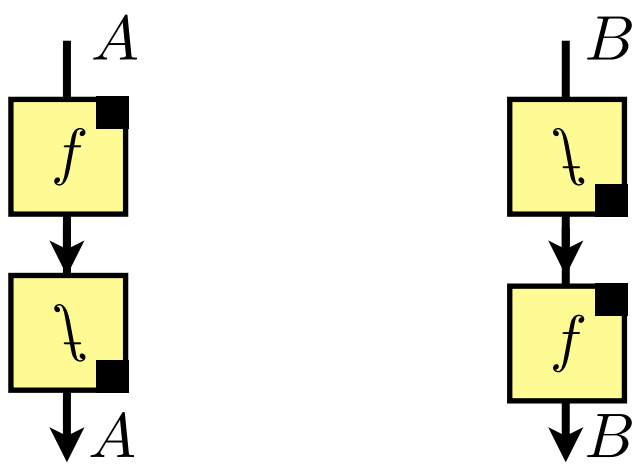
†-Categories

A category is a *†-category* if it is equipped with an involutive functor, $(\cdot)^\dagger$ which reverses the arrows while leaving the objects unchanged.

$$f : A \to B \qquad f^\dagger : B \to A$$

†-Categories

An arrow $f : A \to B$ is called *unitary* when:



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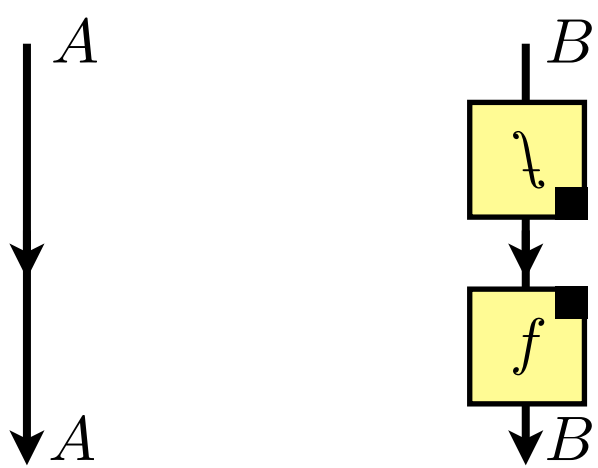

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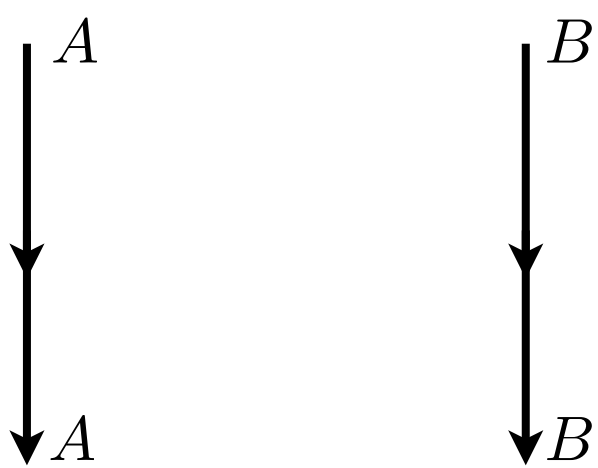
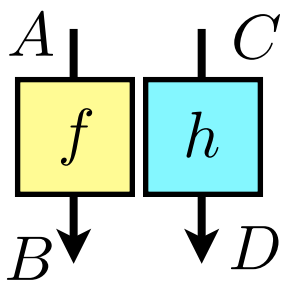


Monoidal Categories

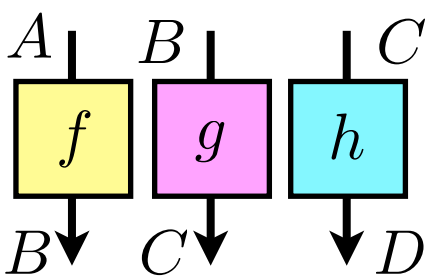
A *strict monoidal category* is a category equipped with a tensor product on both objects and arrows:

$$f \otimes h : A \otimes C \to B \otimes D$$



The tensor is associative:

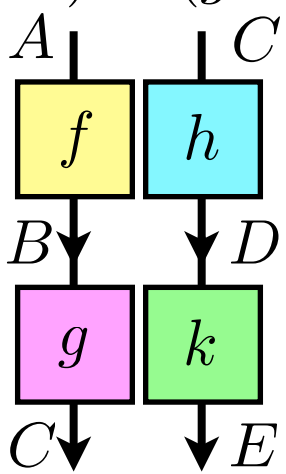
$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$



Monoidal Categories

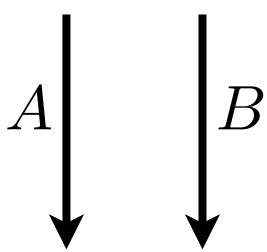
The tensor product is *bifunctorial*, meaning that it preserves composition:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$



and identities:

$$\mathrm{id}_{A \otimes B} = \mathrm{id}_A \otimes \mathrm{id}_B$$



Monoidal Categories

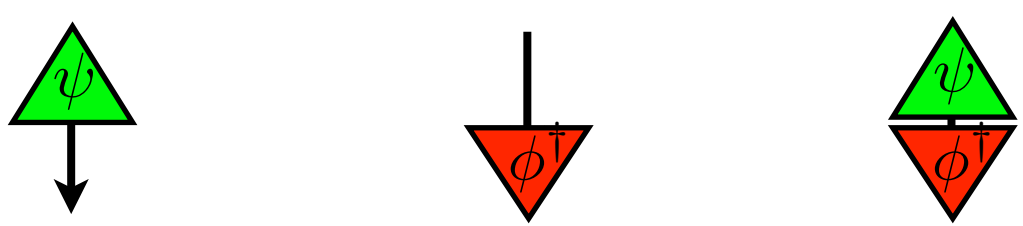
Monoidal categories have a special *unit* object called I which is a left and right identity for the tensor:

$$I \otimes A = A = A \otimes I$$

$$\mathrm{id}_I \otimes f = f = f \otimes \mathrm{id}_I$$

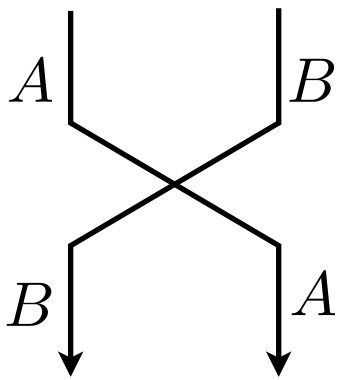
No lines are drawn for I in the graphical notation:

$\psi : I \to A$ $\qquad$ $\phi^\dagger : A \to I$ $\qquad$ $\phi^\dagger \circ \psi : I \to I$



Symmetric Monoidal Categories

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$



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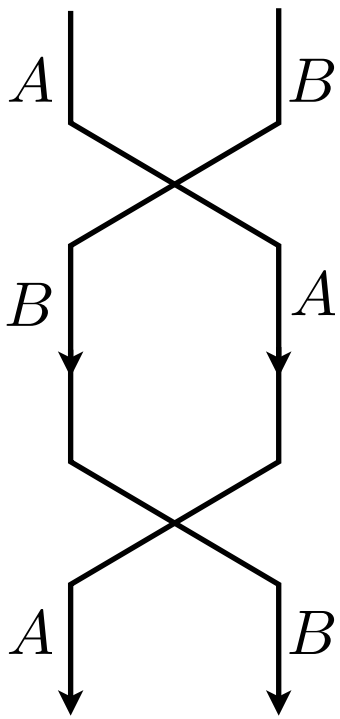

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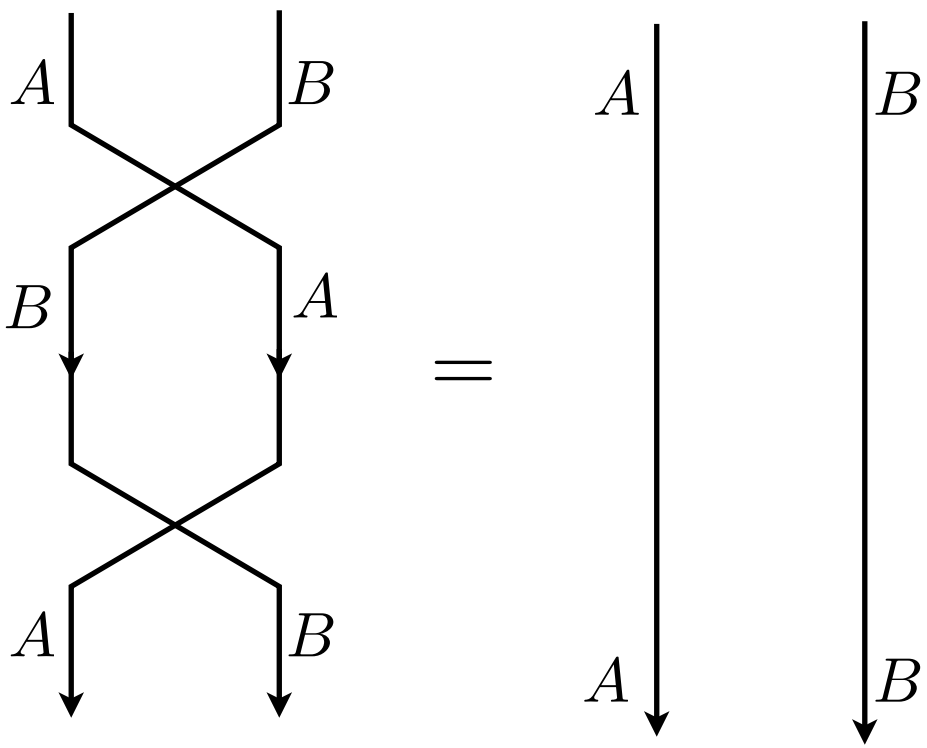

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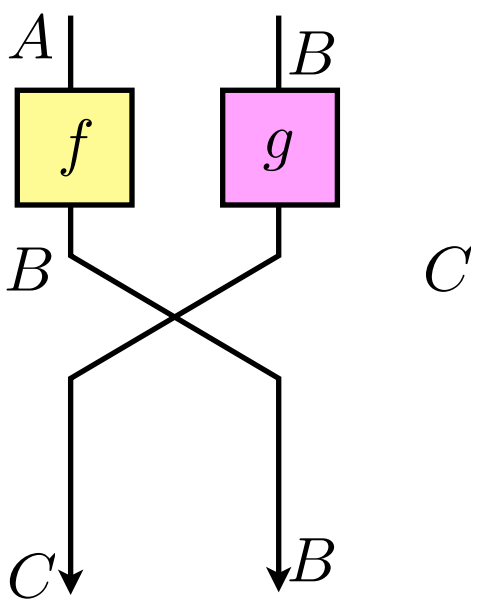

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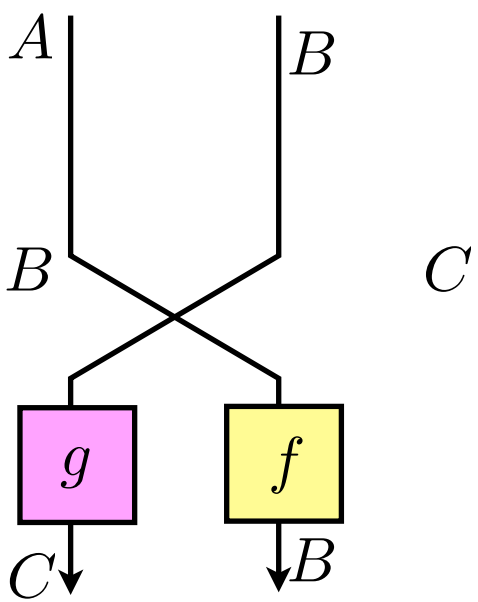
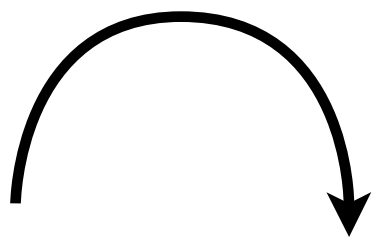
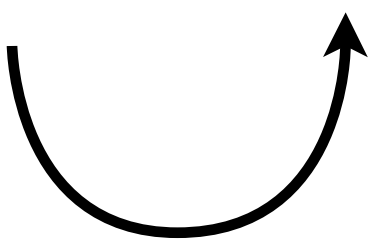


Compact Closure

A symmetric monoidal category is called *compact* if, for every object A, there exists a *dual* object $A*$ and arrows:

$$d : I \to A^* \otimes A \qquad e : A \otimes A^* \to I$$





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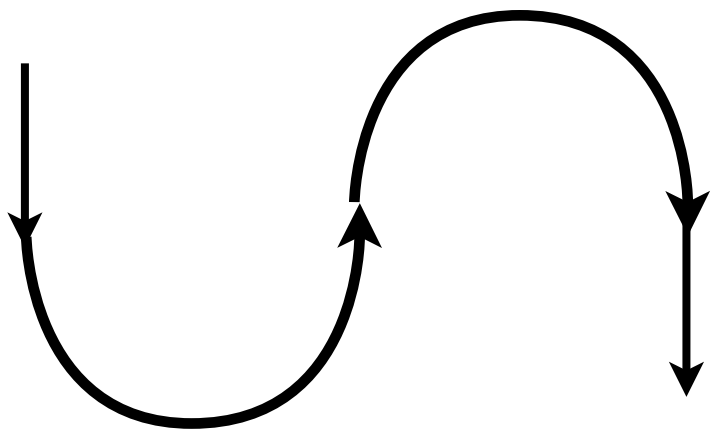

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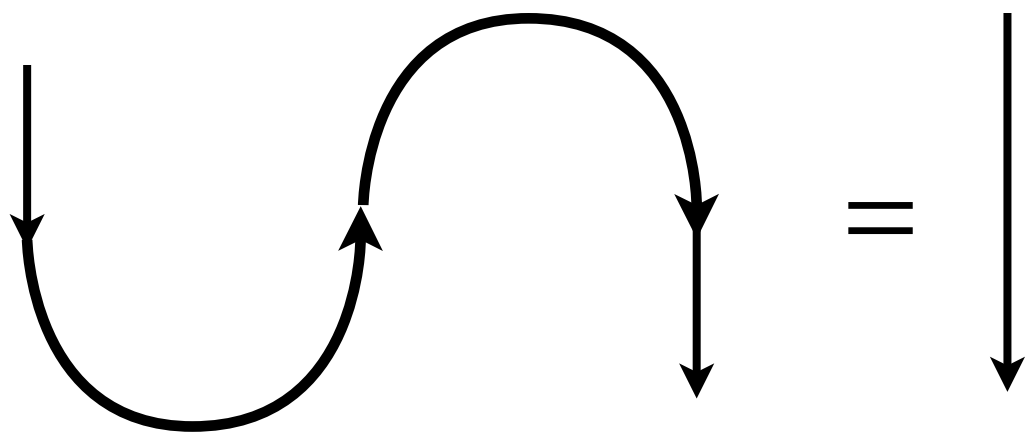

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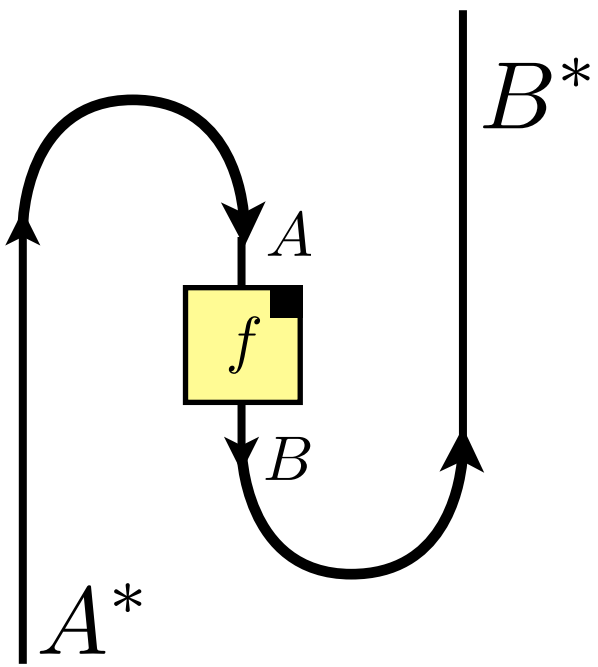
Every arrow f in a compact category has a dual f^* defined by

$f^* =$



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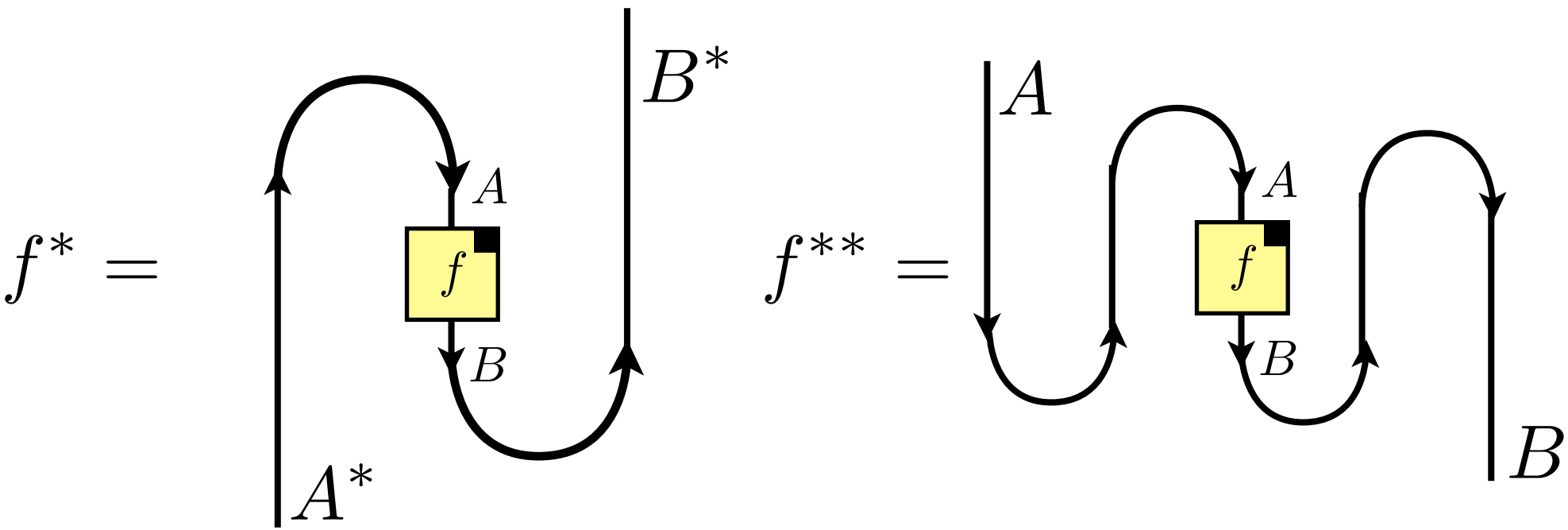

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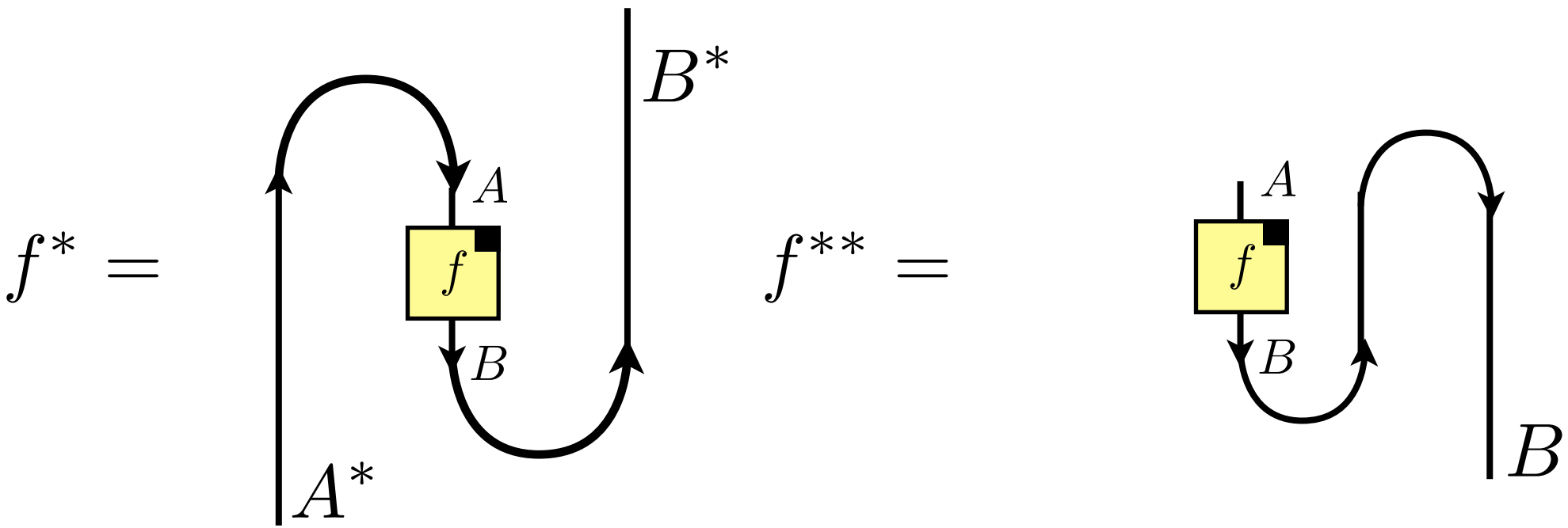

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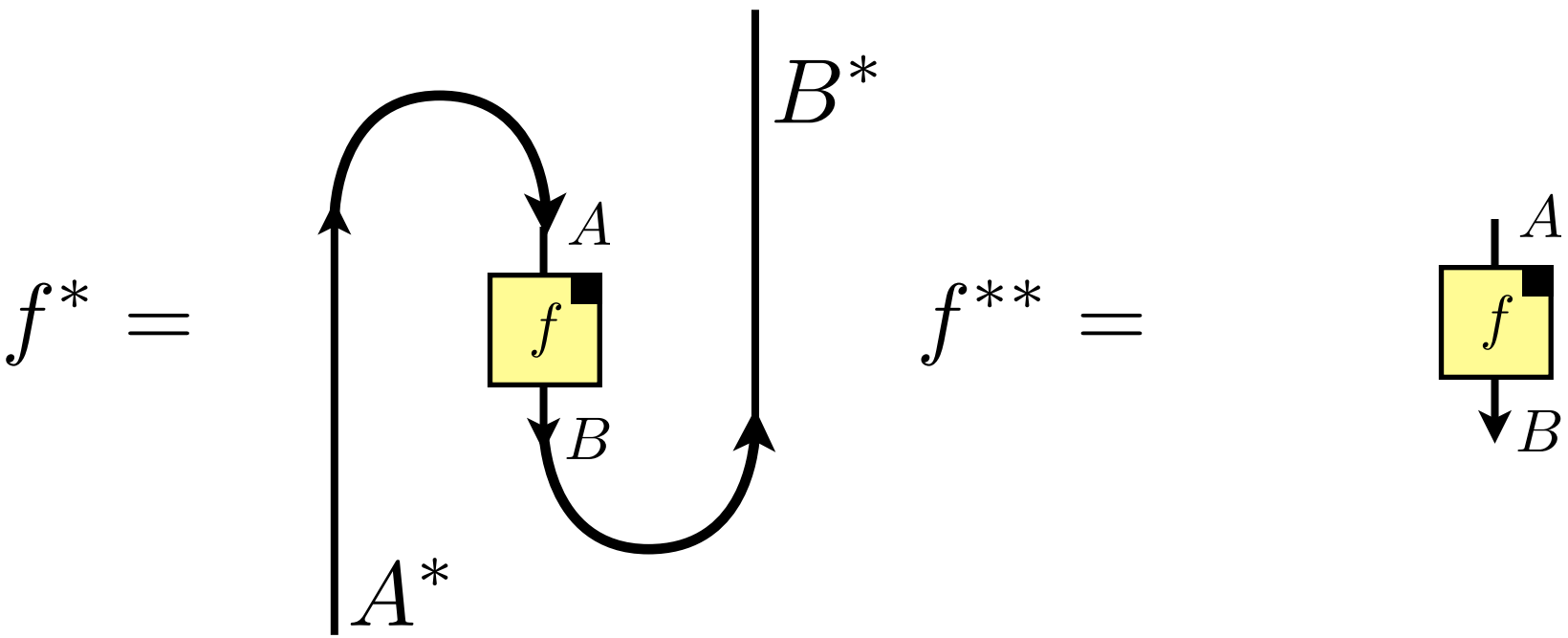

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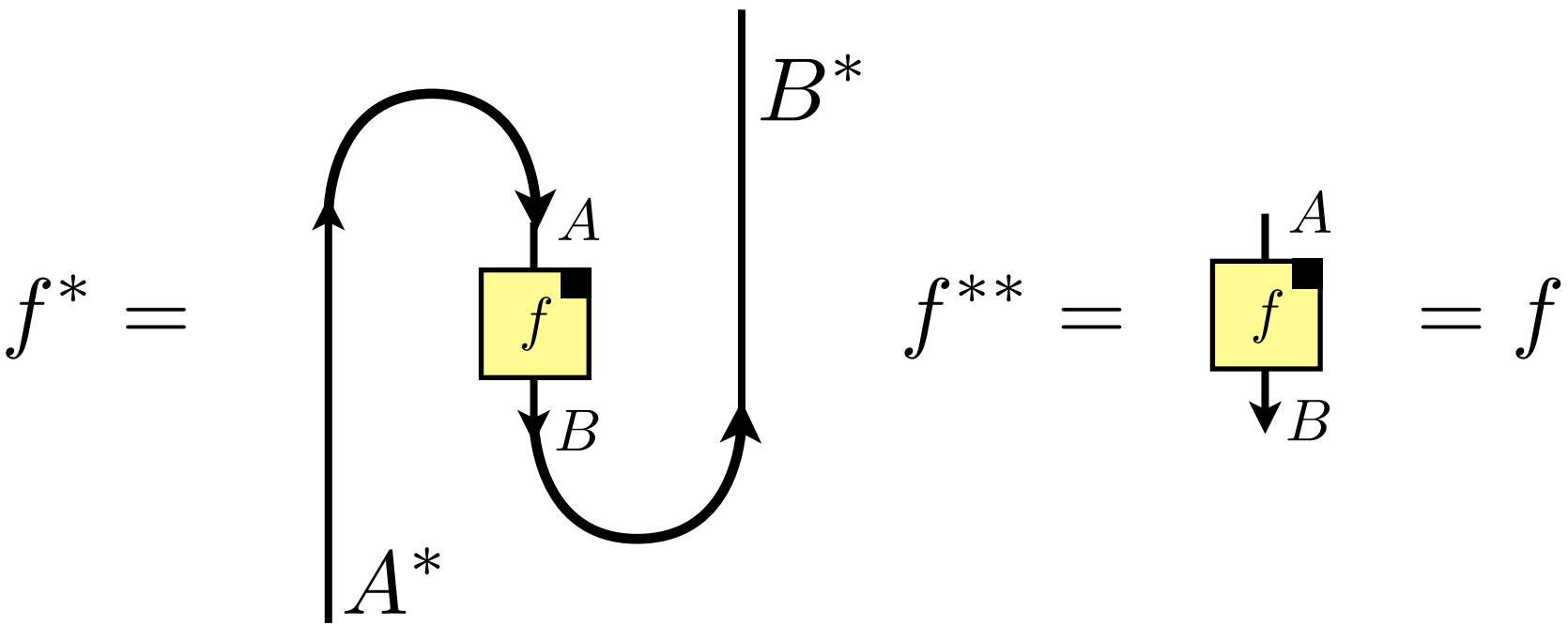

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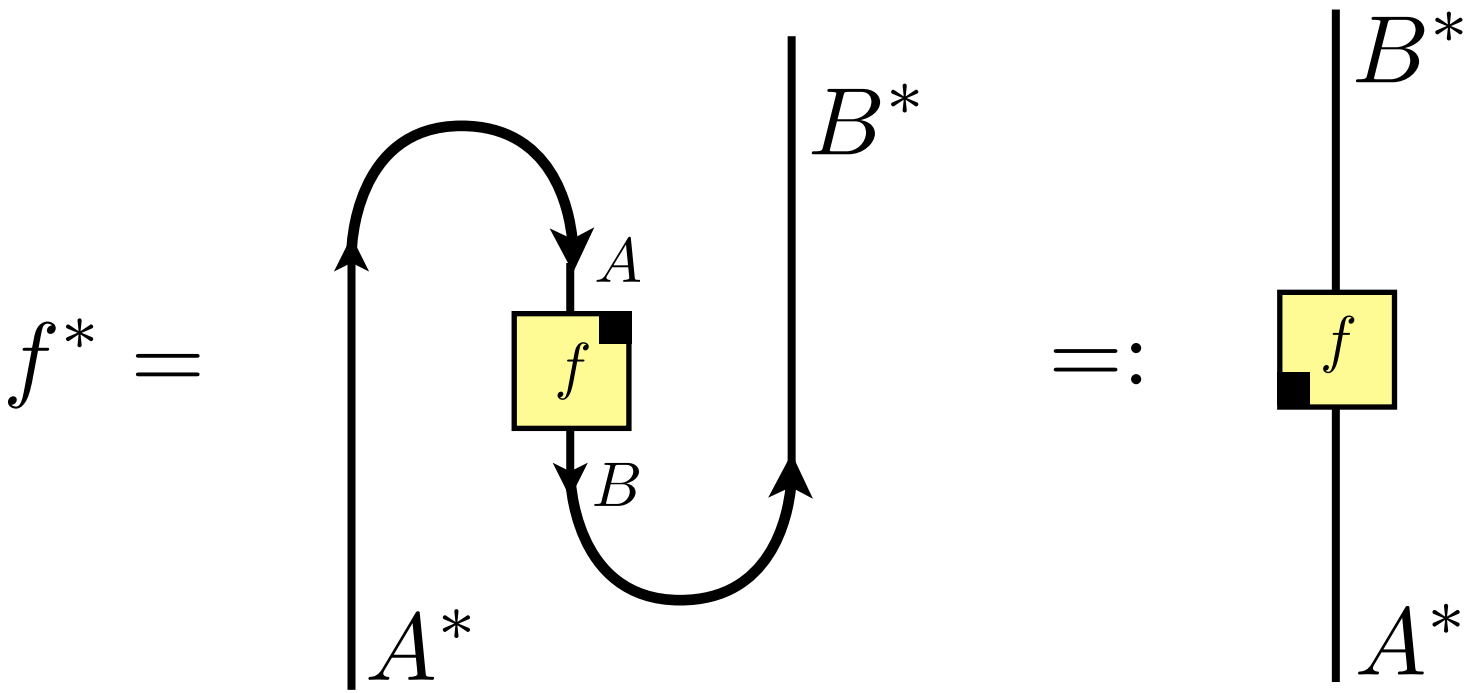

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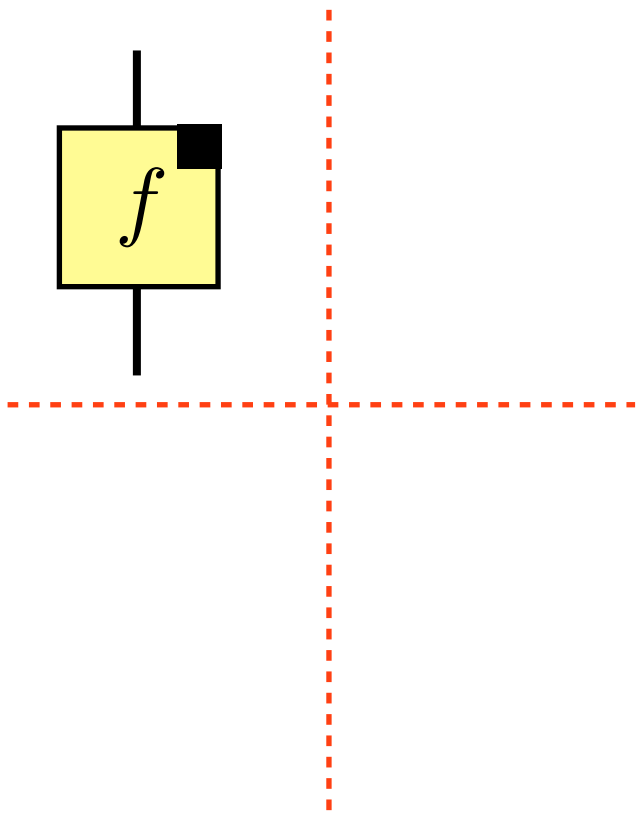
$f =$ 

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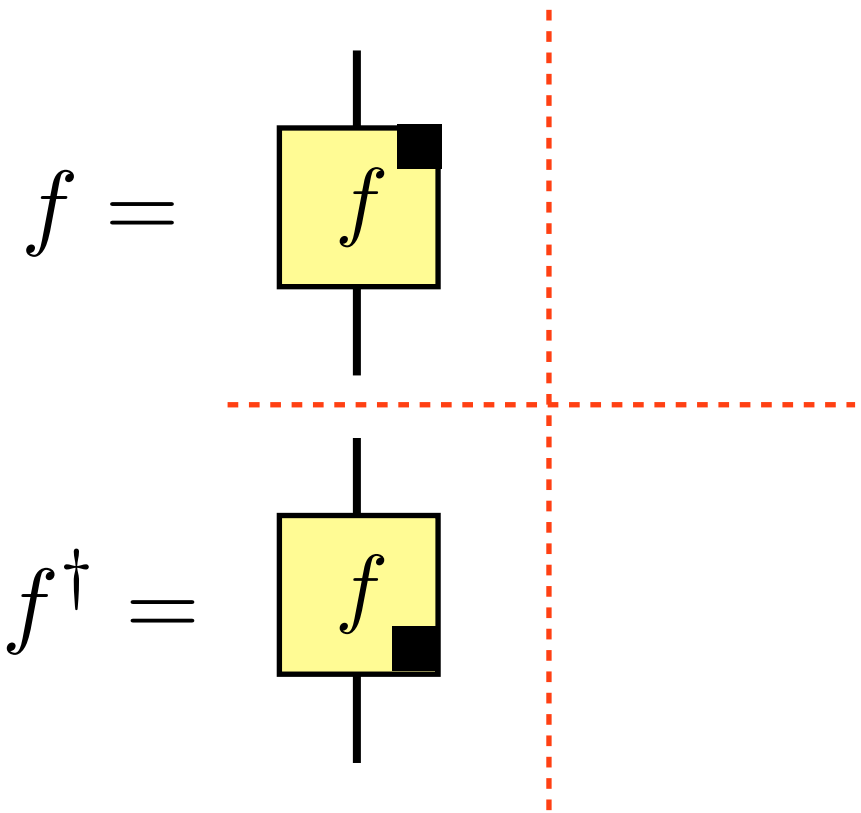

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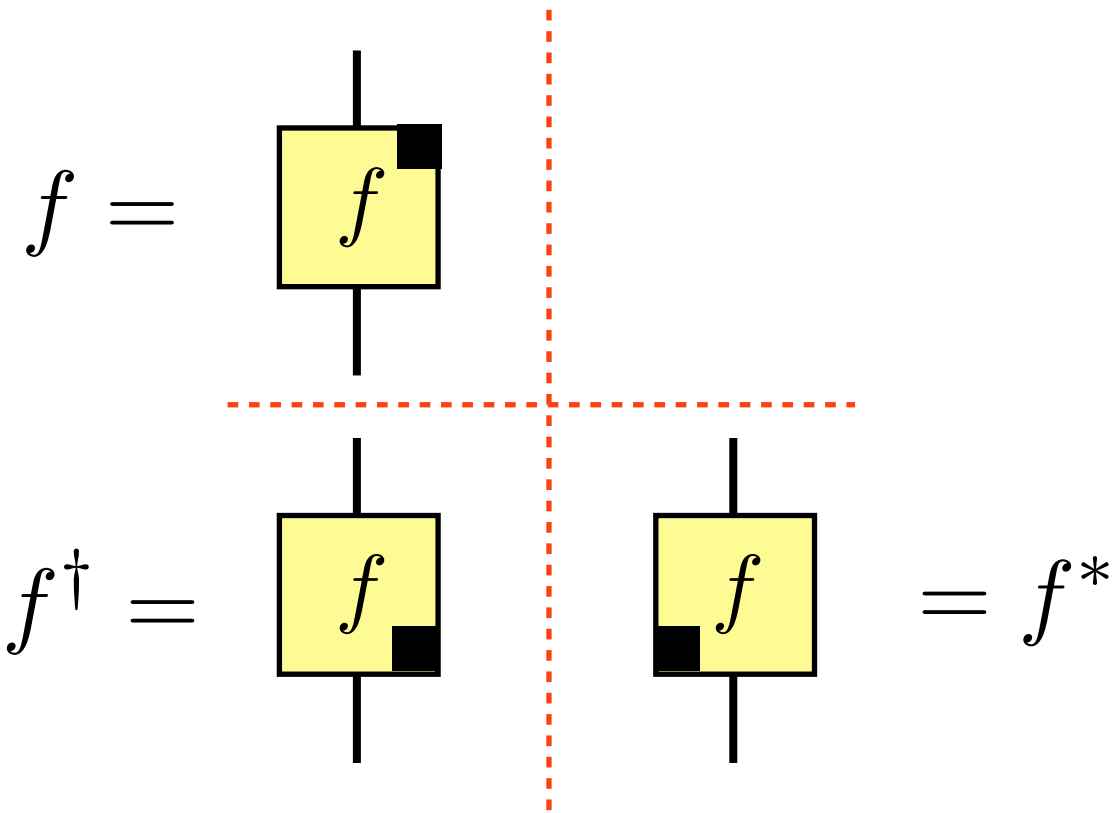

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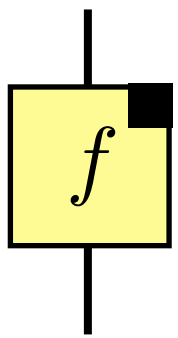
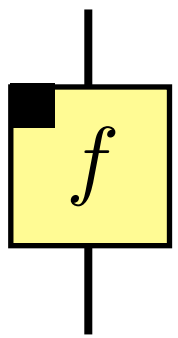
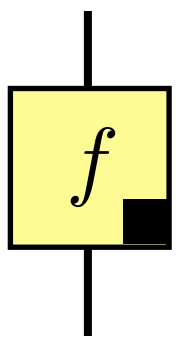
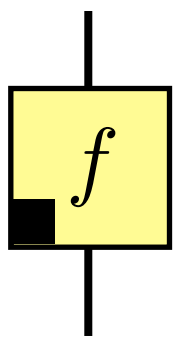
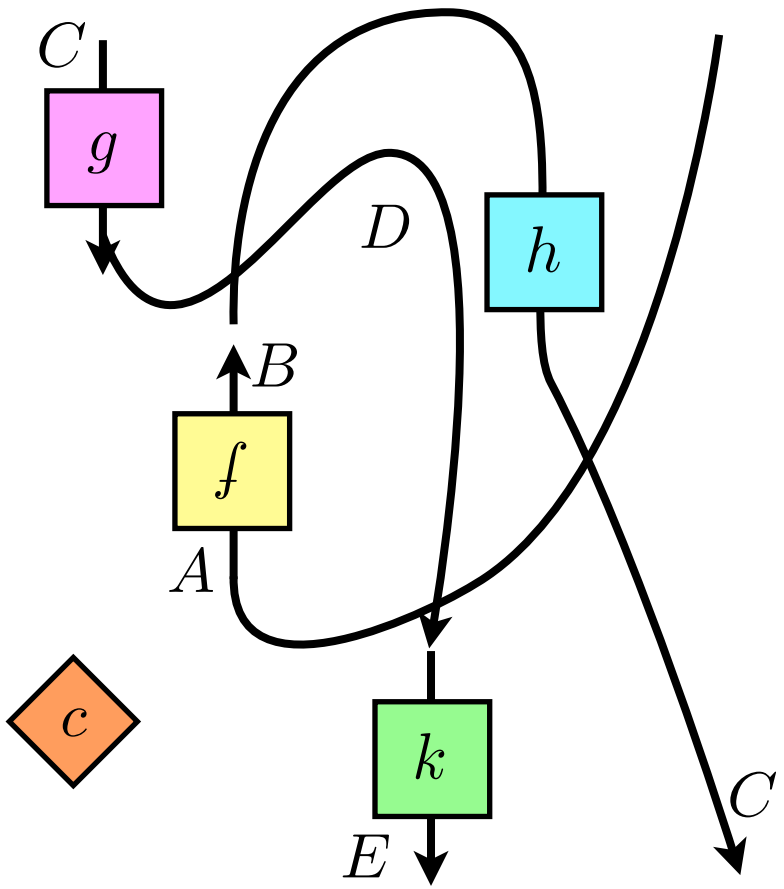
$f =$   $= f^{*\dagger} =: f_*$

$f^\dagger =$   $= f^*$

Graphical Calculus Theorem

Thm: one diagram can be deformed to another if and only if their denotations are equal by the structural equations of the category.



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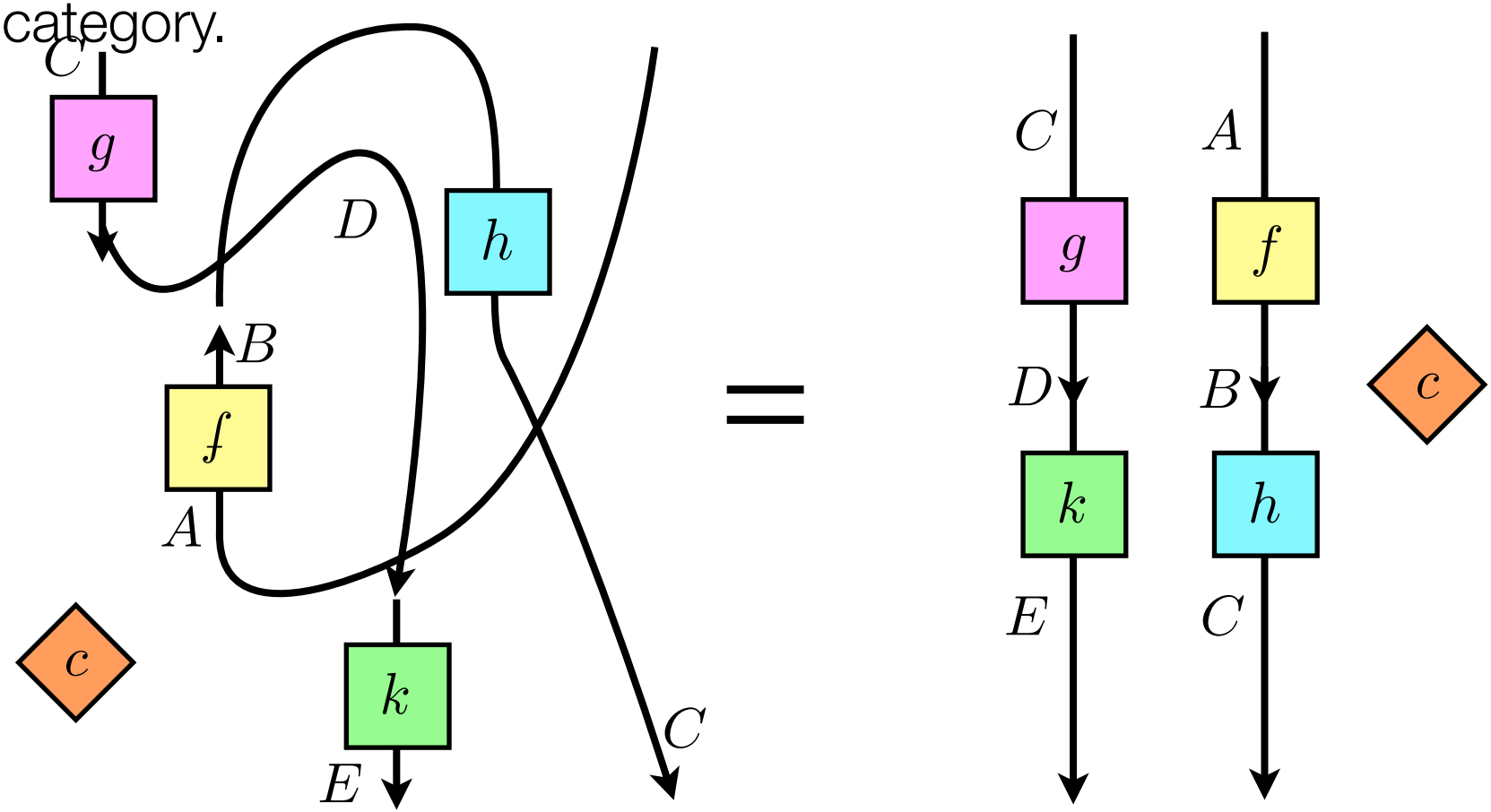


The Category **FRel**

FRel is the category of finite relations. It is †-compact with the following structure.

- Objects: finite sets, $X, Y, Z,$ etc
- Arrows: Relations $R \subseteq X \times Y$
- Tensor: Cartesian product; $X \otimes Y := X \times Y, \quad I = \{*\}$
- $f^\dagger$ is relational converse: $(x, y) \in f \Leftrightarrow (y, x) \in f^\dagger$

A relation $R \subseteq \{*\} \times X$ is simply a *subset* of X ; its converse is also a subset.

Hence $S^\dagger \circ R : I \to I$ is non-empty iff $R \cap S \neq \emptyset$.

The Category **FDHilb**

FDHilb is the category of finite dimensional complex Hilbert spaces. It is †-monoidal with the following structure.

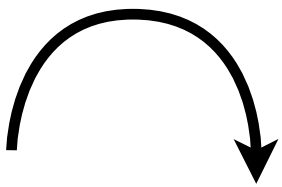
- Objects: finite dimensional Hilbert spaces, A, B, C, etc
- Arrows: all linear maps
- Tensor: usual (Kronecker) tensor product; $I = \mathbb{C}$
- $f^\dagger$ is the usual adjoint (conjugate transpose)

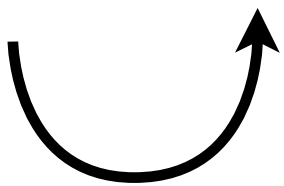
A linear map $\psi : I \to A$ picks out exactly one vector. It is a ket and $\psi^\dagger : A \to I$ is the corresponding bra.

Hence $\psi^\dagger \circ \phi : I \to I$ is the inner product $\langle \psi \mid \phi \rangle$

Compact Structure of **FDHilb**

In $\mathbf{FDHilb}$ the compact structure is given by the maps:





whenever $\{a_i\}_i$ is a basis for A and $\{\overline{a_i}\}_i$ is the corresponding basis for the dual space A^*

Compact Structure of **FDHilb**

In $\mathbf{FDHilb}$ the compact structure is given by the maps:

$$d : 1 \mapsto \sum_i a_i \otimes \overline{a_i} \qquad\qquad e : \overline{a_i} \otimes a_i \mapsto 1$$

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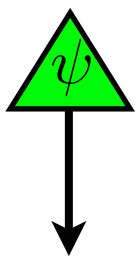
In the case of $\mathbb{C}^2$ the map d picks out the Bell state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

which is the simplest example of *quantum entanglement*.

Example: Quantum Teleportation

Aleks



Bob

Bennett et al 1993; Abramsky & Coecke 2004

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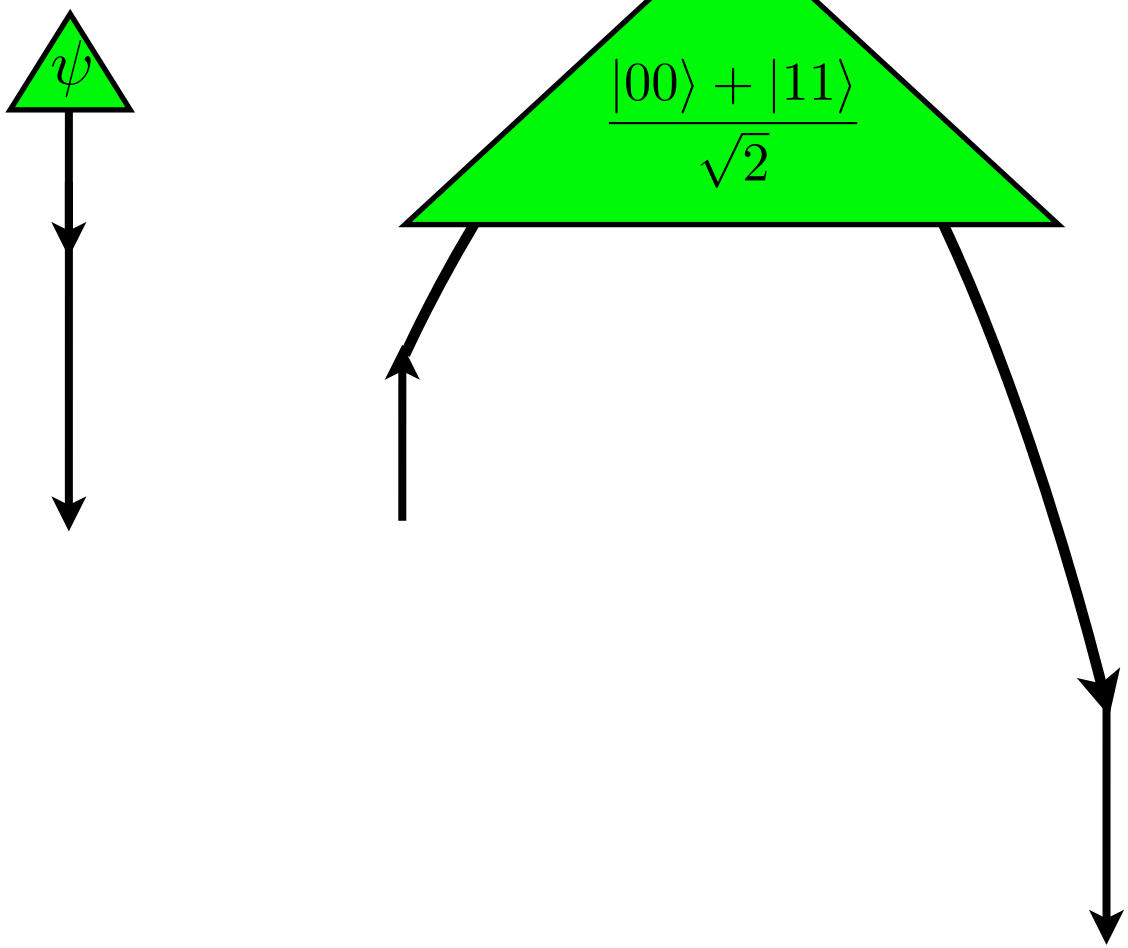

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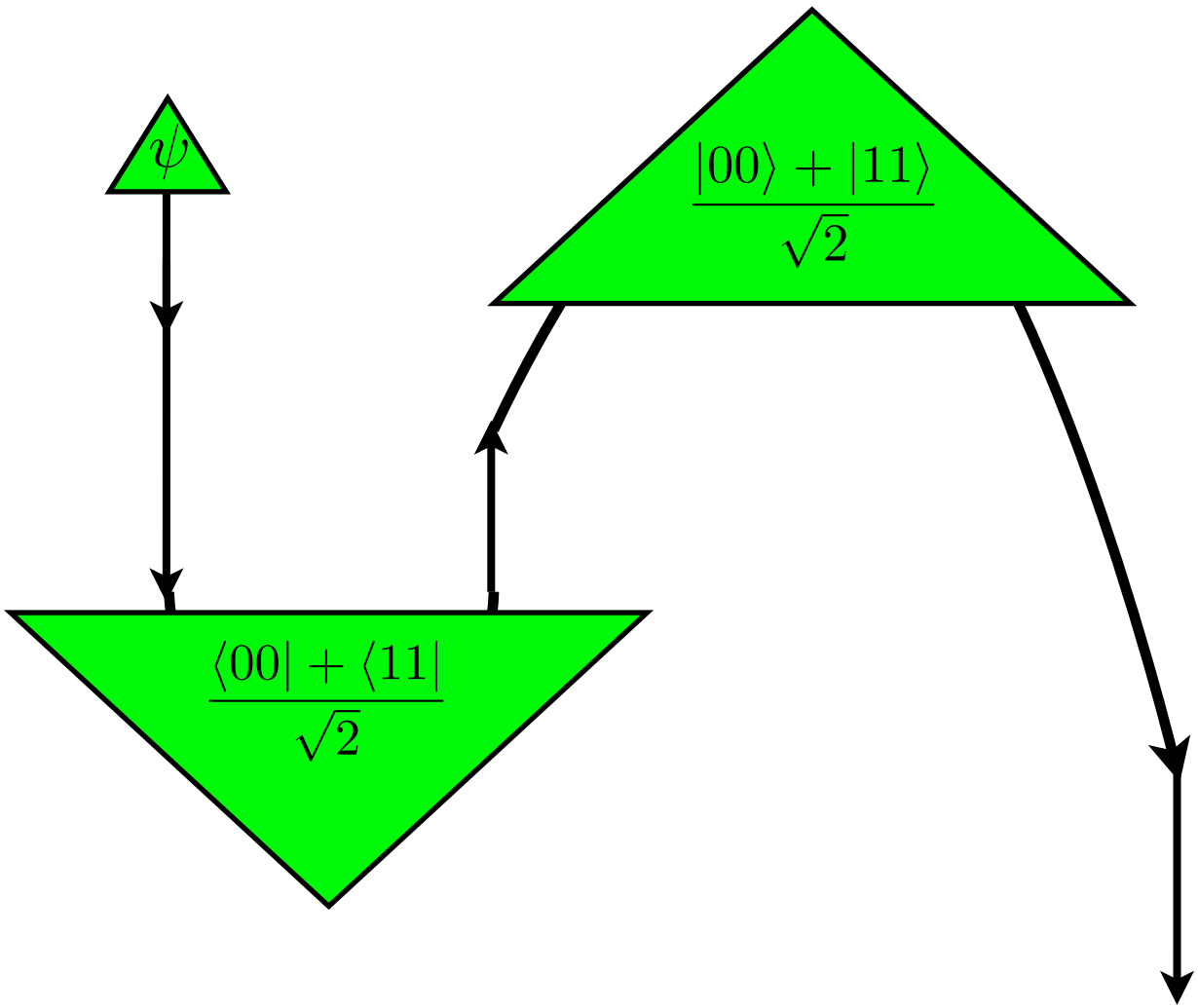

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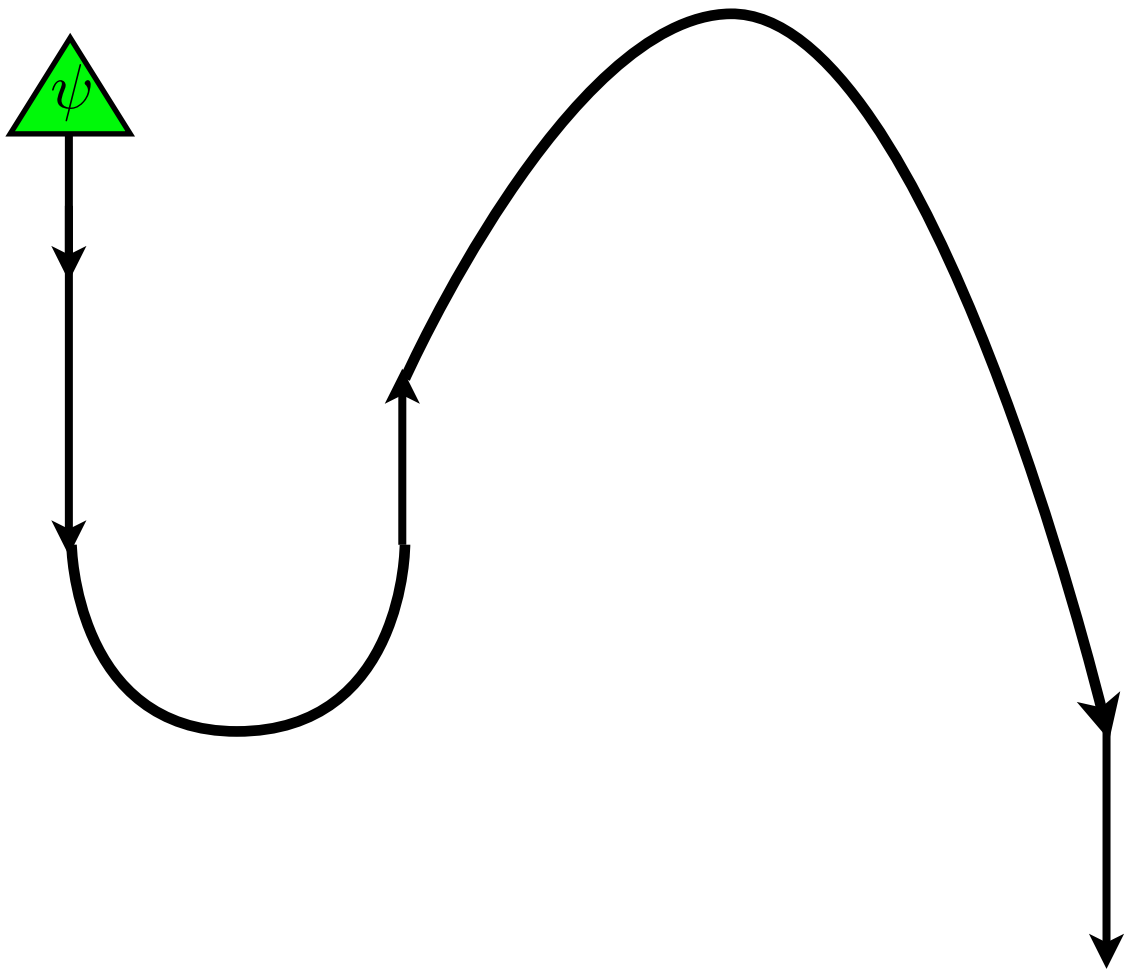

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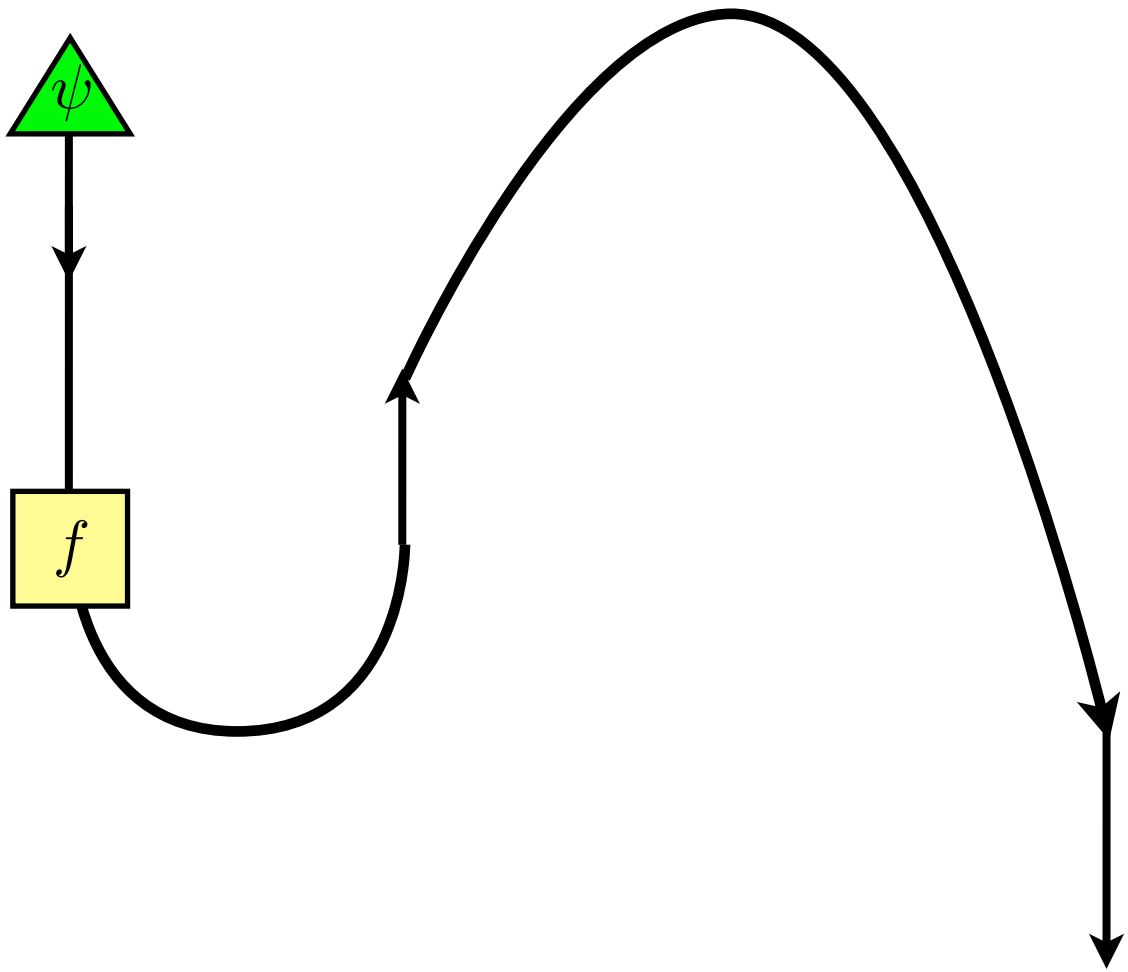

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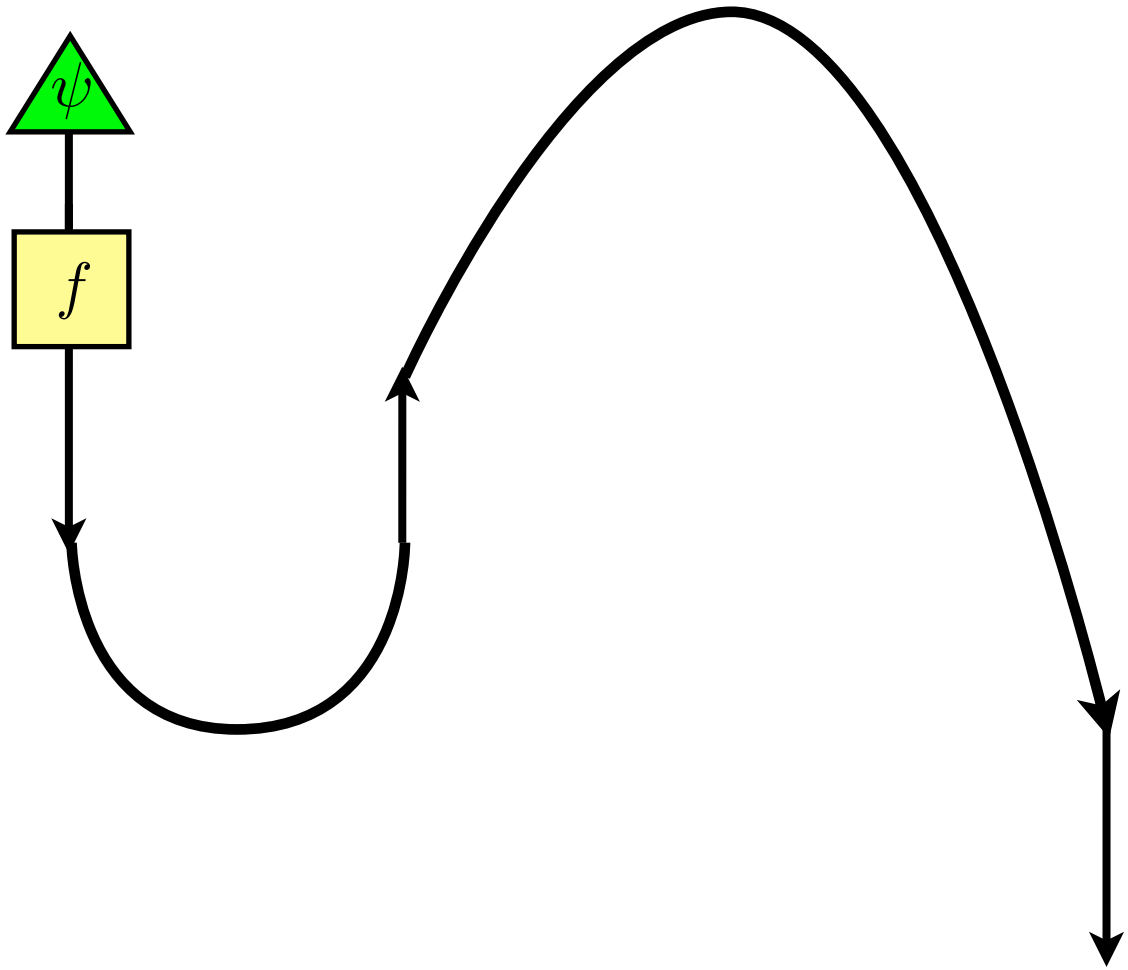

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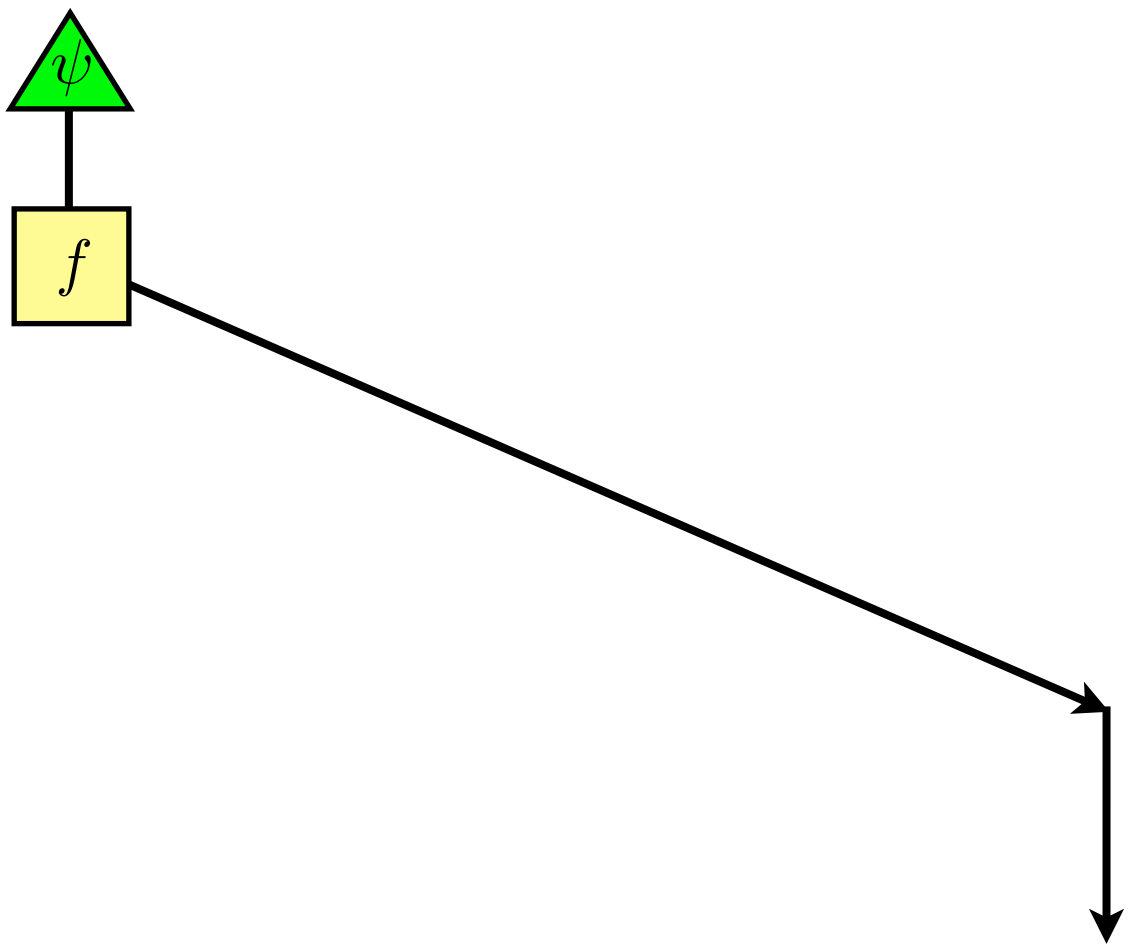

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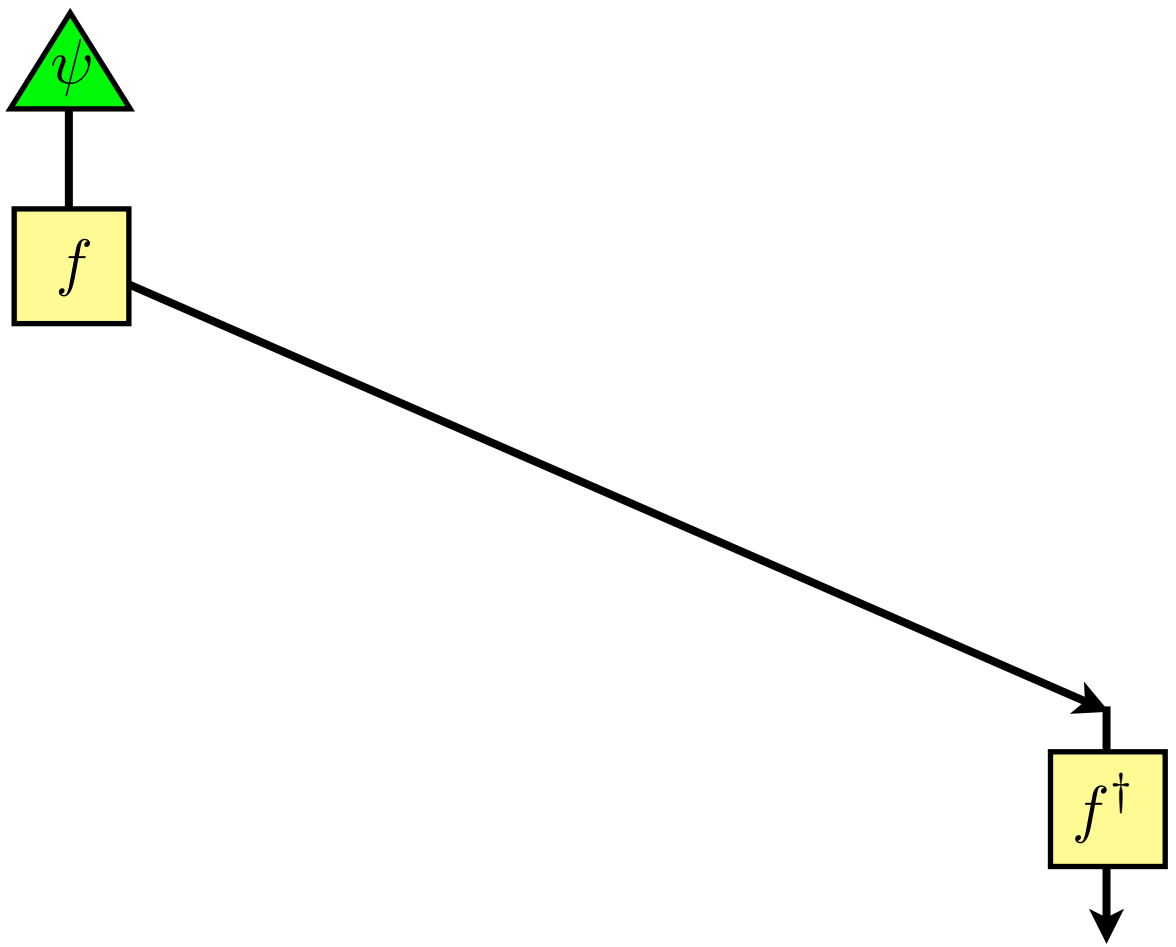

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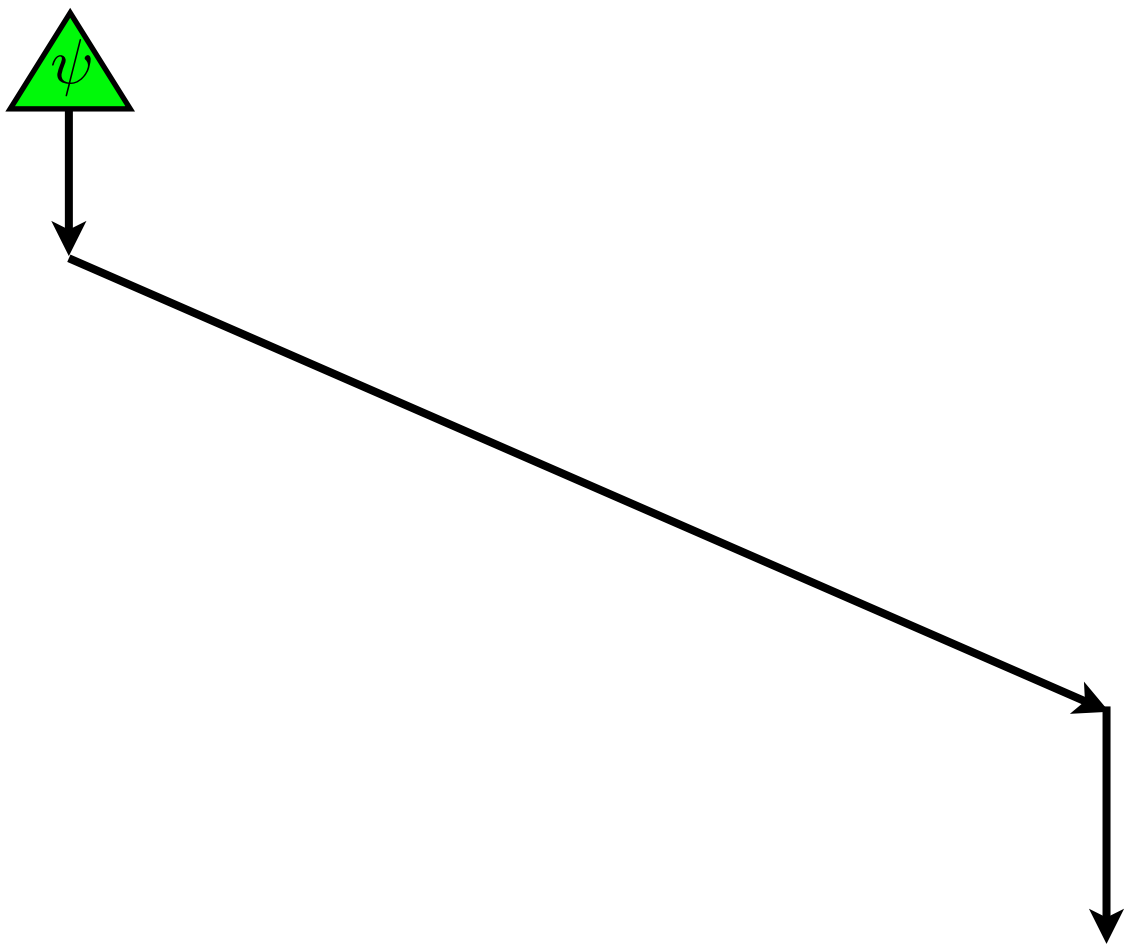

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A compact caveat

If compact structure exists, *it need not be unique*.

For example, in $\mathbf{FDHilb}$, we have compact structures :

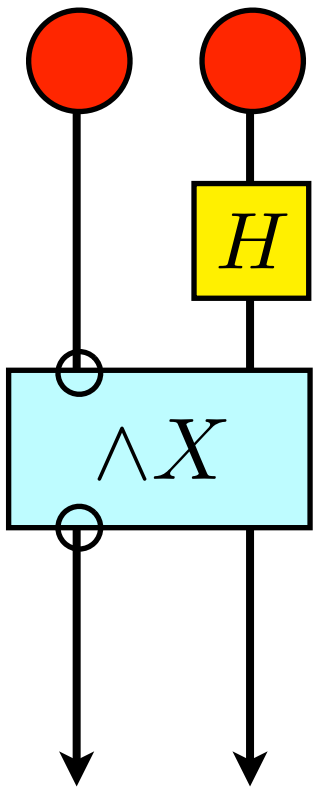
$$d_Z = 1 \mapsto |00\rangle + |11\rangle$$

and

$$d_Y = 1 \mapsto |00\rangle - |11\rangle$$

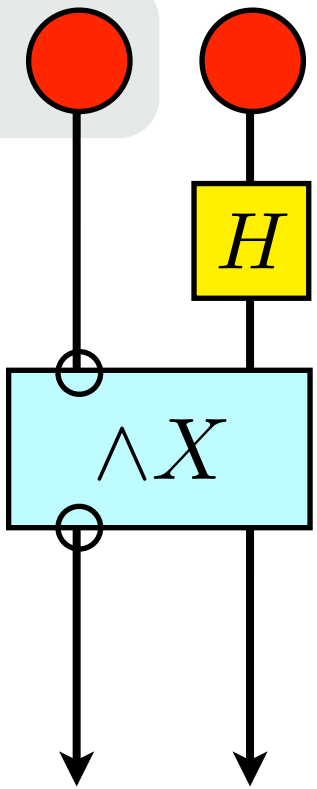
Therefore the $\big(\cdot\big)^*$ and $\big(\cdot\big)_*$ operations are not defined uniquely either.

Example: Preparing a Bell state



Example: Preparing a Bell state

$$|0\rangle : I \to Q$$



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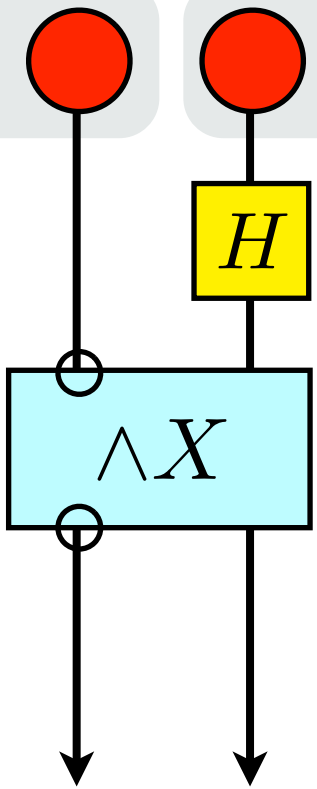

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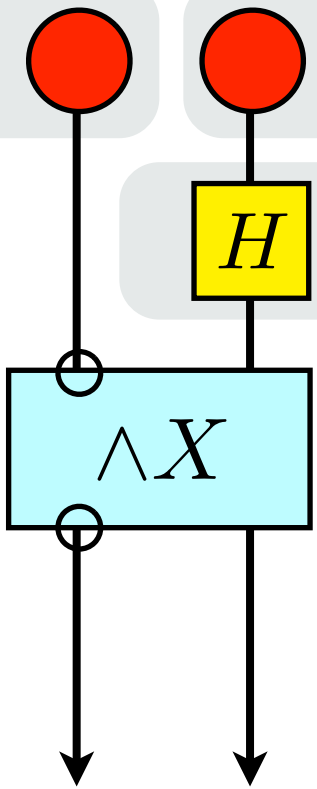

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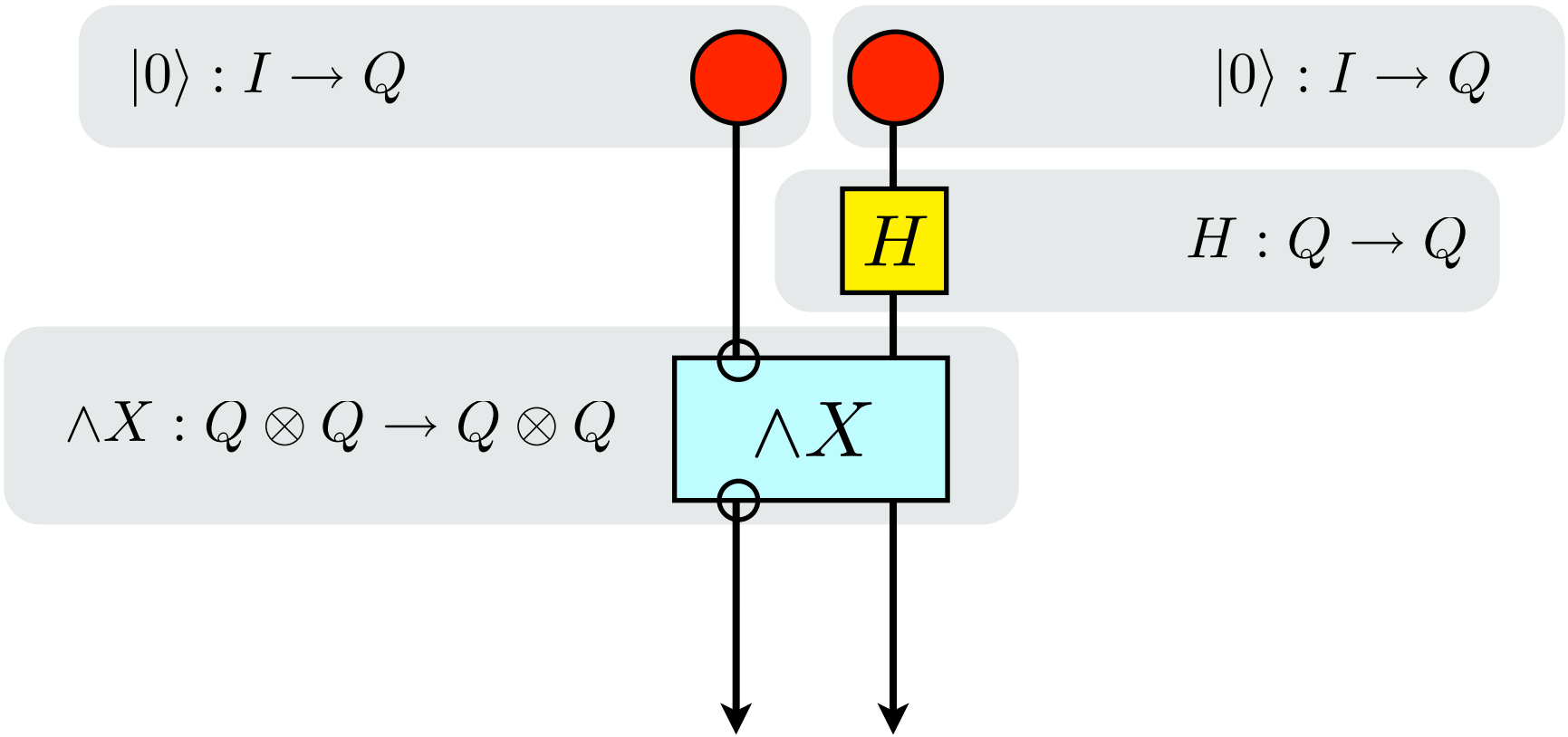
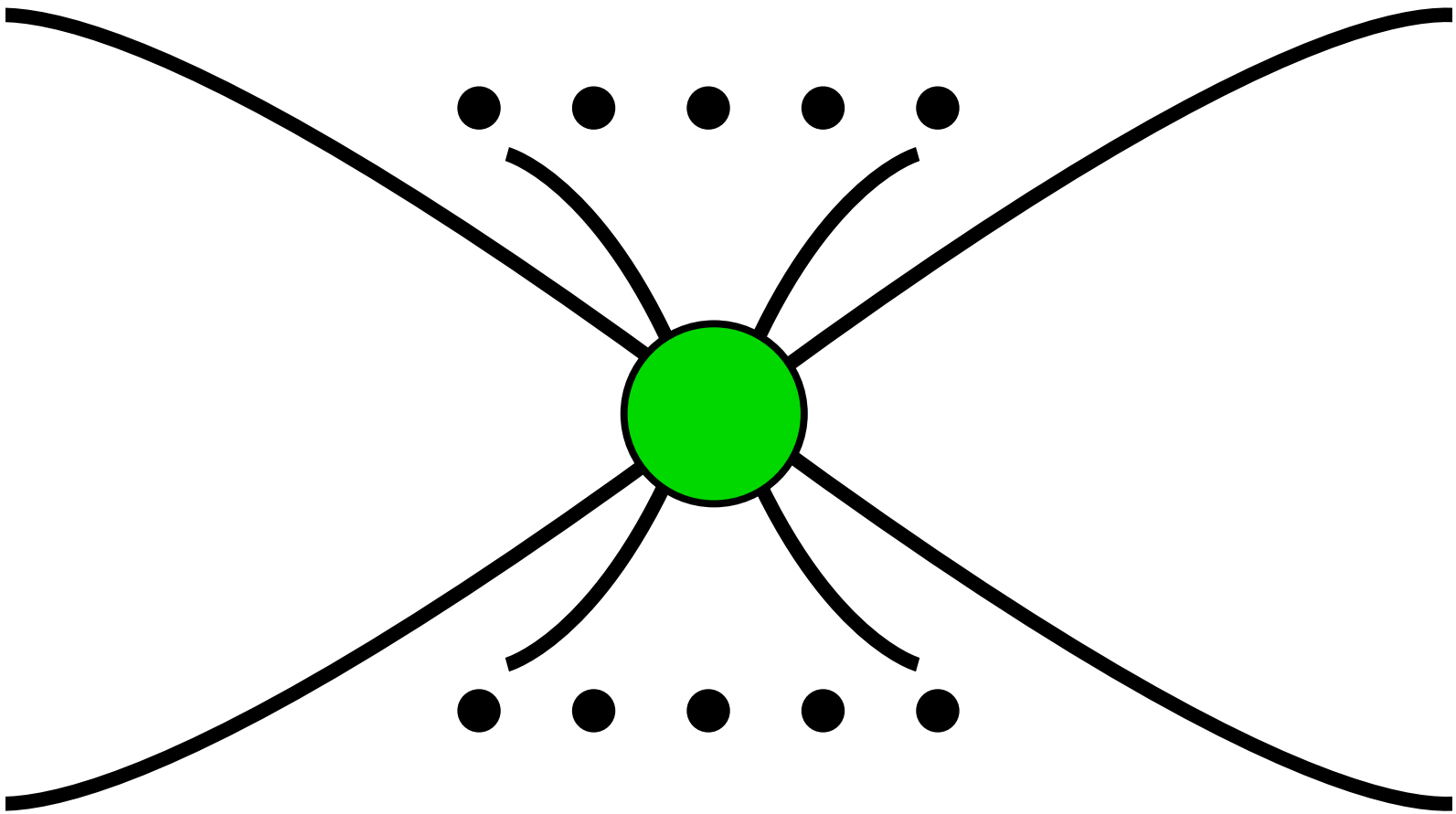


Classical Structure



Copying, deleting, and all that

No-Cloning and No-Deleting

Theorem: There are no unitary operations D such that

$$D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$$
$$D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no unitary operations E such that

$$E : |\psi\rangle \mapsto |0\rangle$$
$$E : |\phi\rangle \mapsto |0\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]

No-Cloning and No-Deleting, abstractly

Theorem: if a †-compact category has natural transformations

$$\delta : - \Rightarrow - \otimes -$$

$$\epsilon : - \Rightarrow I$$

then the category collapses [Abramsky 2007]

(Translation: in our abstract setting there are no universal cloning or deleting operations, just like in quantum mechanics)

“Classical” Quantum States

When can a quantum state be treated as if classical?

- no-go theorems allow copying and deleting of *orthogonal* states;

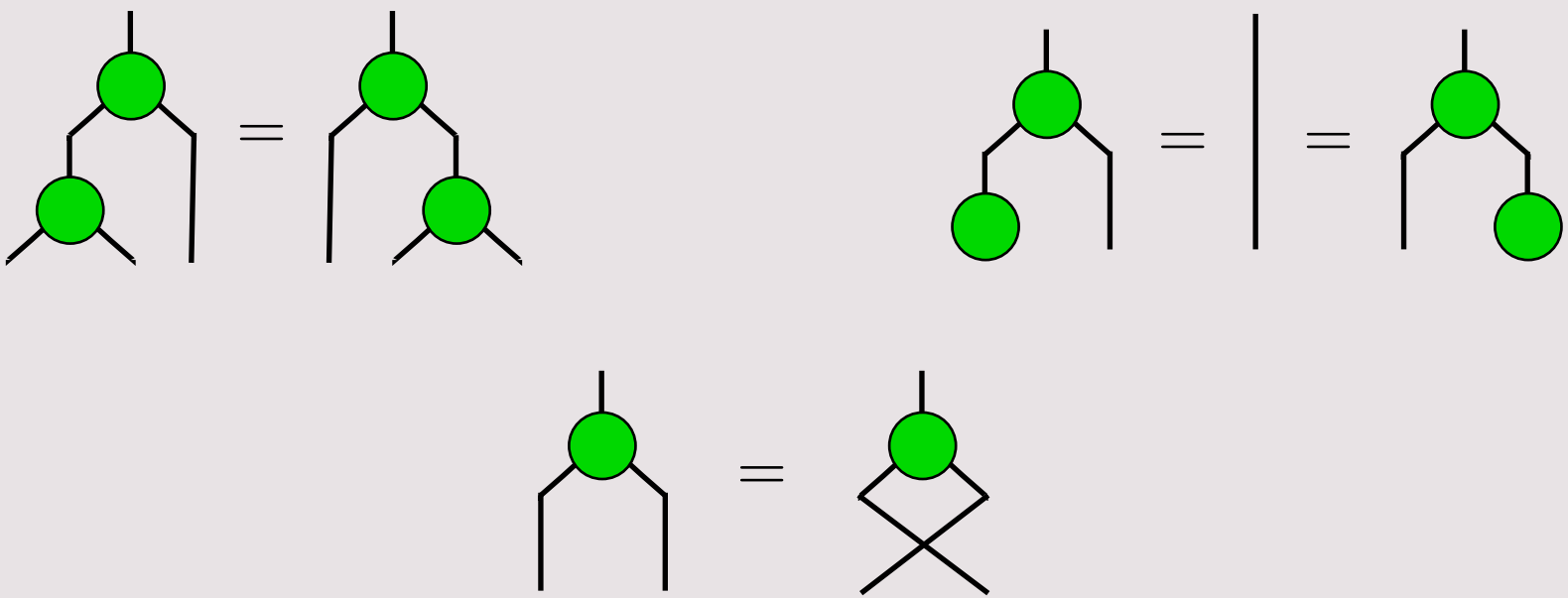
In other words:

- A quantum state may be copied and deleted if it is an eigenstate of some *known observable.*

We’ll use this property to formalise *observables* in terms of *copying* and *deleting* operations.

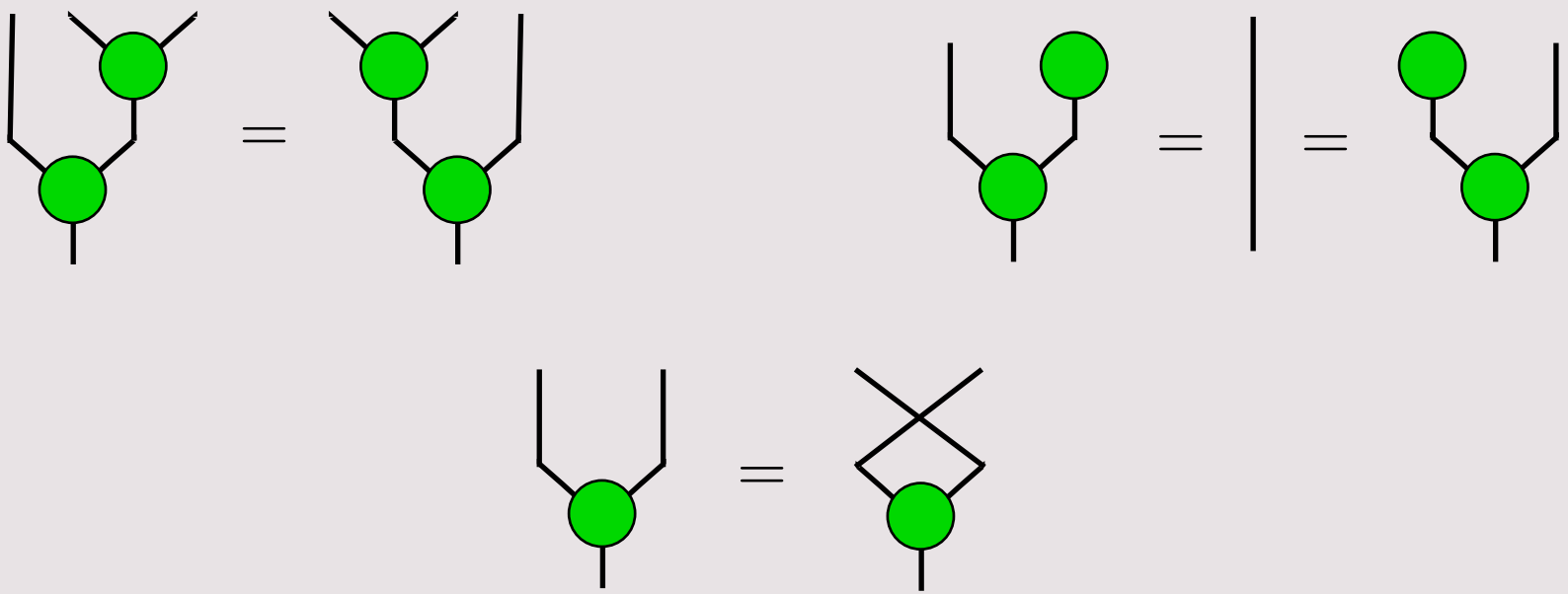
Classical Structures

Comonoid Laws

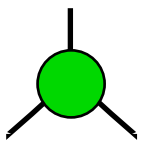
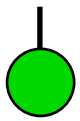
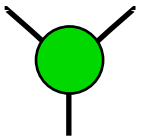
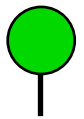


Classical Structures

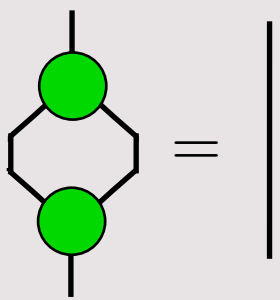
Monoid Laws



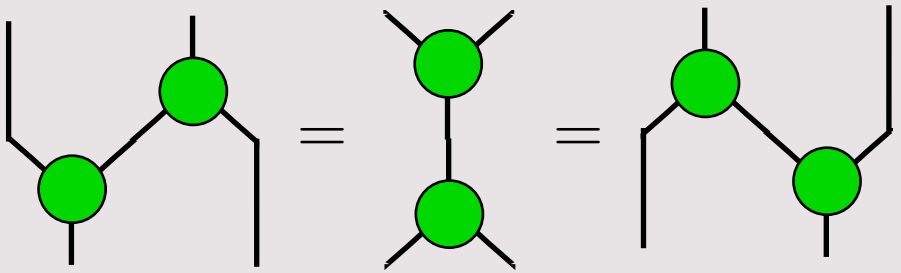
Classical Structures

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

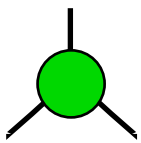
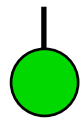
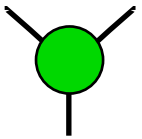
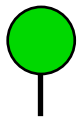
Isometry Law



Frobenius Law



Classical Structures

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

In other words: a classical structure is a *special commutative †-Frobenius algebra*

Classical Structures

Given any finite dimensional Hilbert space we can define a classical structure by

$$\delta : A \to A \otimes A :: a_i \mapsto a_i \otimes a_i$$
$$\epsilon : A \to I :: \sum_i a_i \mapsto 1$$

Example: $\delta : \begin{array}{l} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{array}$ $\qquad \epsilon : |0\rangle + |1\rangle \mapsto 1$

define a classical structure over qubits; the standard basis is copied and erased. Note however that:

$$\delta(|+\rangle) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

showing that not every state can be cloned.

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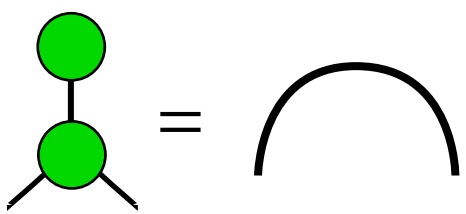

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Each (well behaved) observable defines a basis, hence each observable defines a classical structure!

Example: $\mathbf{FRel}$

For any set X we can define a classical structure by:

$$D : x \sim (x, x) \qquad \forall x \in X$$
$$E : x \sim *$$

Perhaps surprisingly, there are others. Let $X = \{0,1\}$ then we have:

$$D_X : \quad 0 \sim \{(0,0),(1,1)\}$$
$$1 \sim \{(0,1),(1,0)\}$$
$$E_X : \quad 0 \sim *$$

Spider Theorem

Theorem: any maps constructed from δ and ε , and their adjoints, whose graph is connected, is determined uniquely by the number of inputs and outputs.

Coecke & Paquette 2006

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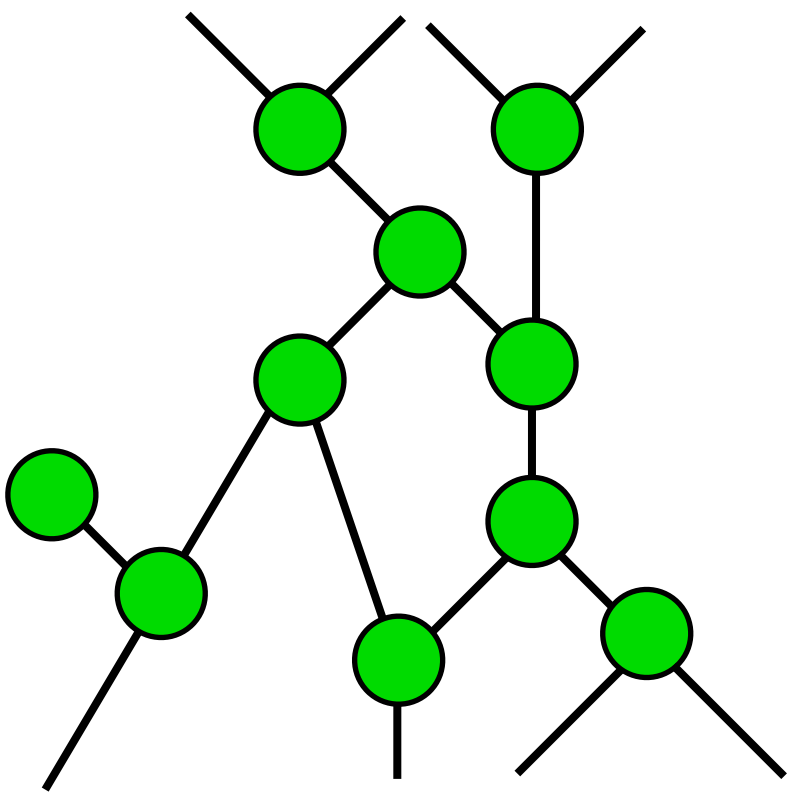

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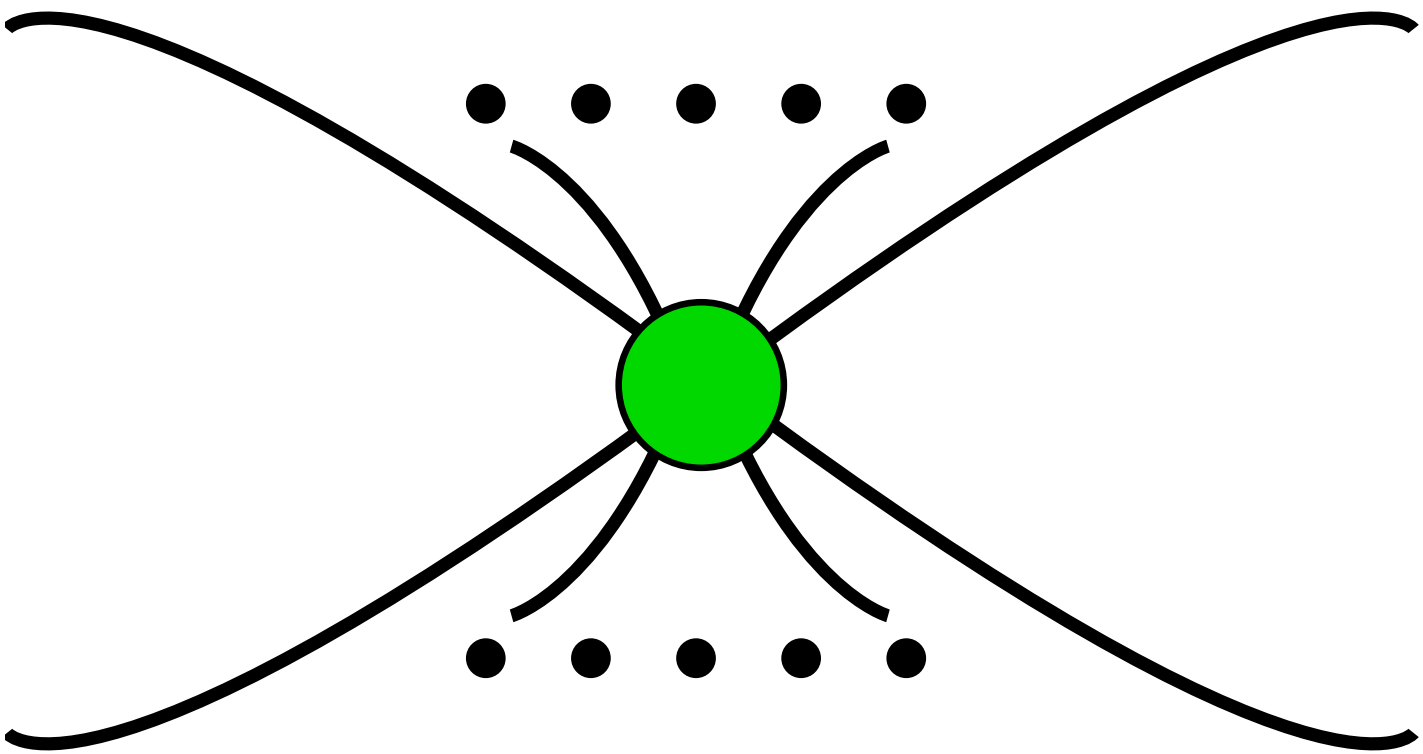
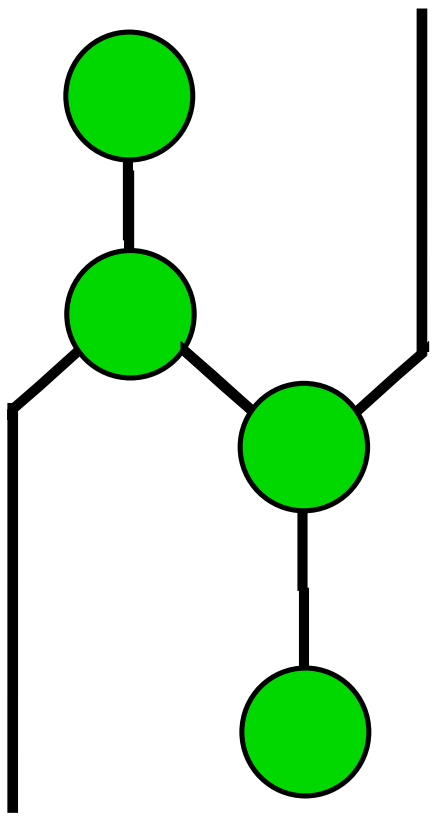


Coecke & Paquette 2006

Classical Structure begets Compact Structure



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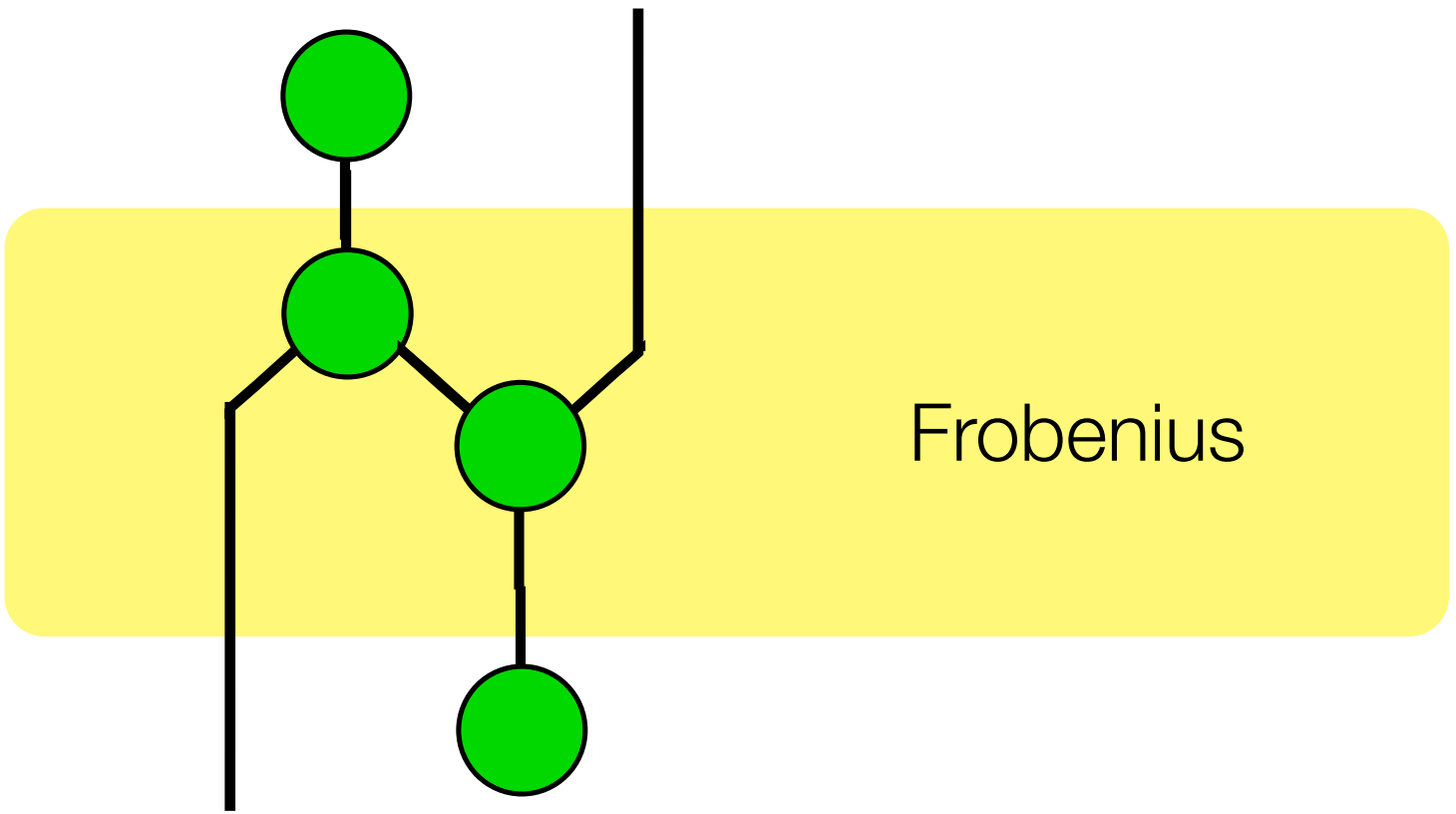

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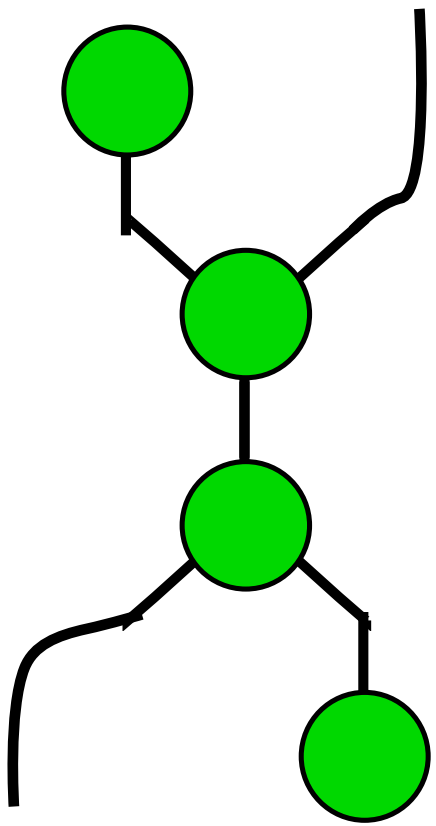

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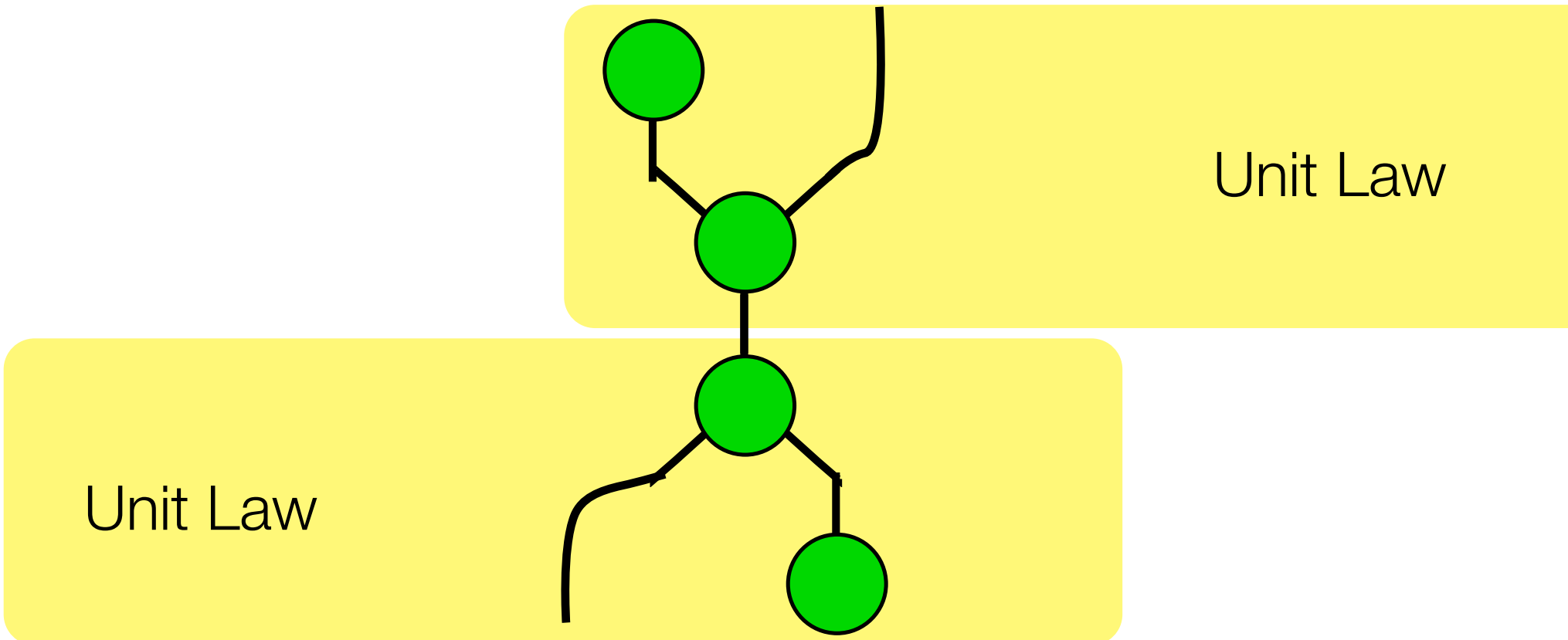

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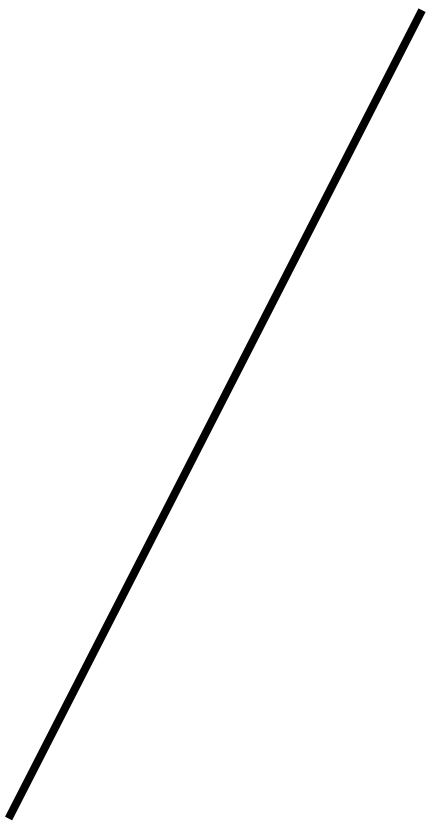

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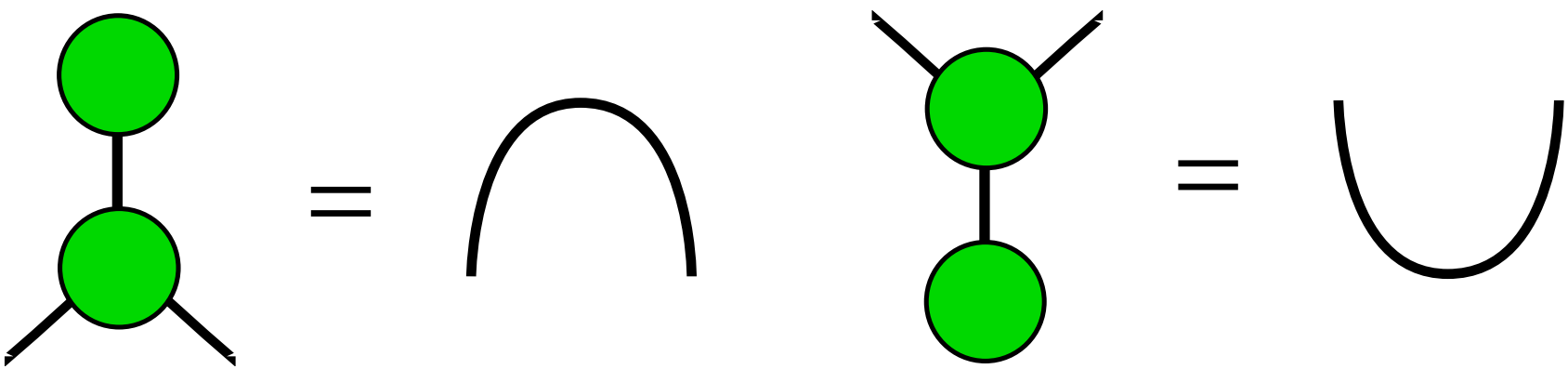
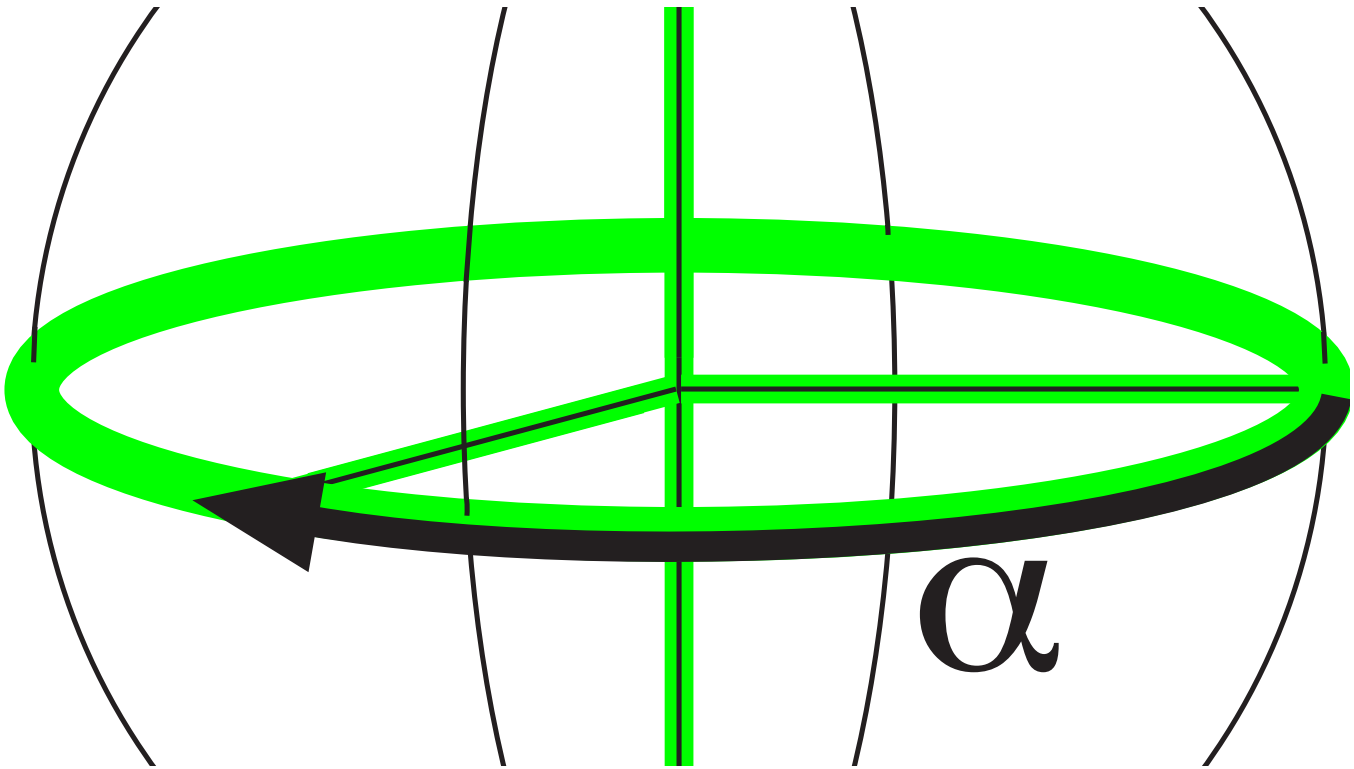


Each classical structure induces a self-dual compact structure.

Self duality is addressed in Coecke, Paquette, & Perdrix 2008

Phase Maps

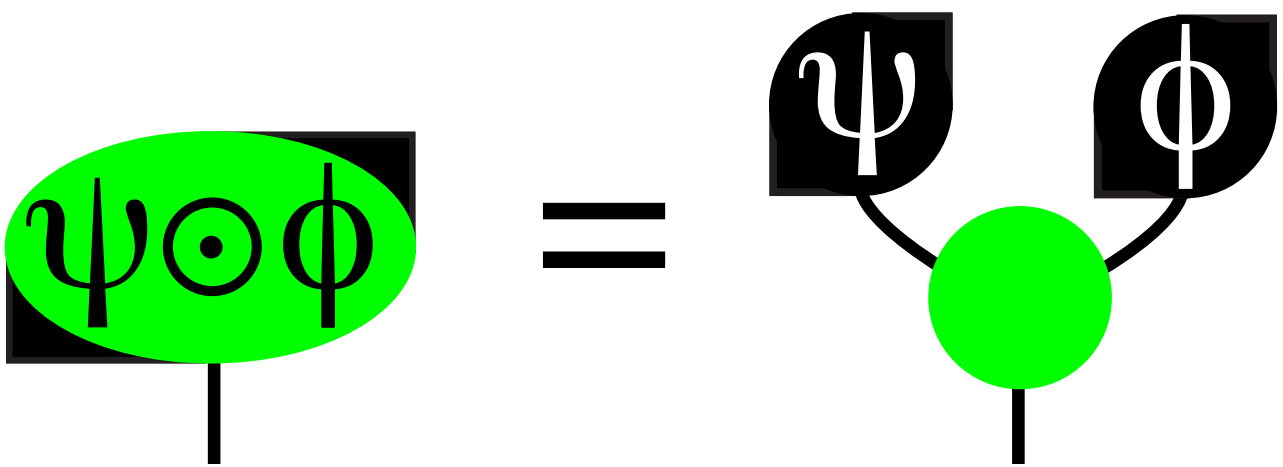


Spinning around an observable

Using the monoid operation

Let $\psi, \phi : I \to A$ be points of A ; we can combine them using the monoid operation $\delta^\dagger : A \otimes A \to A$

$$\psi \odot \phi := \delta^\dagger \circ (\psi \otimes \phi)$$



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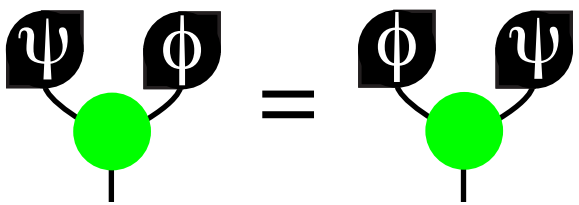

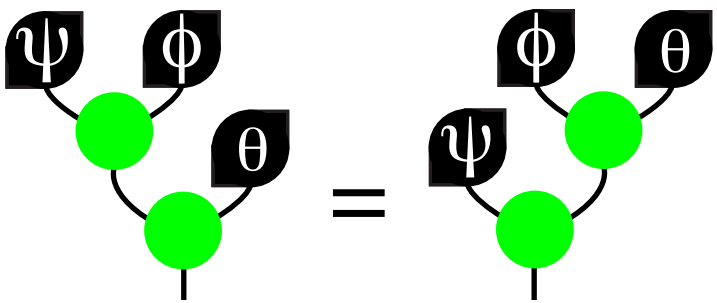

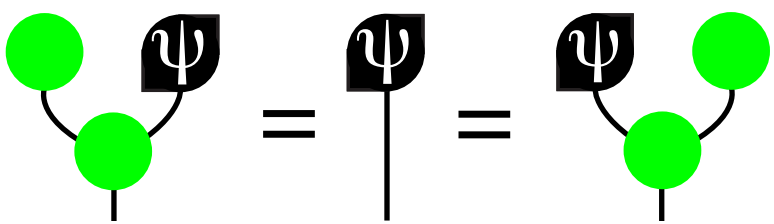

Example: qubits

$$\delta_Z^\dagger : Q \otimes Q \to Q$$

$$|\phi\rangle : I \to Q$$
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$$\delta_Z^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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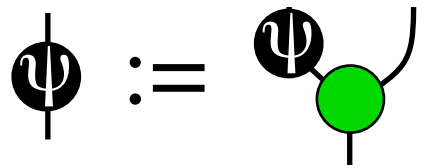
a.k.a. the *Schur product* or *convolution product*

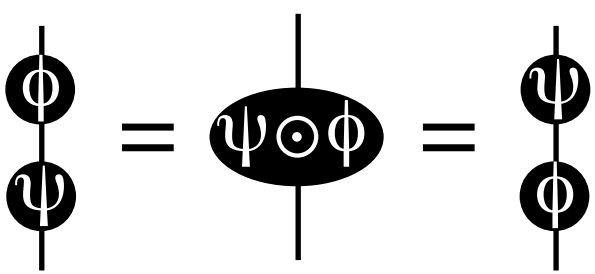
Using the monoid operation

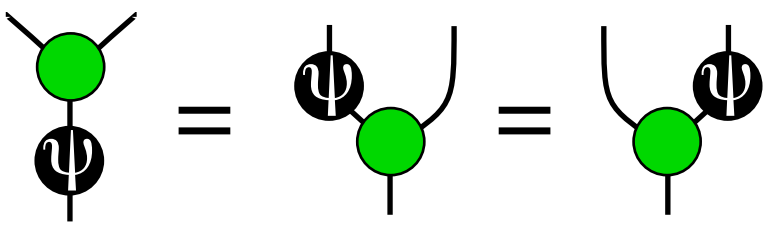
Moreover, each point $\psi : I \to A$ can be lifted to an endomorphism $\Lambda(\psi) : A \to A$

$$\Lambda(\psi) := \delta^\dagger \circ (\psi \otimes \mathrm{id}_A)$$



This yields a homomorphism of monoids so we have:





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$$\delta_Z^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

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$$\Lambda^Z(\psi) := \delta_Z^\dagger \circ (\mathrm{id} \otimes |\psi\rangle) = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$$

Generalised Spider Theorem

Theorem: any maps constructed from δ, ε, some points $\psi_i : I \to A$ and their adjoints, whose graph is connected, is determined by the number of inputs and outputs and the product $\bigodot_i \psi_i$.

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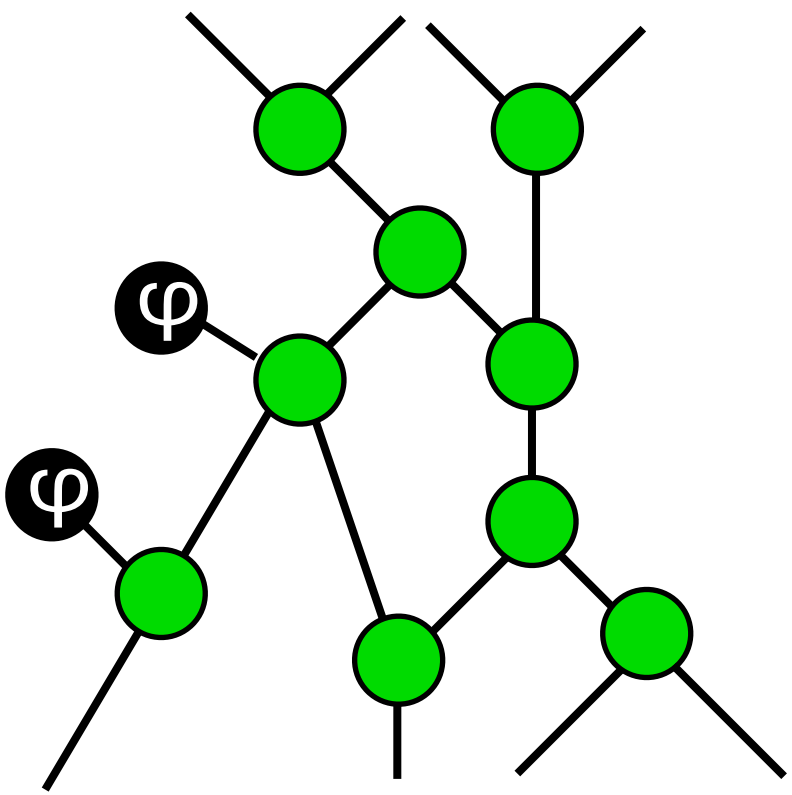

Generalised Spider Theorem

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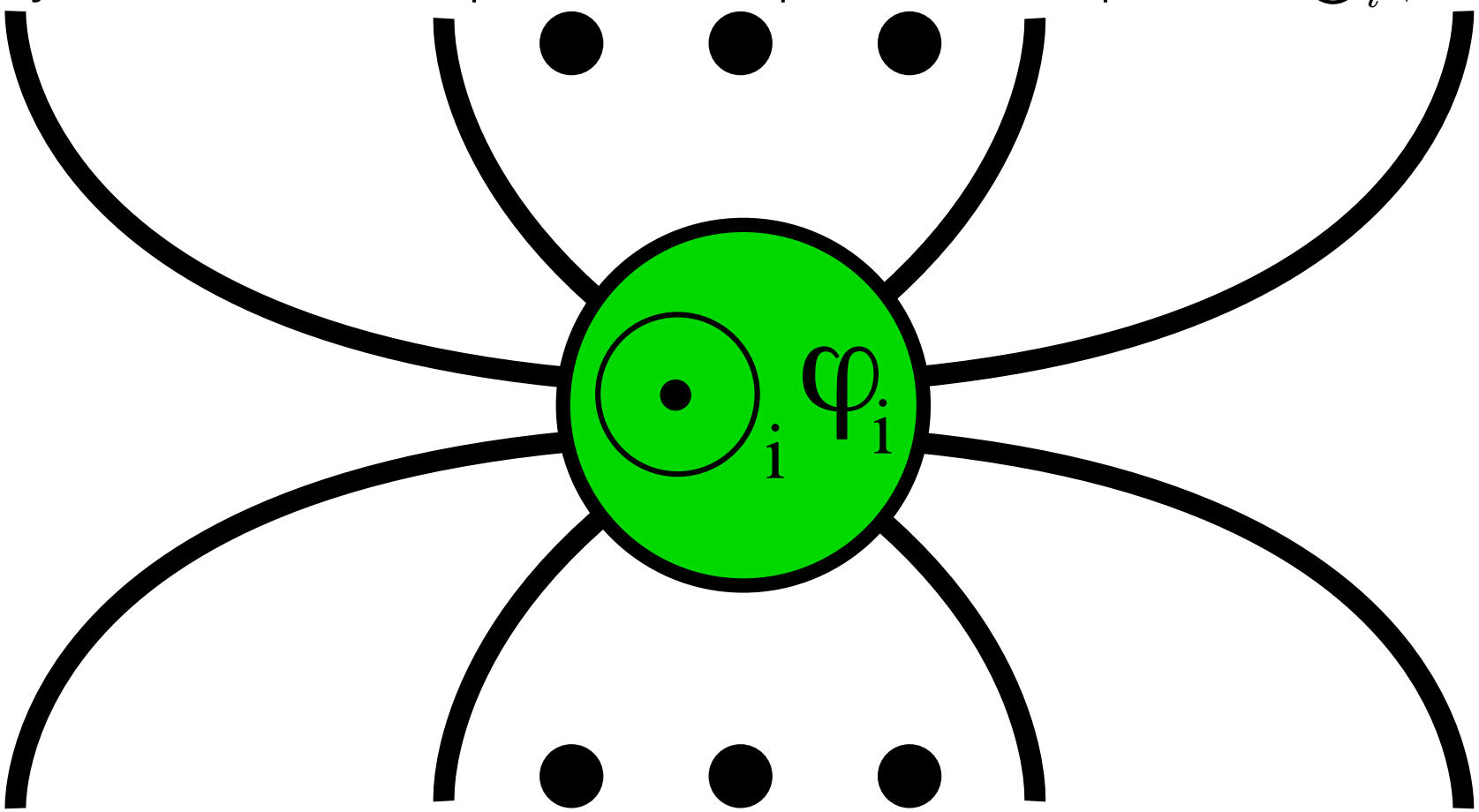
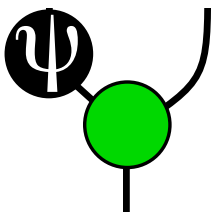
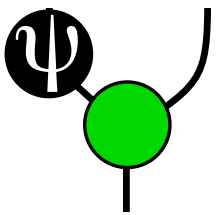


Unbiased Points

Q: When is  unitary?

A: In Hilbert spaces, $\Lambda(\psi)$ is unitary iff $|\psi\rangle$ is unbiased w.r.t. the basis copied by δ.

Unbiased Points

Q: When is 

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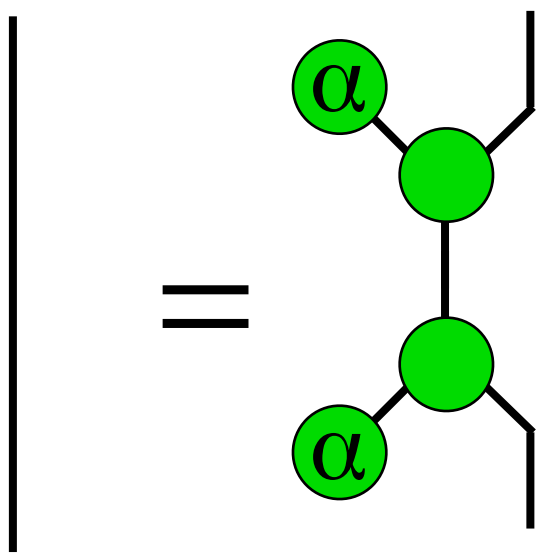

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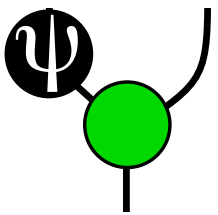
Q: When is 

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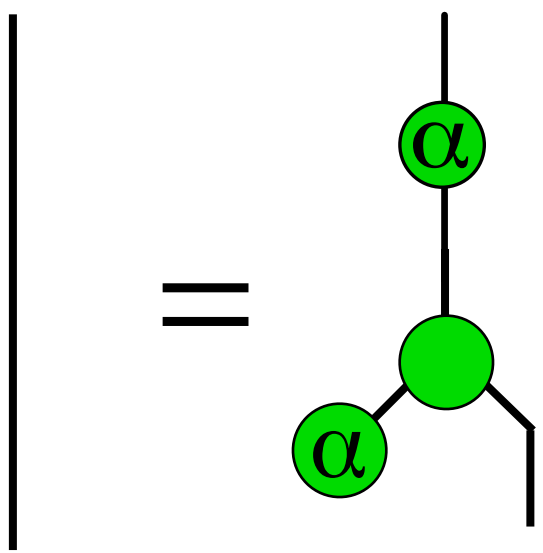

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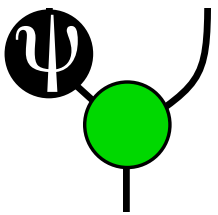
Q: When is 

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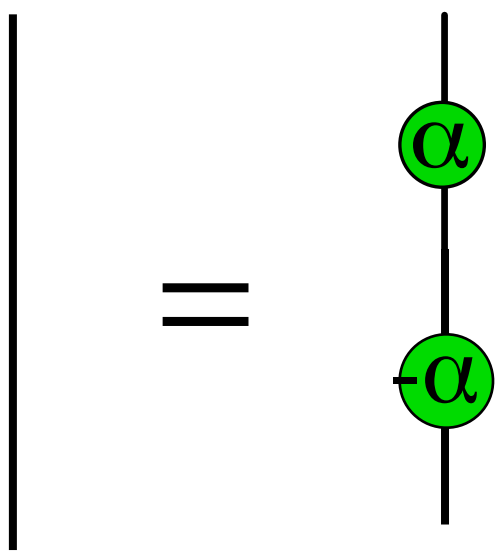

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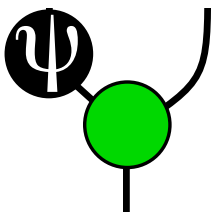
Q: When is  unitary?

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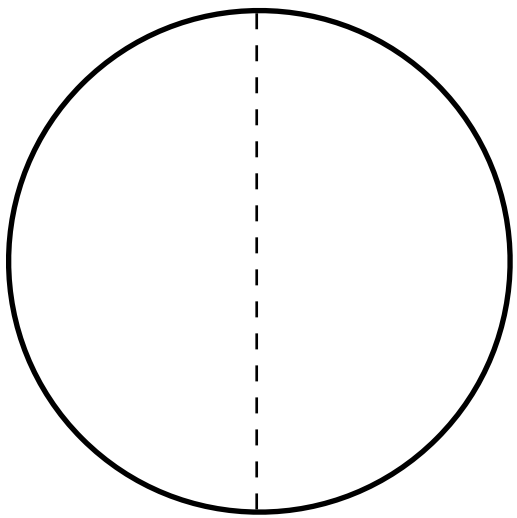
Prop:
1. the unbiased points for (δ, ϵ) form an abelian group w.r.t. $\odot$
2. the arrows generated by the unbiased points form an abelian group w.r.t. composition.

Example: qubits



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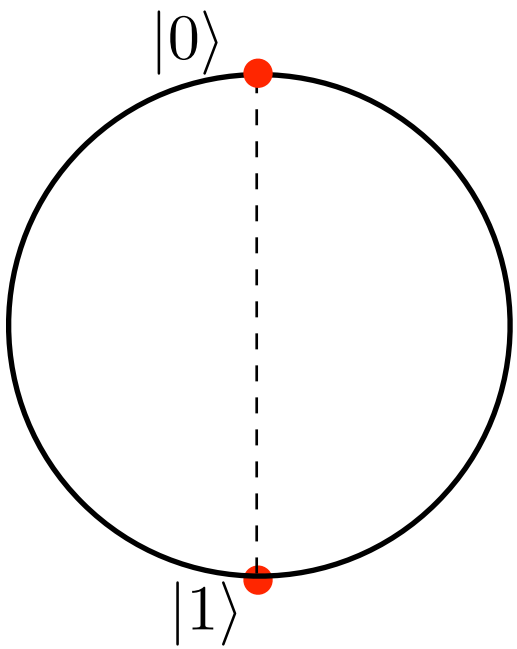

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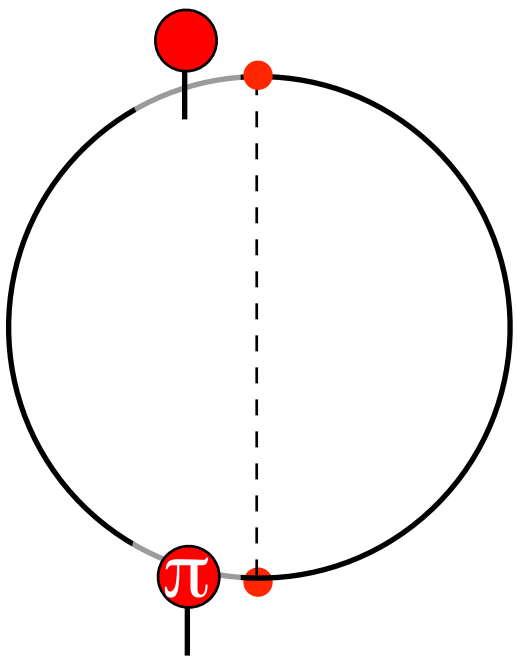

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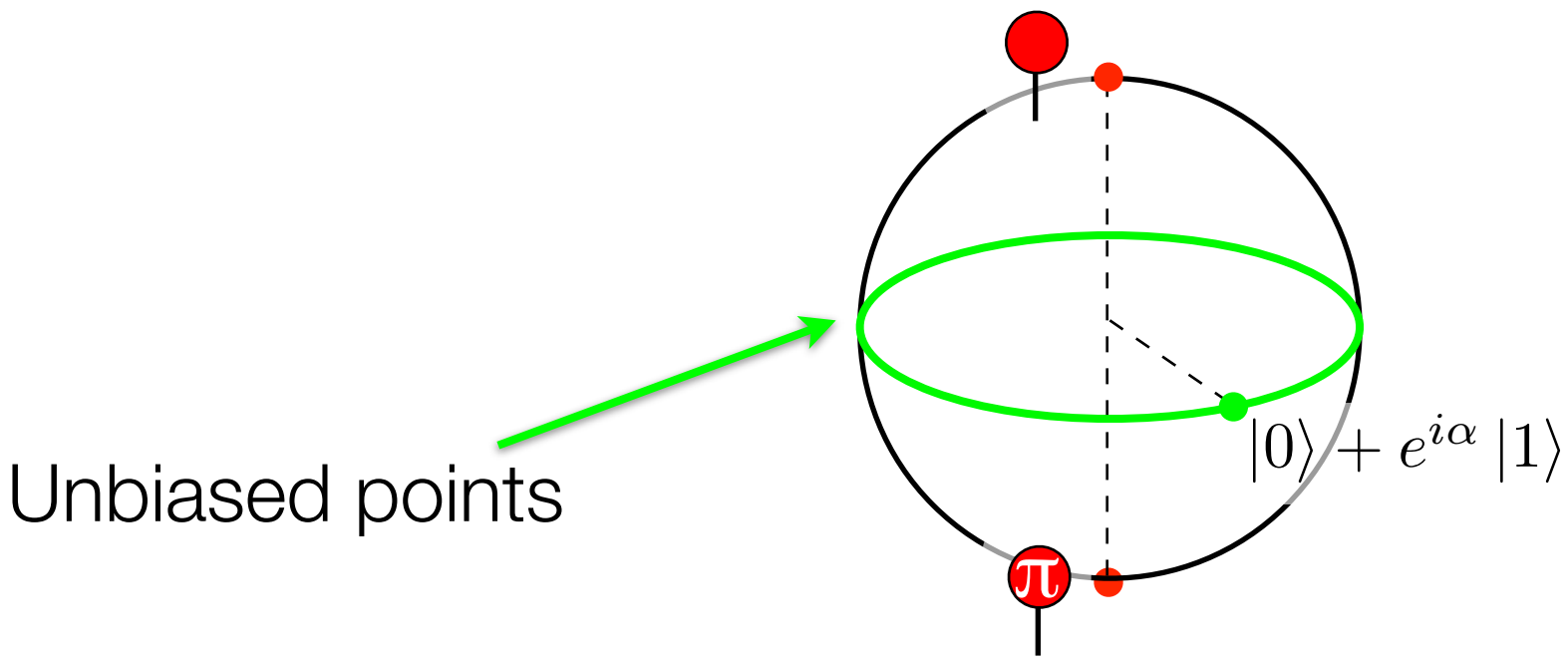
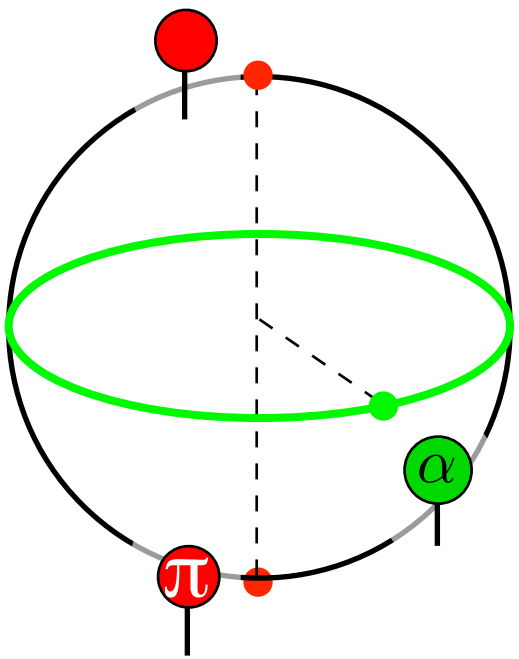


Example: qubits

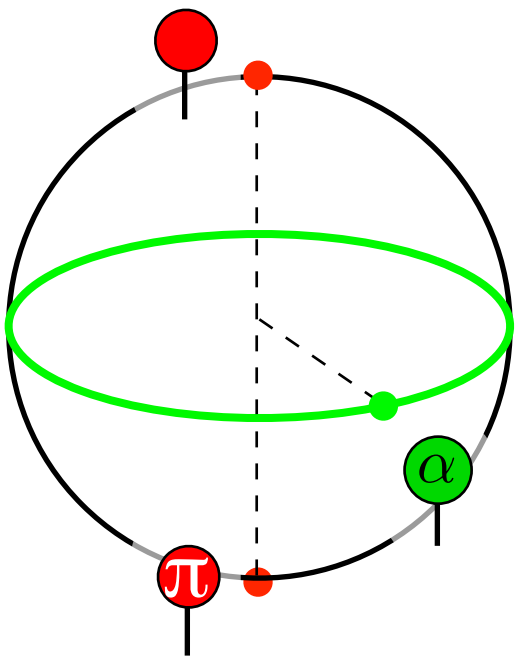
Unbiased points



Example: qubits

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

Unbiased points



Example: $\mathbf{FRel}$

$$D^\dagger : (x, x) \sim x, \forall x \in X$$

$D^\dagger \circ (\mathrm{id} \otimes \psi)$ is unitary iff $\psi = X$

Hence the phase group is trivial.

Example: $\mathbf{FRel}$

Suppose that we are working with a two element set = {0,1}.
Then:

$$D : \begin{array}{l} 0 \sim \{(0,0),(1,1)\} \\ 1 \sim \{(0,1),(1,0)\} \end{array}$$

$$E : 0 \sim *$$

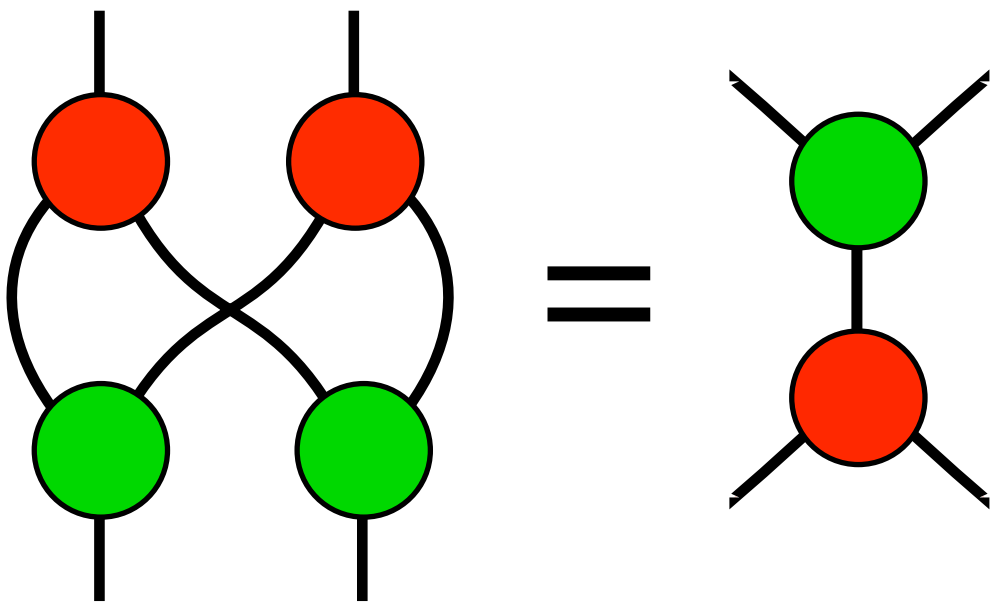
Defines a classical structure whose phase group is $\mathbb{Z}_2$

Example: $\mathbf{FRel}$

Theorem (Pavlovic) : The classical structures of $\mathbf{FRel}$ are exactly biproducts of Abelian groups.

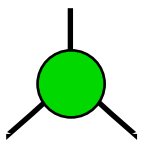
It's easy to check that in this gives a phase group which is just the product group

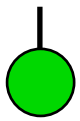
Complementary Observables

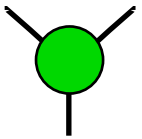


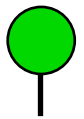
A very general theory of interference

Two kinds of points

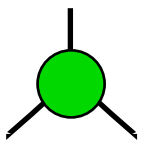
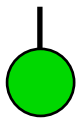
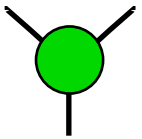
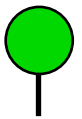
$\delta =$ 

$\epsilon =$ 

$\delta^\dagger =$ 

$\epsilon^\dagger =$ 

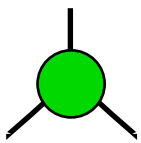
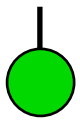
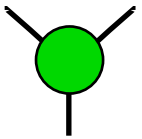
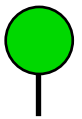
Two kinds of points

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

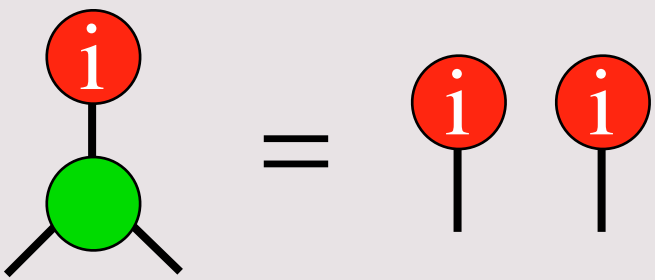
Classical Points

Those points which can be copied by δ

Two kinds of points

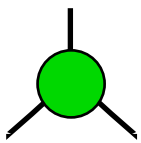
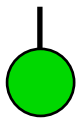
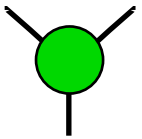
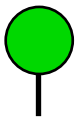
$\delta =$ 
$\epsilon =$ 
$\delta^\dagger =$ 
$\epsilon^\dagger =$ 

Classical Points

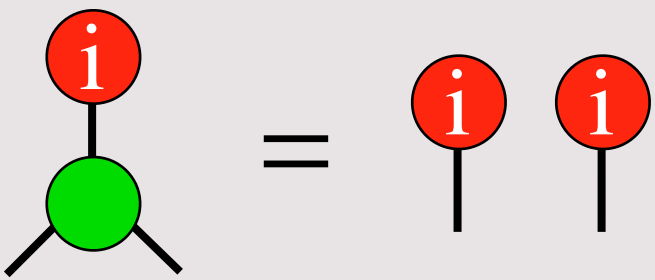


Those points which can be copied by δ

Two kinds of points

$\delta =$ 
$\epsilon =$ 
$\delta^\dagger =$ 
$\epsilon^\dagger =$ 

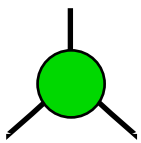
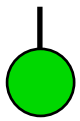
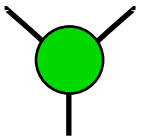
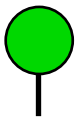
Classical Points



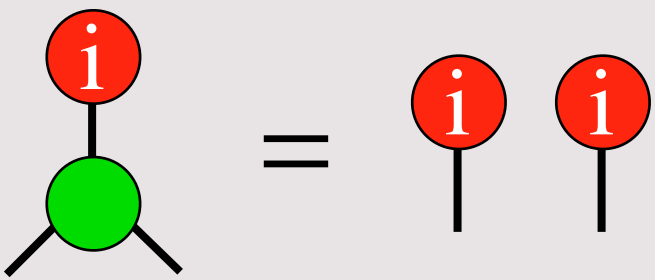
Those points which can be copied by δ

* to keep the pictures tidy, a scalar factor has been omitted (and everywhere from here on)

Two kinds of points

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

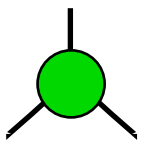
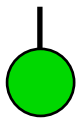
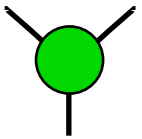
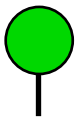
Classical Points



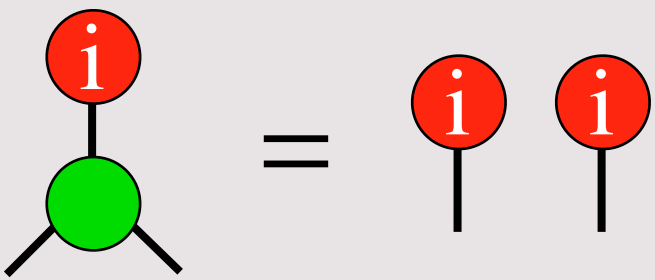
Those points which can be copied by δ

Unbiased Points

Two kinds of points

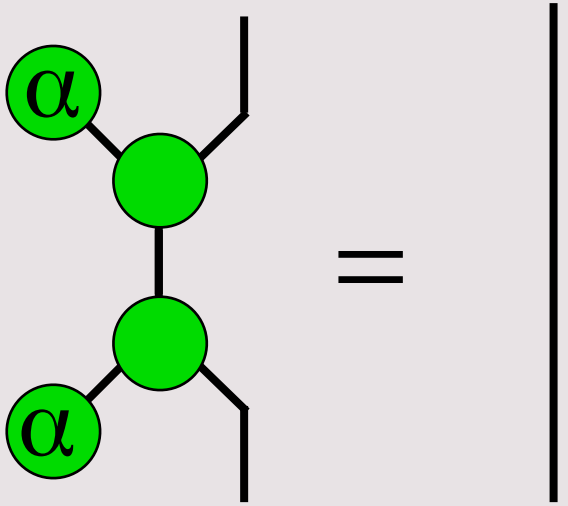
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Classical Points

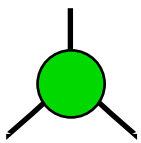
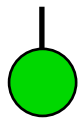
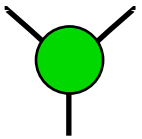
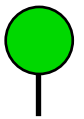


Those points which can be copied by δ

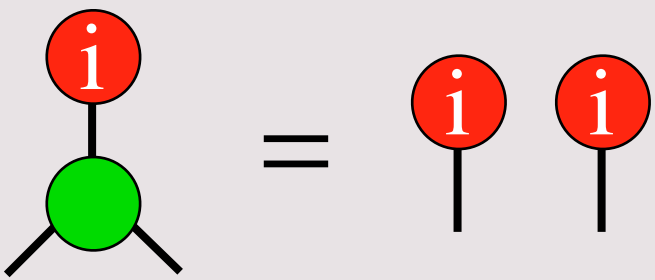
Unbiased Points



Two kinds of points

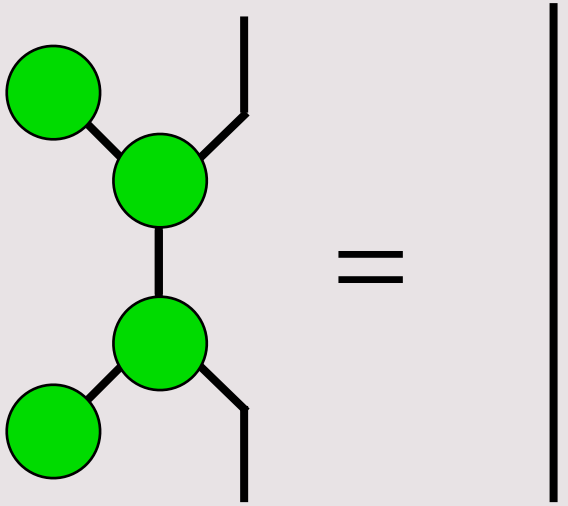
$\delta =$ 
$\epsilon =$ 
$\delta^\dagger =$ 
$\epsilon^\dagger =$ 

Classical Points

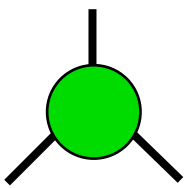
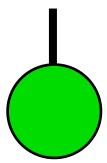


Those points which can be copied by δ

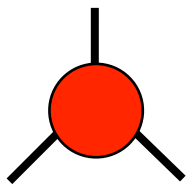
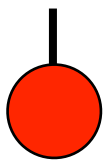
Unbiased Points



Complementary Classical Structures

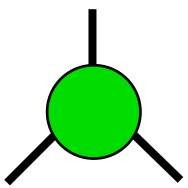
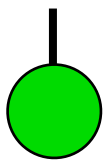
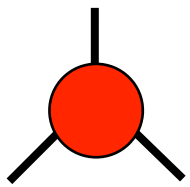
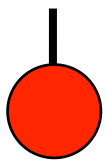
$\delta_Z =$  $\epsilon_Z =$ 

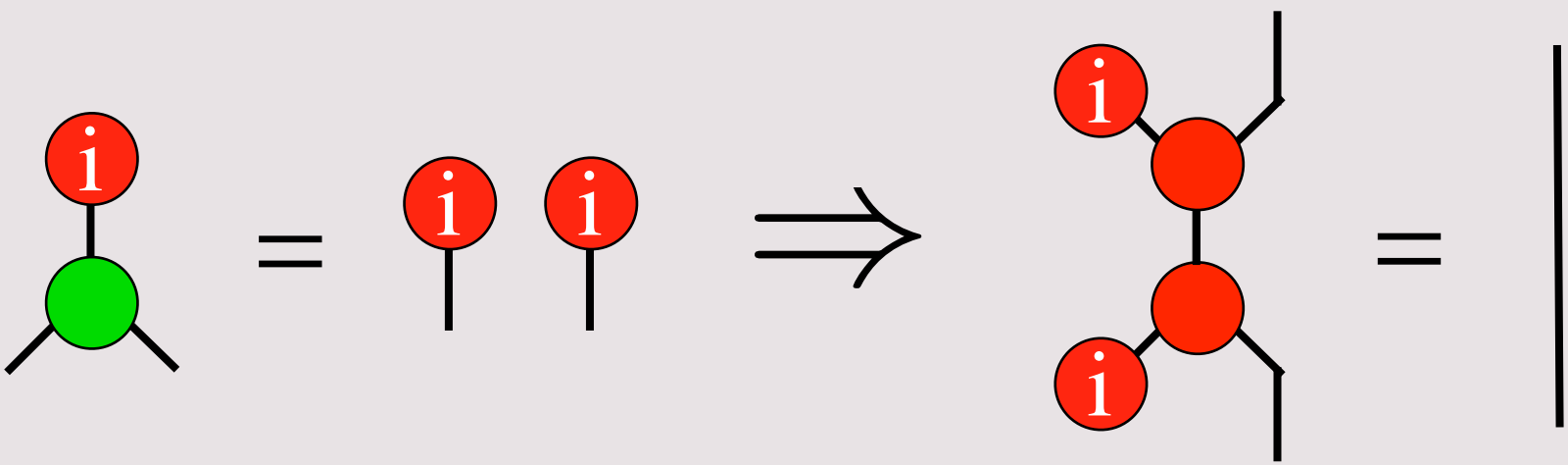
$|i\rangle \mapsto |ii\rangle$ $|+\rangle \mapsto 1$

$\delta_X =$  $\epsilon_X =$ 

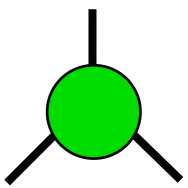
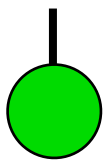
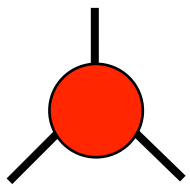
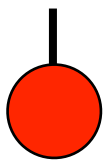
$|\pm\rangle \mapsto |\pm\pm\rangle$ $|0\rangle \mapsto 1$

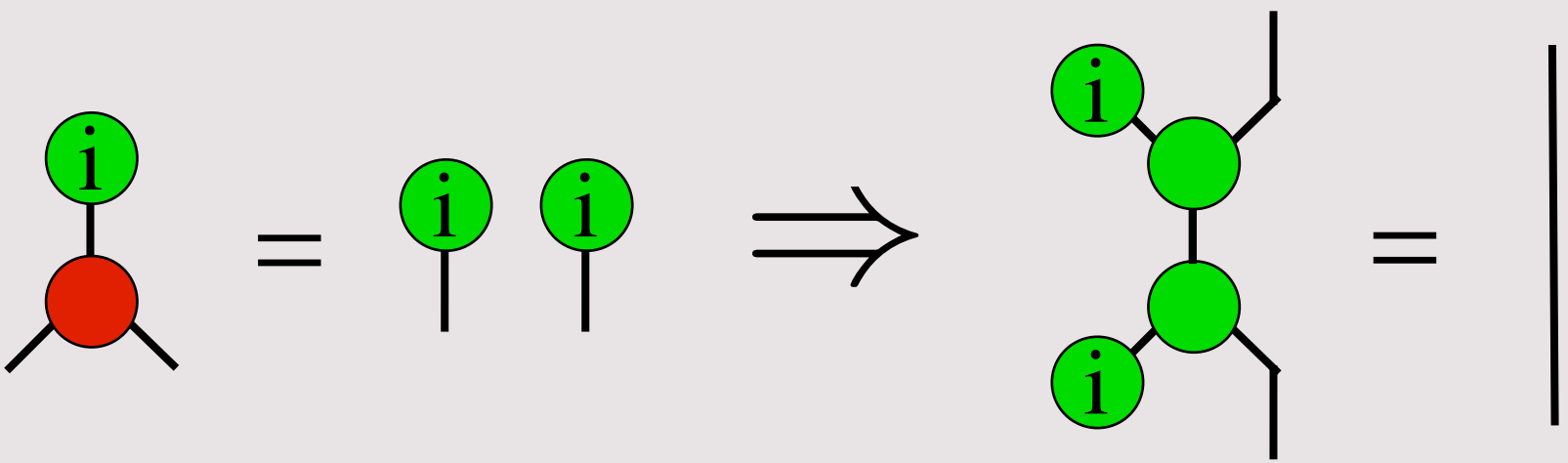
Complementary Classical Structures

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

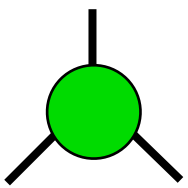
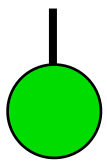


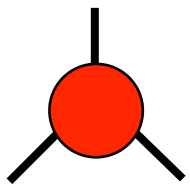
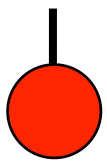
Complementary Classical Structures

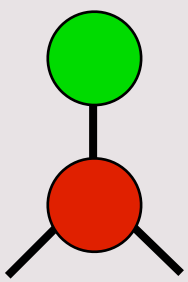
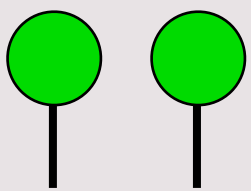
$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

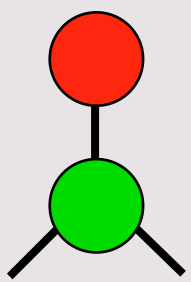
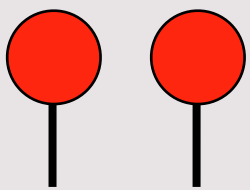


Complementary Classical Structures

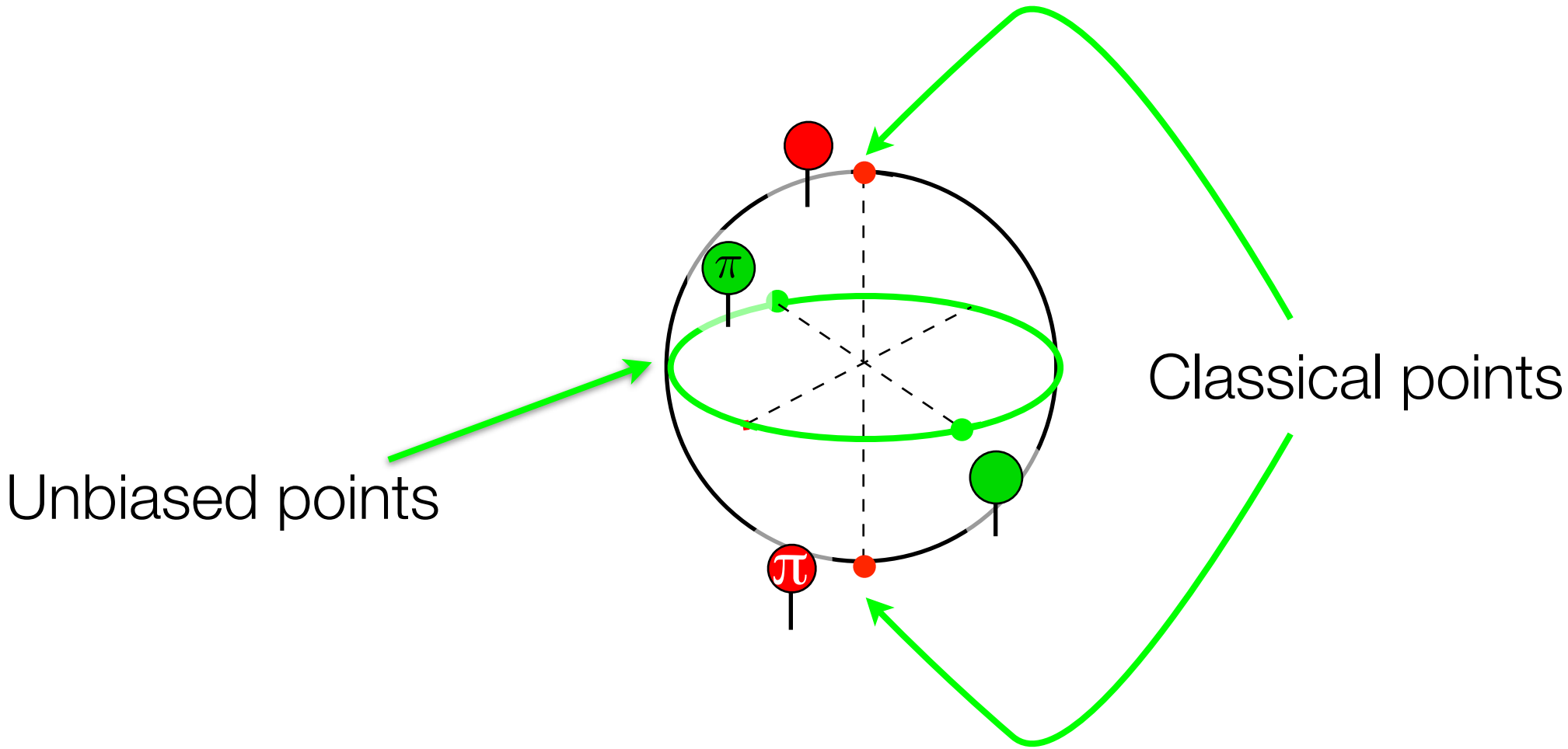
$\delta_Z =$  $\epsilon_Z =$ 

$\delta_X =$  $\epsilon_X =$ 

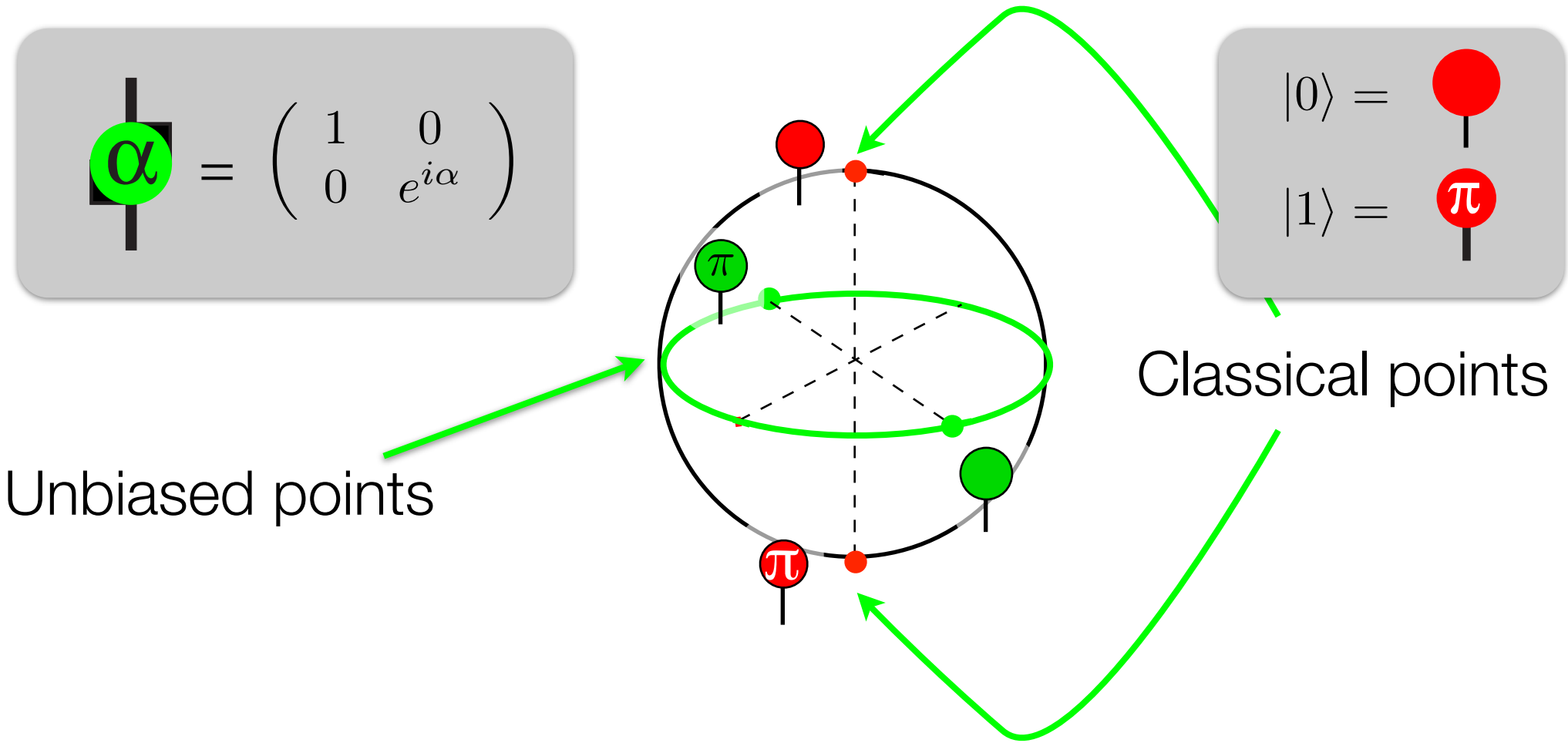
 $=$ 

 $=$ 

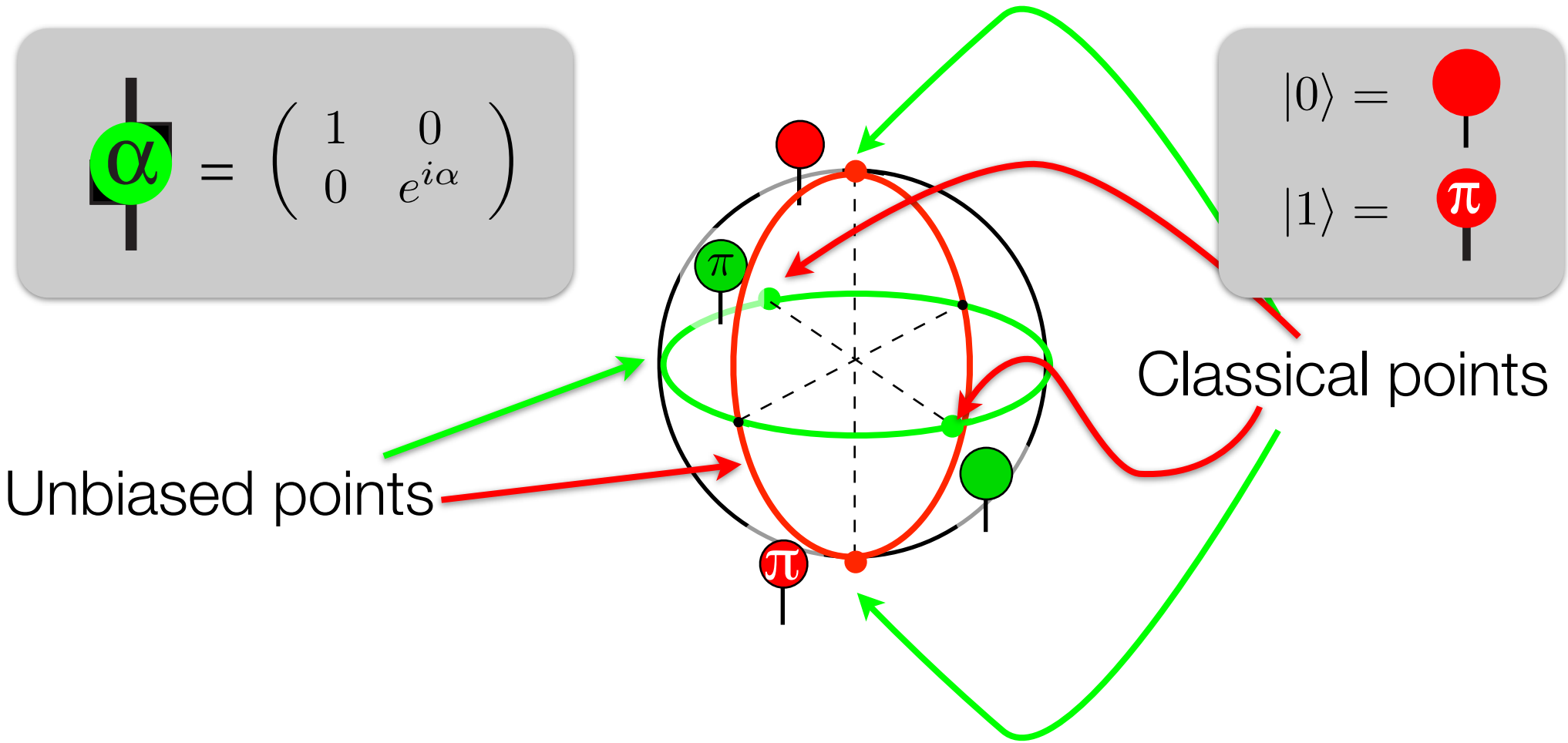
Example: qubits



Example: qubits



Example: qubits

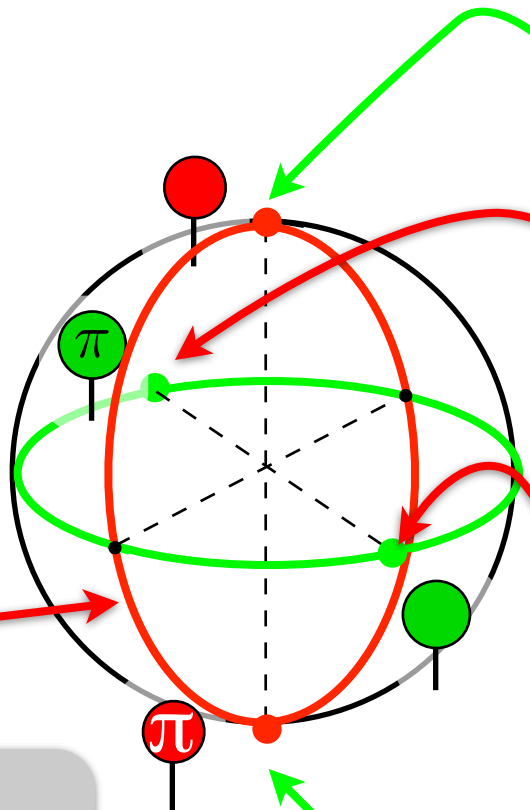


Example: qubits

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$|0\rangle =$$

$$|1\rangle =$$



Classical points

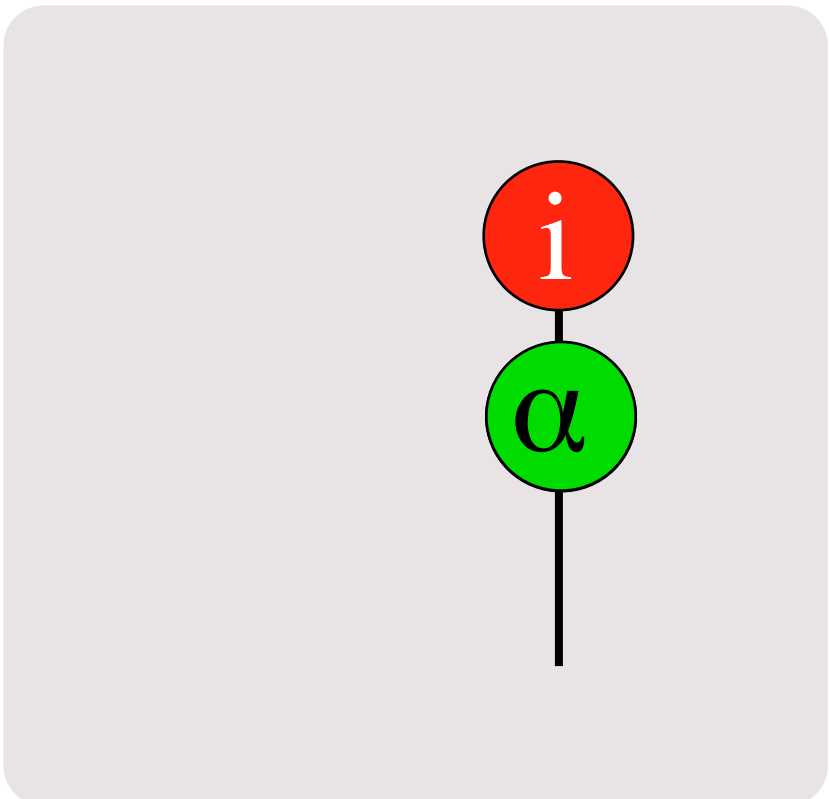
Unbiased points

$$\alpha = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

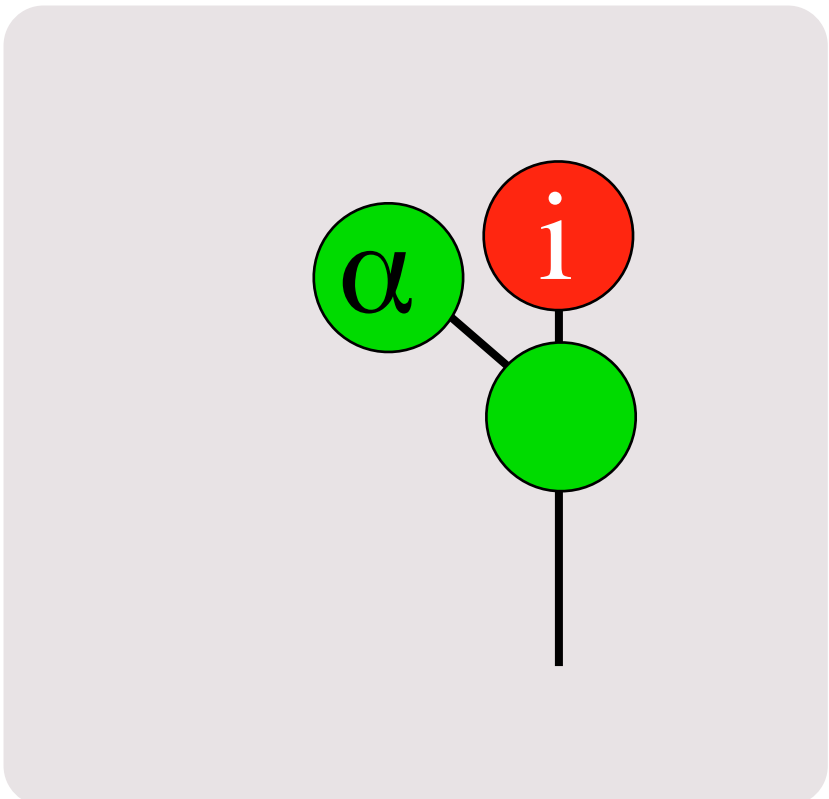
$$|+\rangle =$$

$$|-\rangle =$$

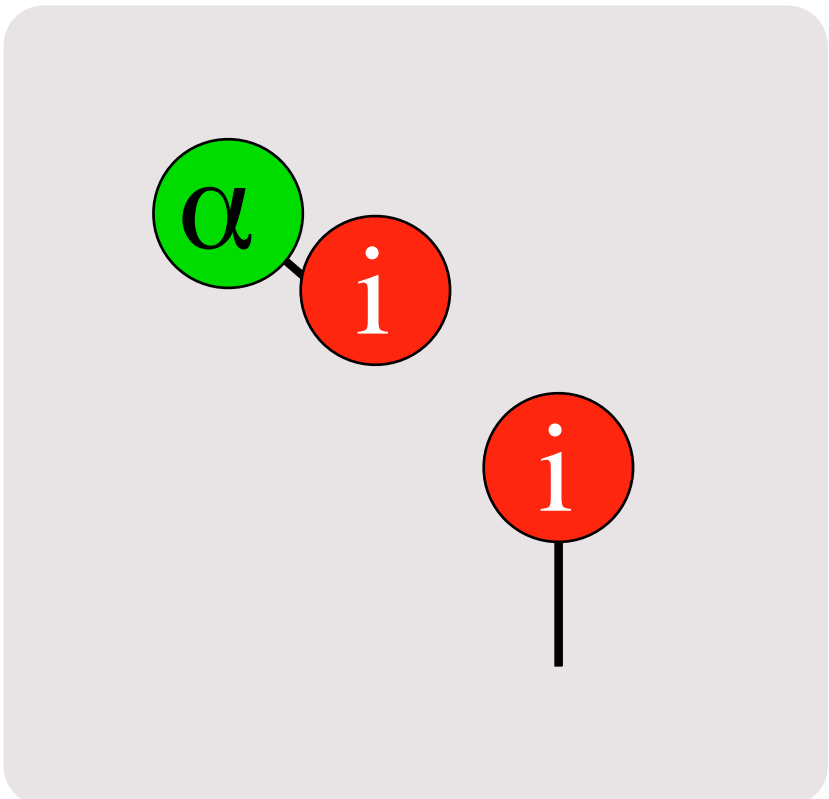
Classical points are eigenvectors



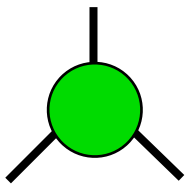
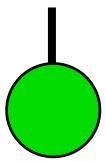
Classical points are eigenvectors

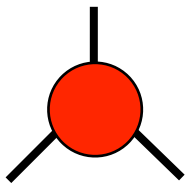
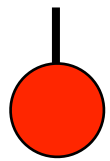


Classical points are eigenvectors

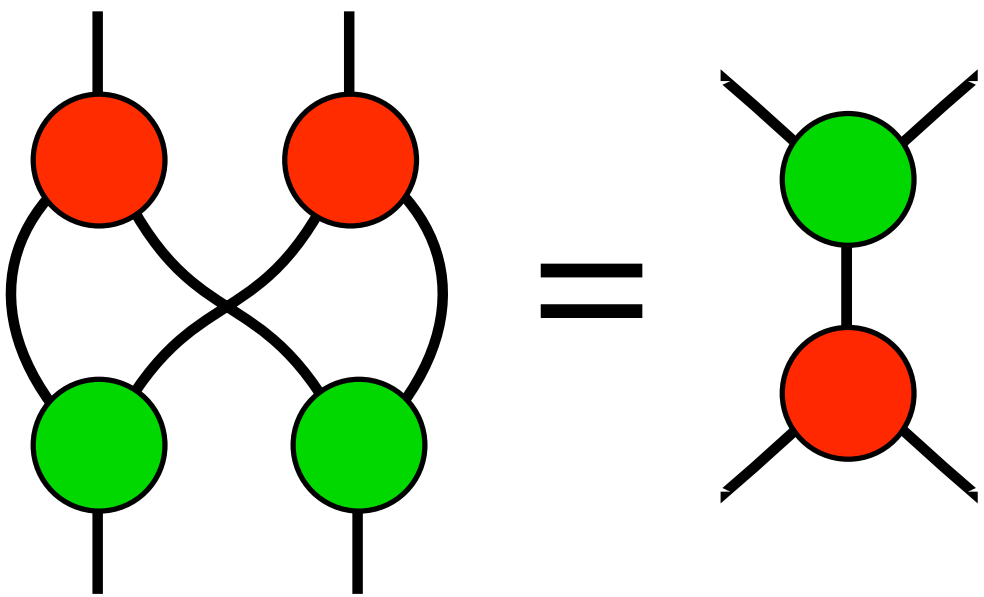


Strong Complementarity

$\delta_Z =$  $\epsilon_Z =$ 

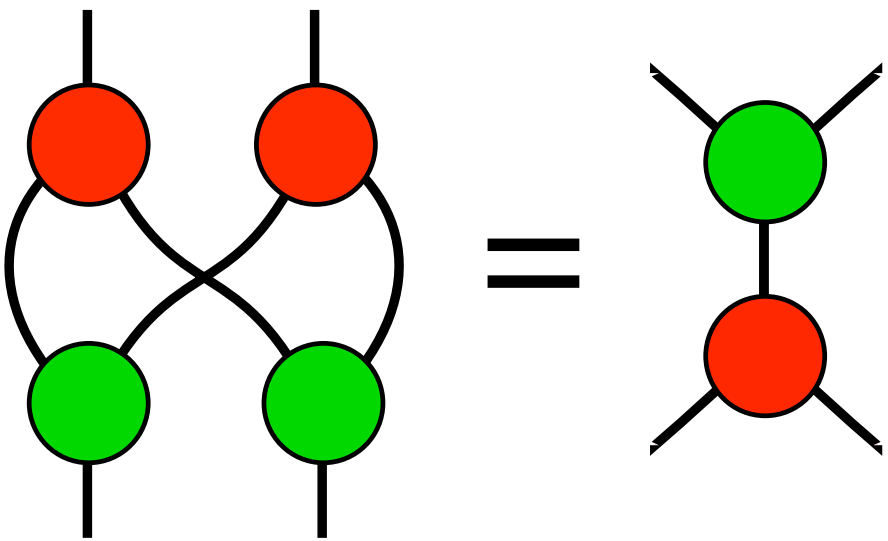
$\delta_X =$  $\epsilon_X =$ 

A pair of complementary observables are called *strongly complementary* when they satisfy this equation:



Strong Complementarity

Corollary: strongly complementary observables form a *bialgebra*



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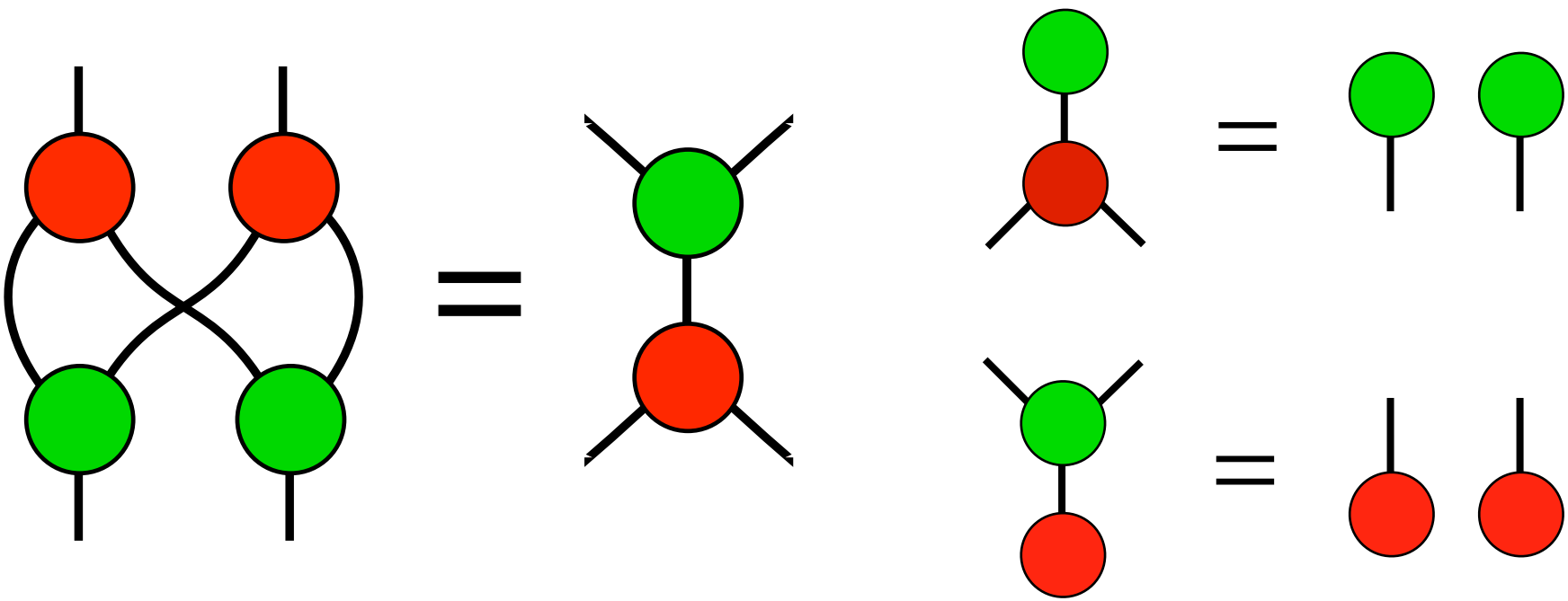
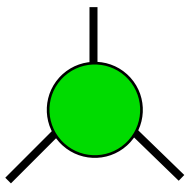
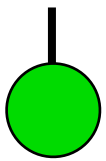
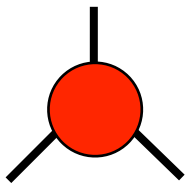
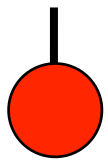
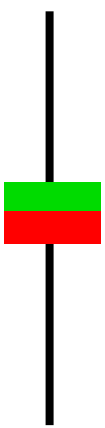
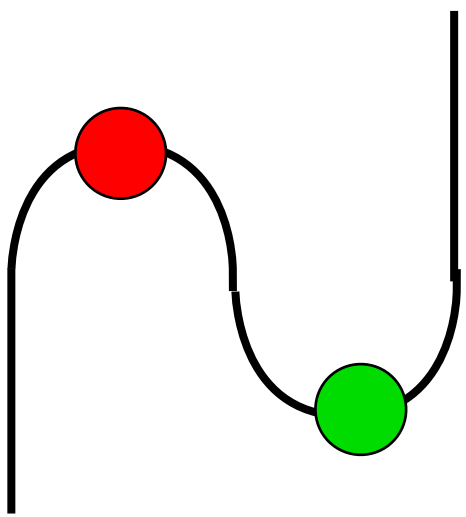


The antipode

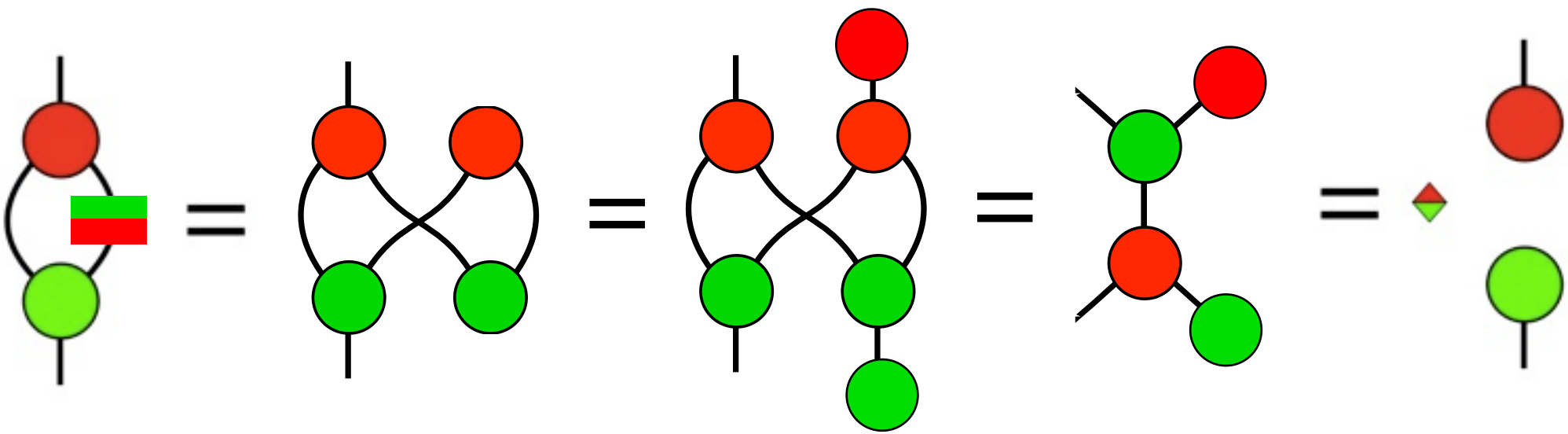
$\delta_Z =$  $\epsilon_Z =$ 

$\delta_X =$  $\epsilon_X =$ 

Define the map S by:

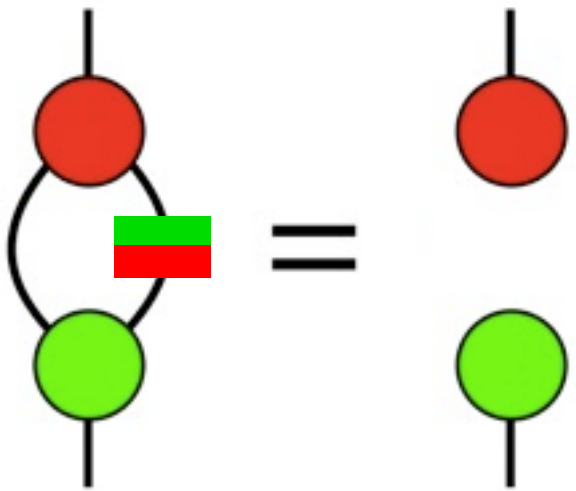
$S =$  $:=$ 

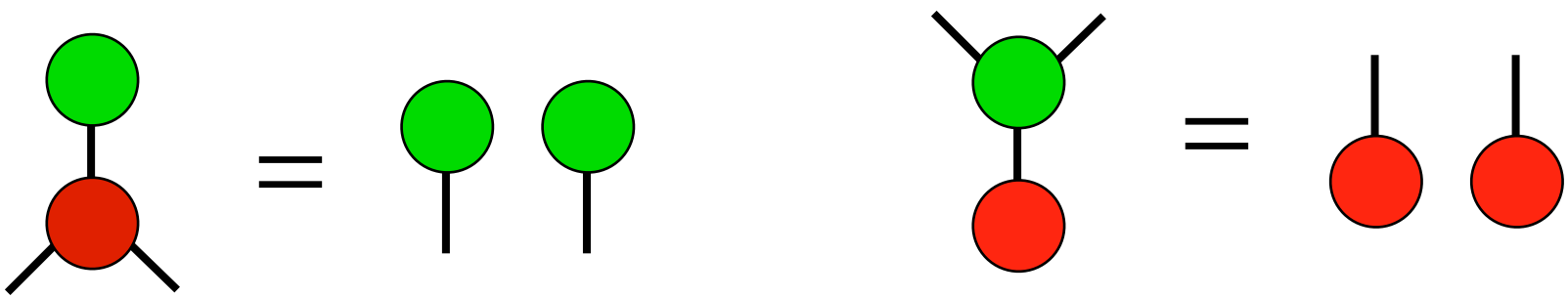
The antipode



Strongly complementary observables form Hopf algebras

Theorem:



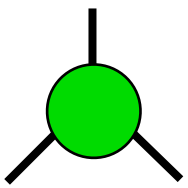
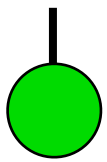


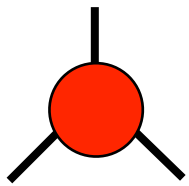
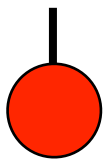
Remark: under the assumption of *enough classical points* the “strong” assumption is not needed; simple complementarity suffices

Strong Complementarity

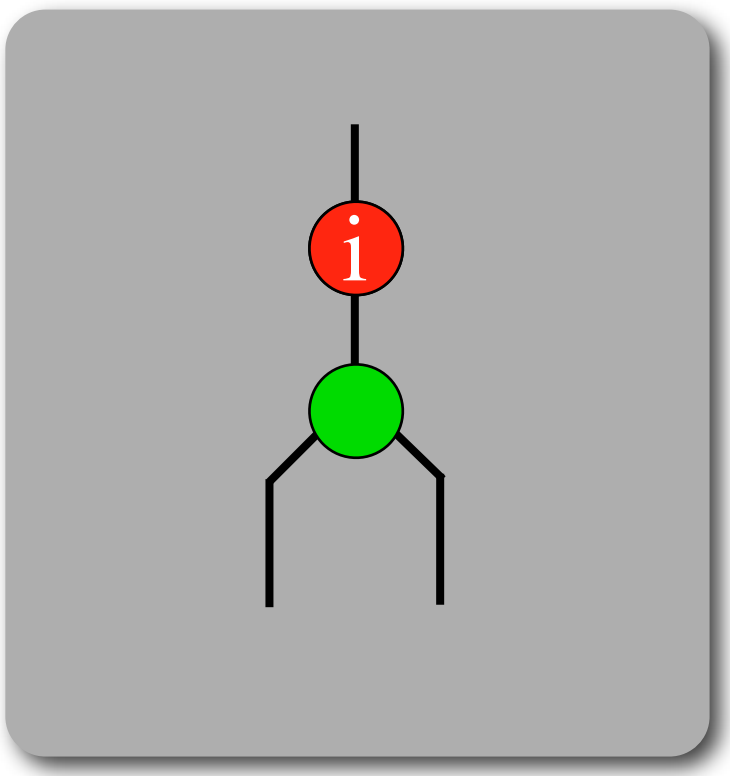
The following are equivalent characterisations of strong complementarity:

Comonoid Homomorphism

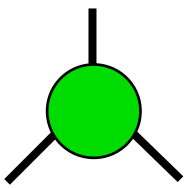
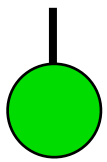
$\delta_Z =$  $\epsilon_Z =$ 

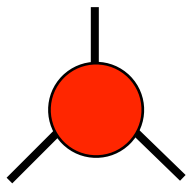
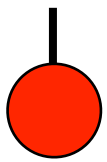
$\delta_X =$  $\epsilon_X =$ 

The classical maps are comonoid homomorphisms



Comonoid Homomorphism

$\delta_Z =$  $\epsilon_Z =$ 

$\delta_X =$  $\epsilon_X =$ 

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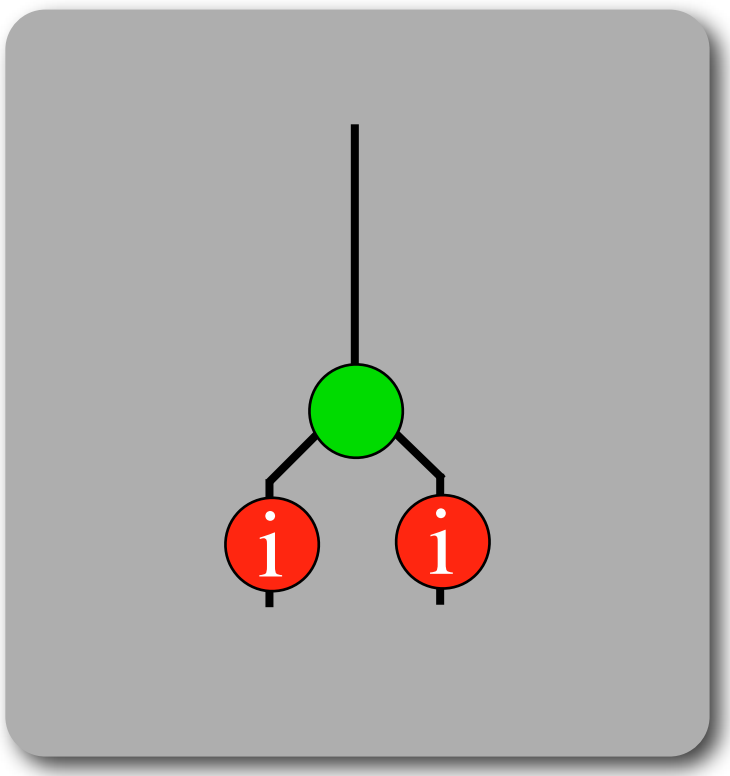
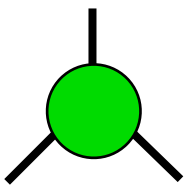
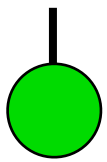
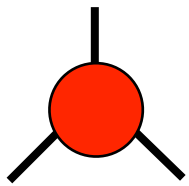
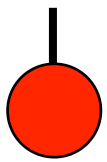
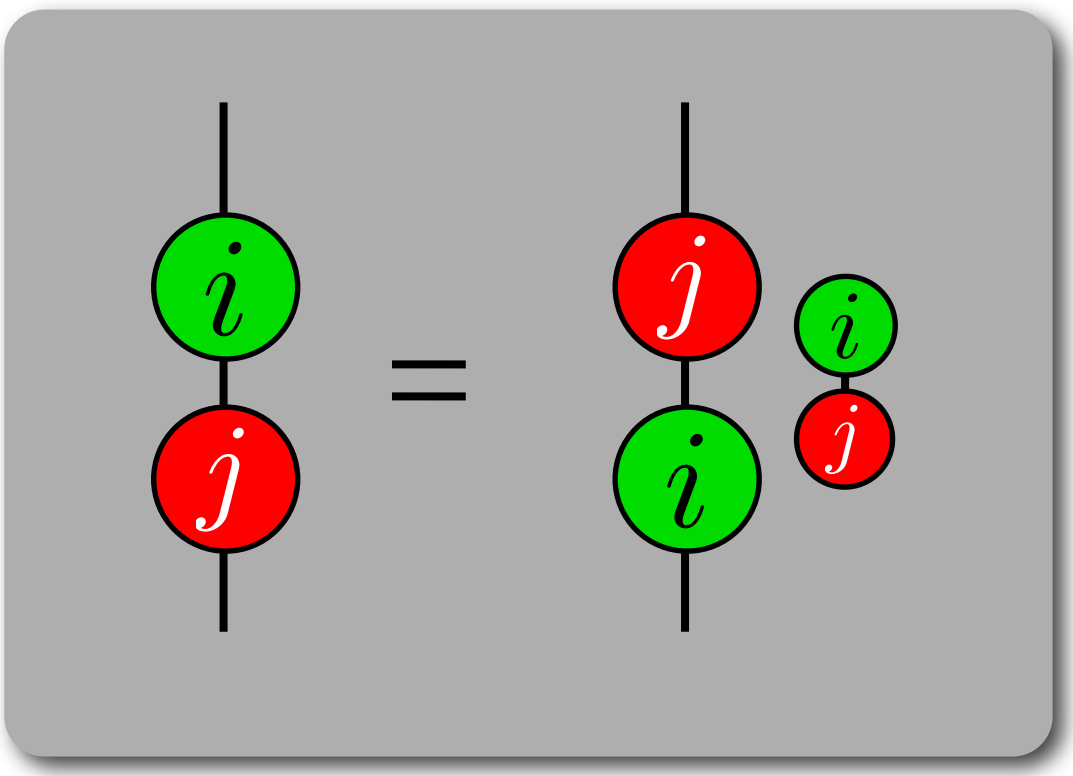


Classical Phases Commute

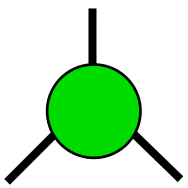
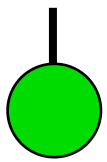
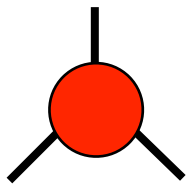
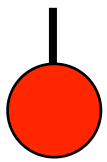
$\delta_Z =$  $\epsilon_Z =$ 

$\delta_X =$  $\epsilon_X =$ 

The classical maps satisfy canonical commutation relations

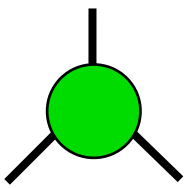
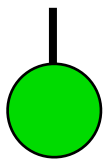
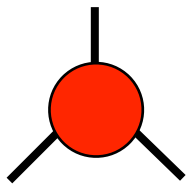
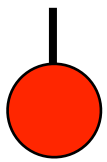


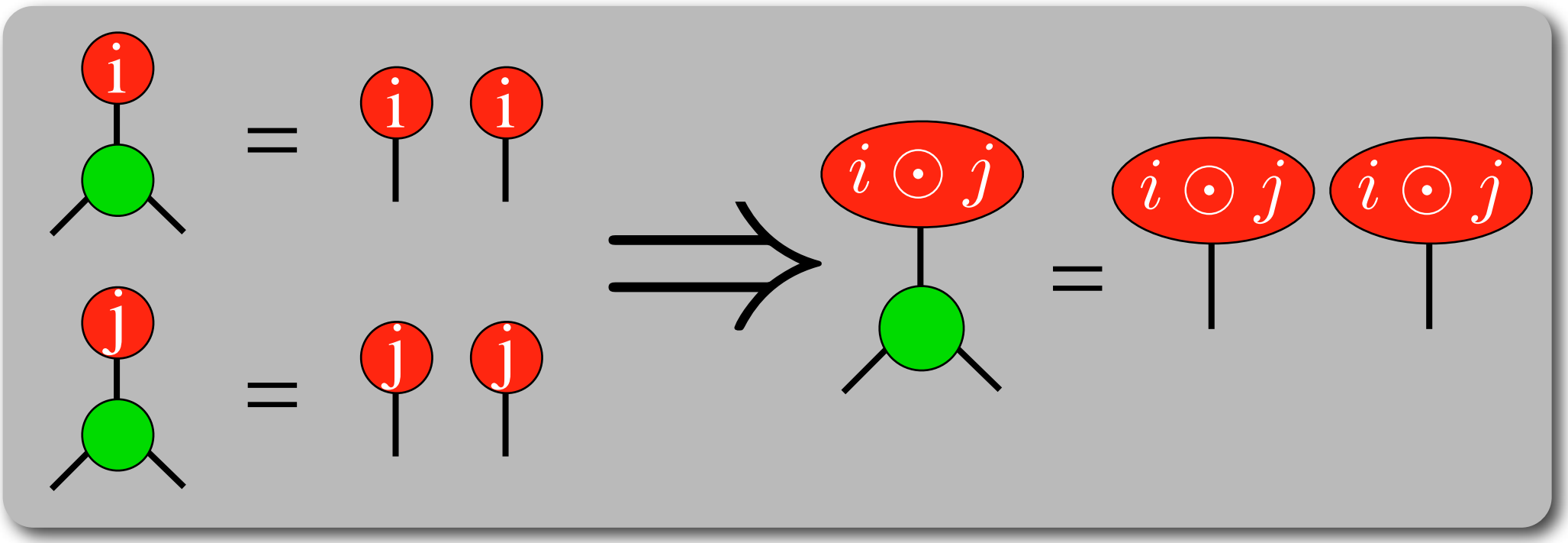
Closedness Property

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

Defn: complentary classical structures are called *closed when...*

Closedness Property

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 



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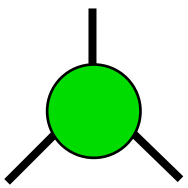
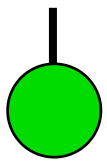
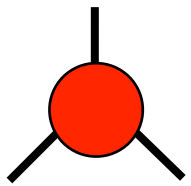
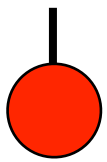
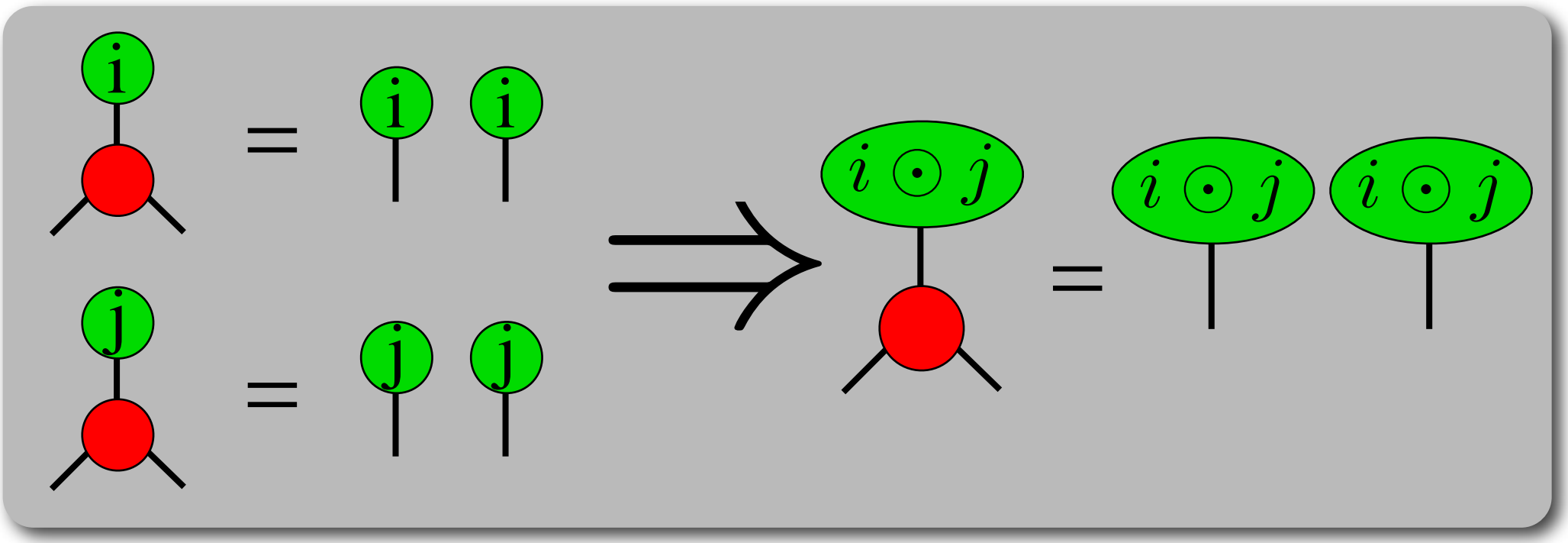
$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 



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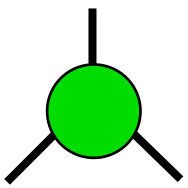
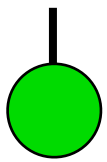
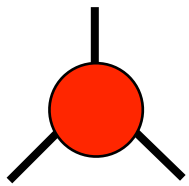
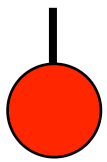
$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

In **fdHilb** it is always possible to construct a pair of strong complementary observables...

... but in big enough dimension, it is possible to construct MUBs which are not closed.

Classical Points form a Subgroup

By the definition of complementarity, we have that:

$$C_X \subseteq U_Z$$
$$C_Z \subseteq U_X$$

Prop: If the observable structure is closed, and there are finitely many classical points, then they form a *subgroup* of the unbiased points, i.e.

$$(C_Z, \odot_X) \leq (U_X, \odot_X)$$
$$(C_X, \odot_Z) \leq (U_Z, \odot_Z)$$

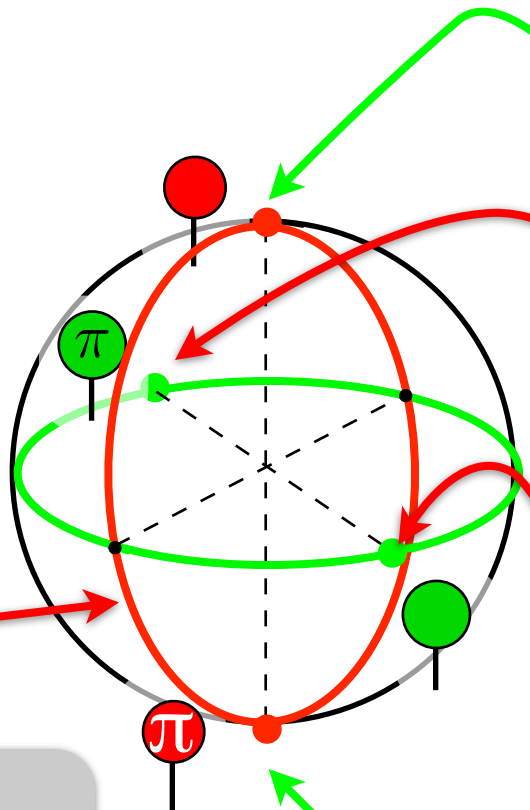
In particular, each $\Lambda^X(z_i)$ is a permutation on C_Z.

Example: qubits

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$|0\rangle =$$

$$|1\rangle =$$



Classical points

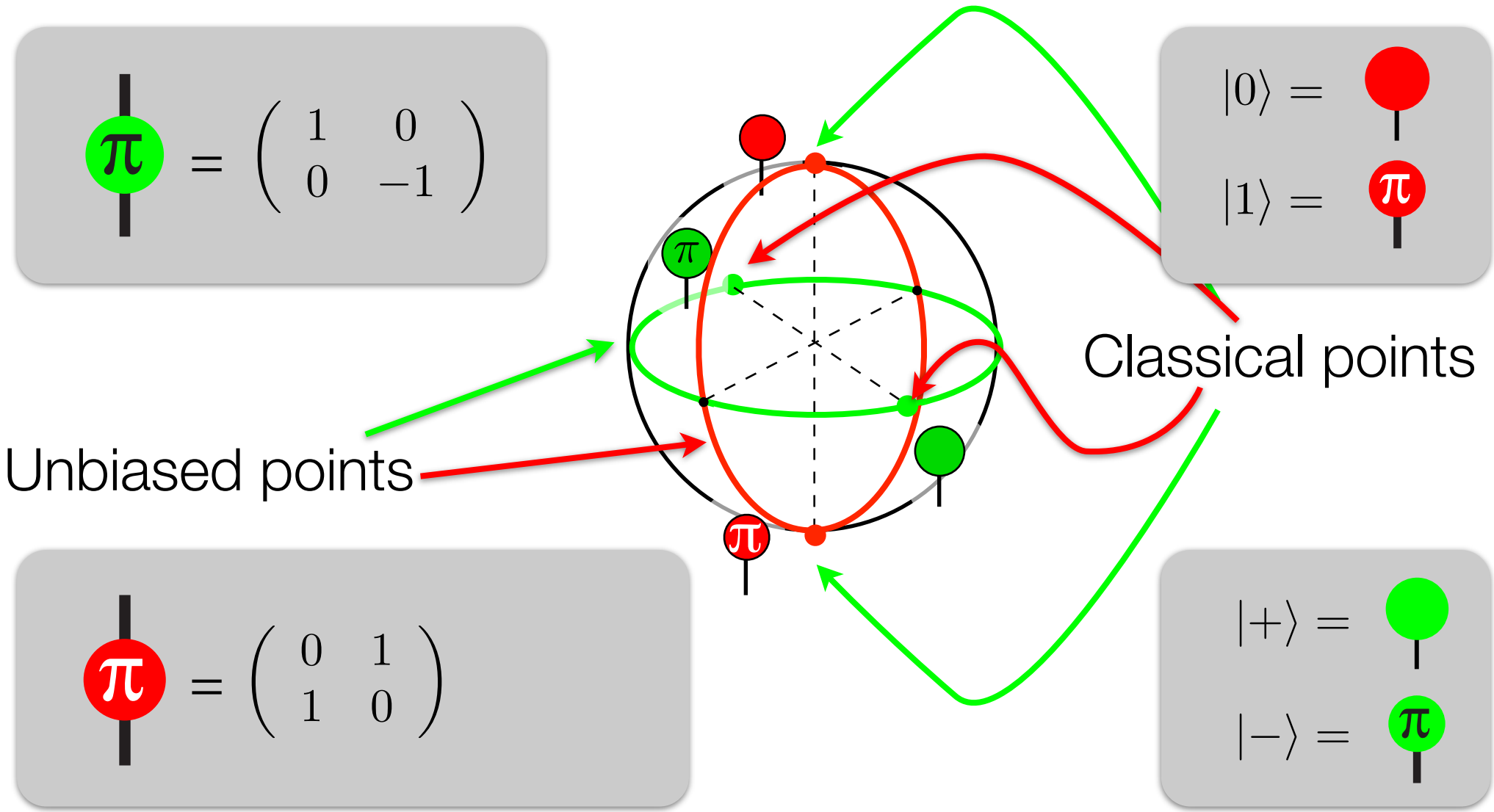
Unbiased points

$$\alpha = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$|+\rangle =$$

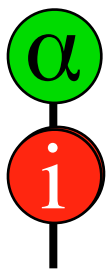
$$|-\rangle =$$

Example: qubits



Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:



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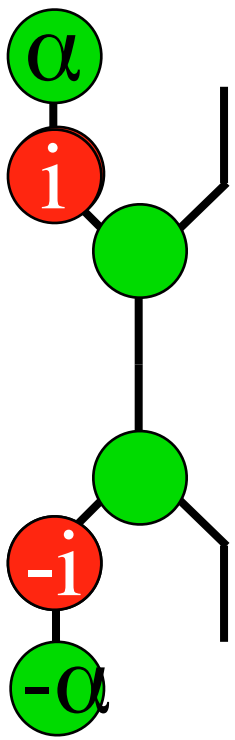

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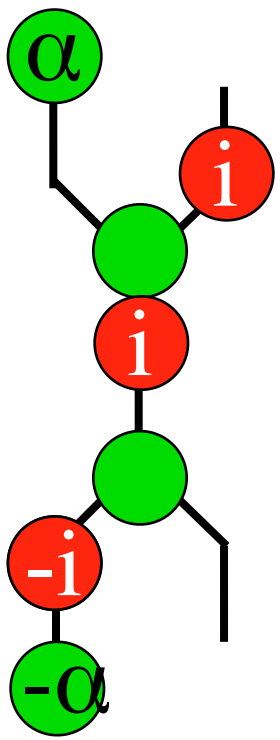

Automorphism Action

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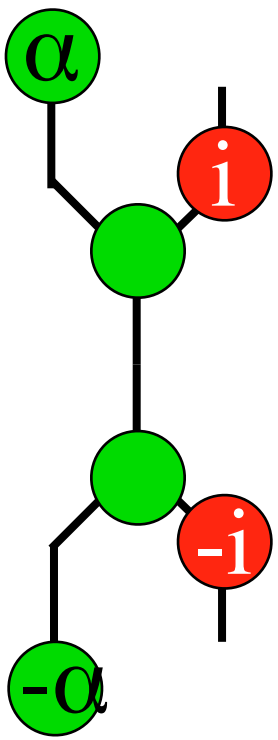

Automorphism Action

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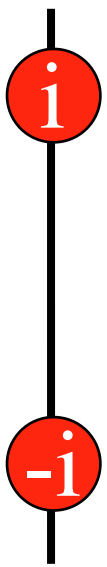

Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:

Automorphism Action

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Automorphism Action

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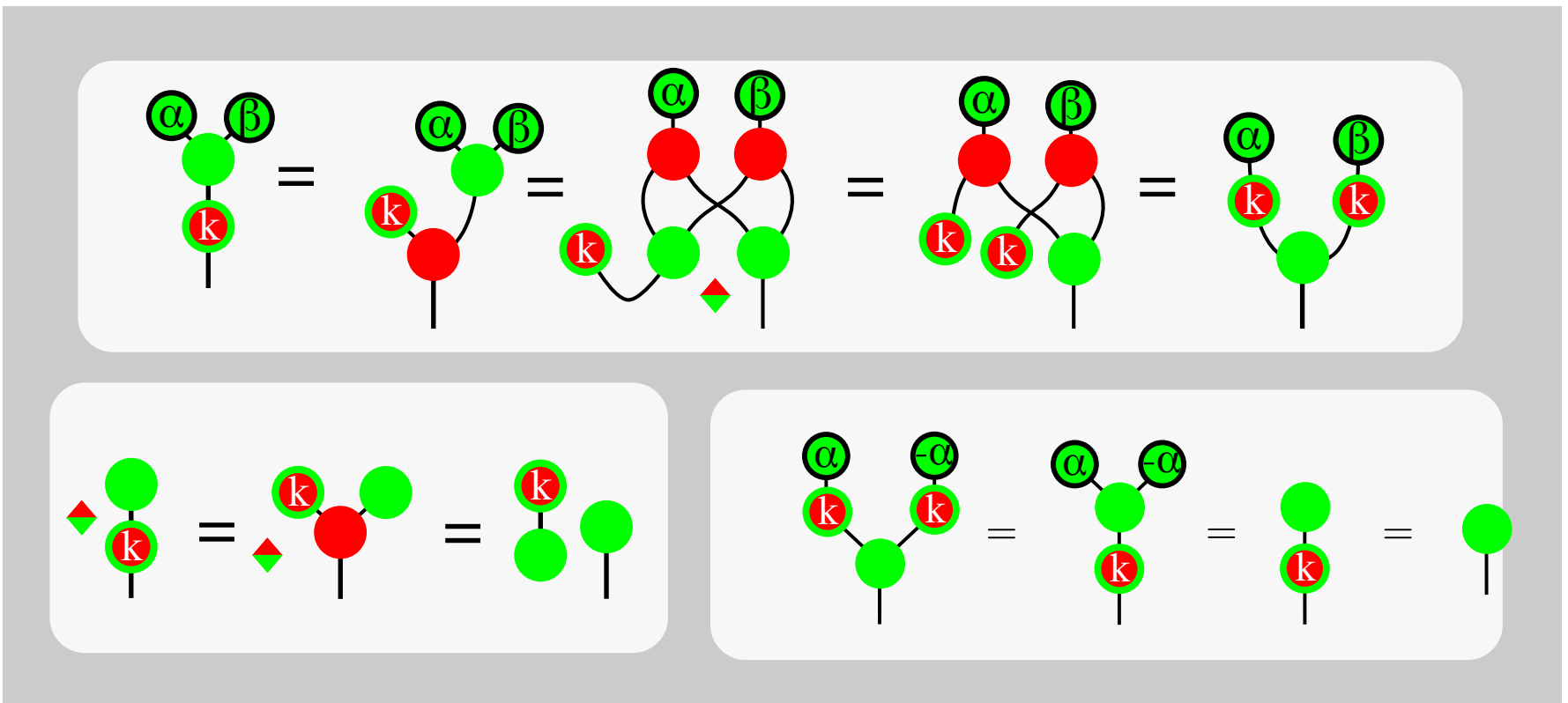


Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:

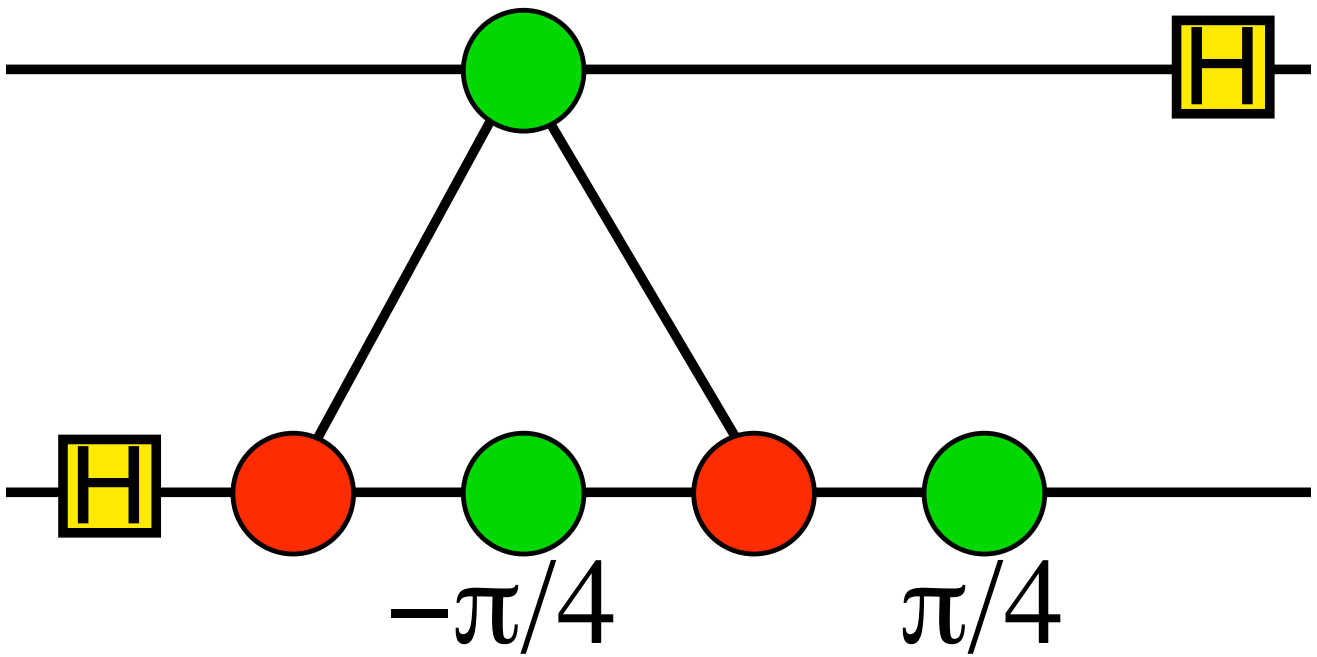
Classical points are symmetries of the phase group.

Why care about phase groups?

The phase group seems like a rather random structure. Why do we care? The phase group:

- ... determines the underlying *arithmetic*
- ... fixes the possibility for *interference*
- ... determines the possible results of CHSH-type experiments, and hence the possibility for non-locality.

The ZX-calculus

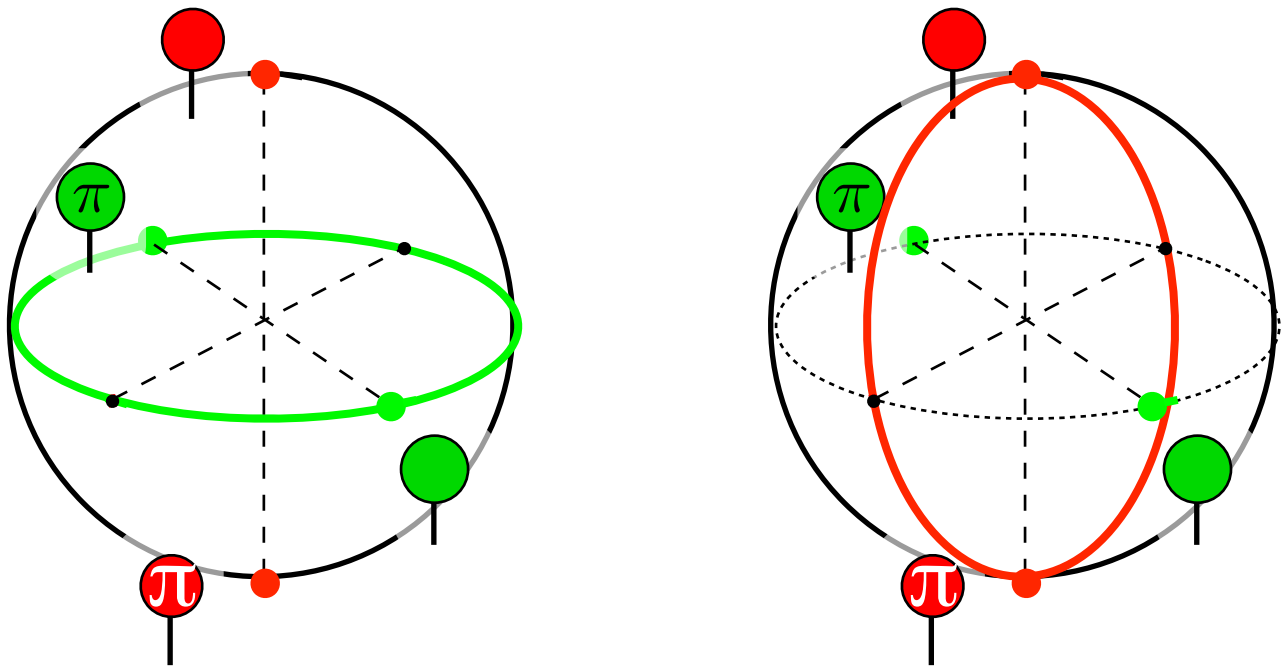


Quantum processes, diagrammatically

Z and X Observables

The Pauli **Z** and **X** observables are strongly complementary, with some additional features:

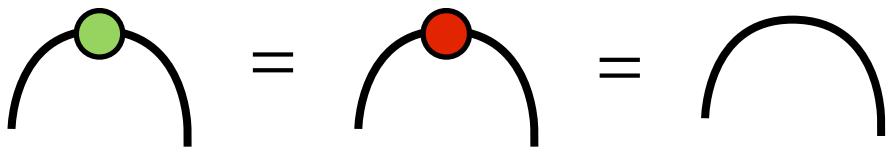
- The phase group is $[0,2\pi)$



- Dimension 2 implies two classical points
- The action of the classical group is $\alpha \mapsto -\alpha$

Z and X Observables

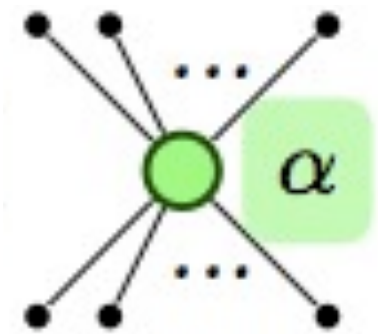
- Both observables generate the same compact structure

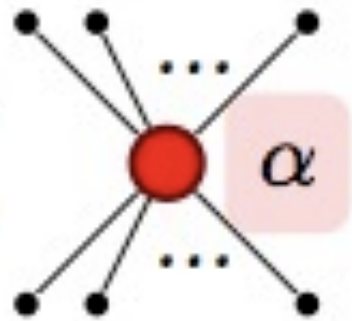


- can just treat the diagram as an undirected graph

- the Z and X are related by a definable unitary
  - gives rise to colour change rule

ZX-calculus syntax



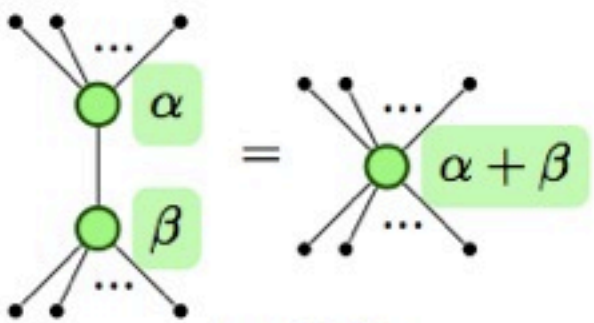


Defn: A *diagram* is an undirected open graph generated by the above vertices.

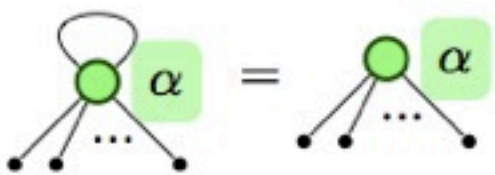
Defn: let $\mathbb{D}$ be the dagger compact category of diagrams s.t.

$$(\cdot)\dagger : \alpha \mapsto -\alpha$$

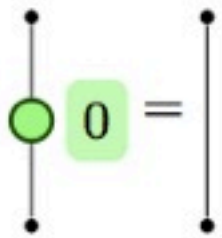
Equations



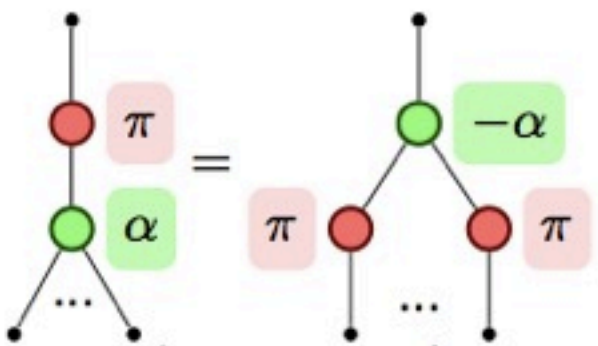
(spider)



(anti-loop)

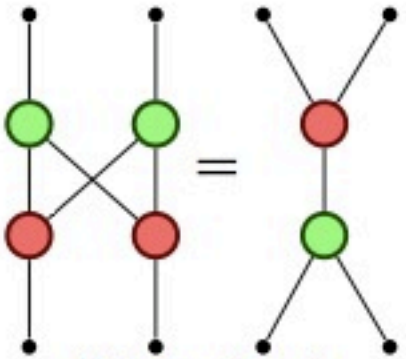


(identity)

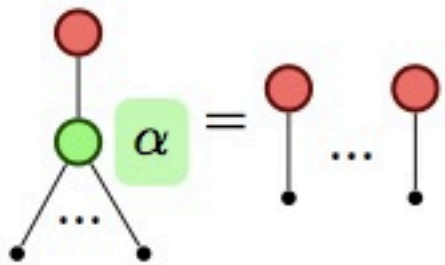


(π-commute)

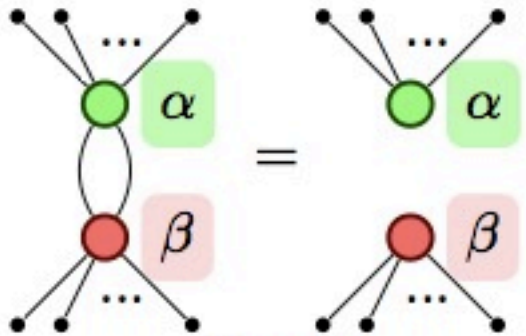
Equations



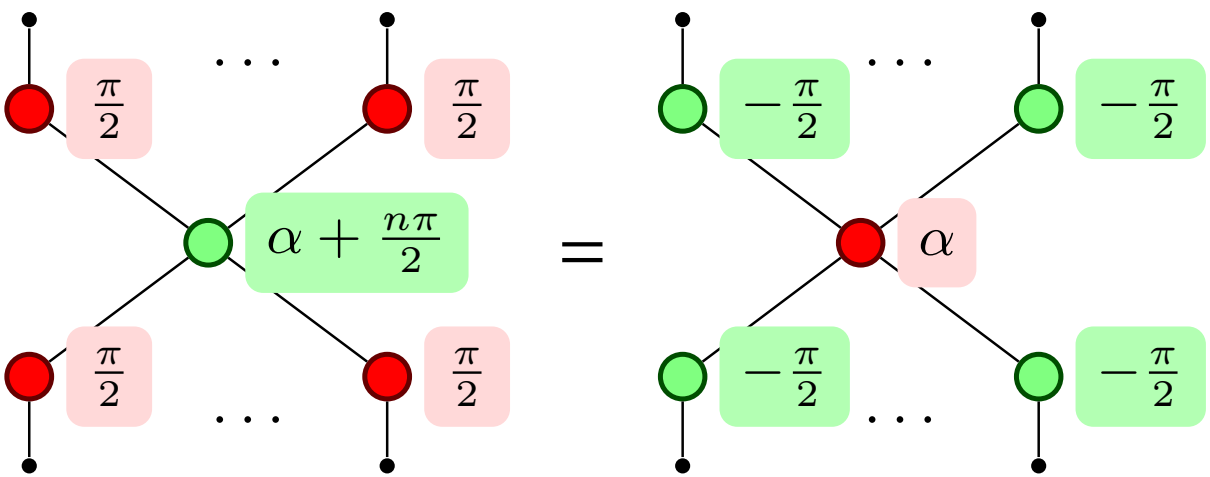
(bialgebra)



(copying)



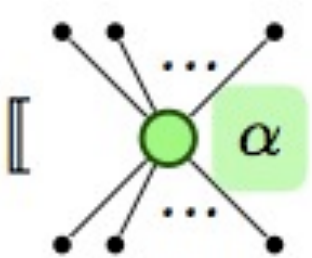
(hopf)



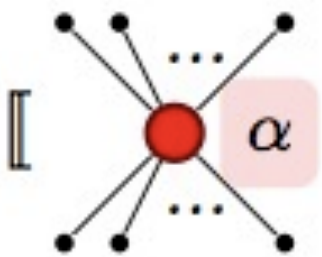
(colour change)

ZX-calculus semantics

Defn: Let $[\![\cdot]\!] : \mathbb{D} \to \boldsymbol{FDHilb}$ be the traced monoidal functor such and that $[\![A]\!] = \mathbb{C}^{2^n}$ and define its action on generators by:



$$[\![\quad]\!] = \begin{cases} |0\rangle^{\otimes m} \mapsto |0\rangle^{\otimes n} \\ |1\rangle^{\otimes m} \mapsto e^{i\alpha} |1\rangle^{\otimes n} \end{cases}$$



$$[\![\quad]\!] = \begin{cases} |+\rangle^{\otimes m} \mapsto |+\rangle^{\otimes n} \\ |-\rangle^{\otimes m} \mapsto e^{i\alpha} |-\rangle^{\otimes n} \end{cases}$$

Representing Qubits

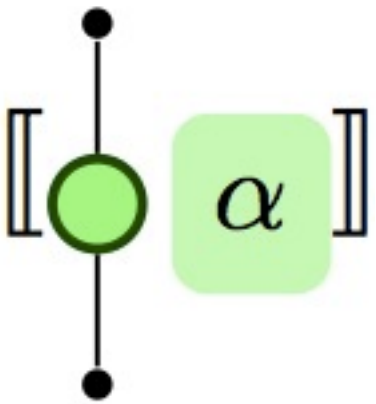
$$[\![\; \overset{\bullet}{|} \;]\!] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$[\![\; \overset{\bullet}{|}\,\pi \;]\!] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

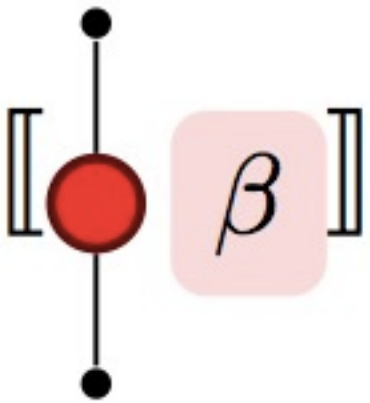
$$[\![\; \overset{\bullet}{|} \;]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

$$[\![\; \overset{\bullet}{|}\,\pi \;]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$$

Representing Phase shifts

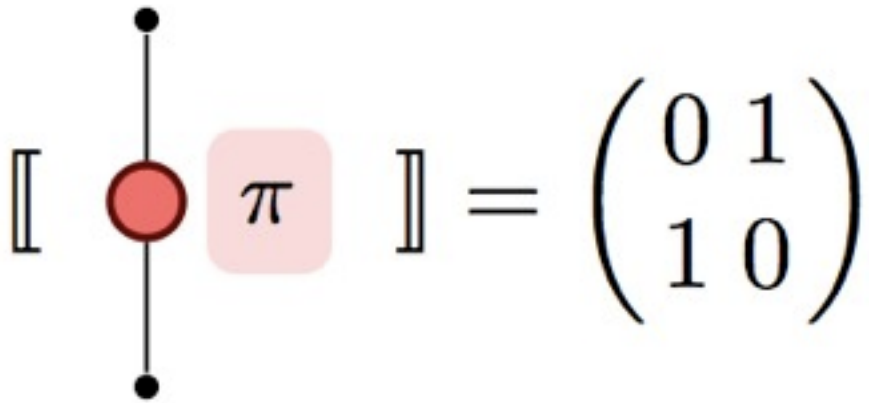


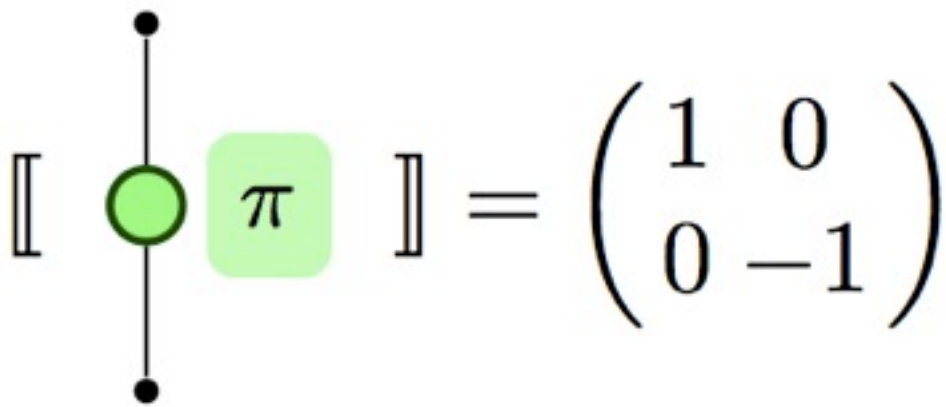
$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



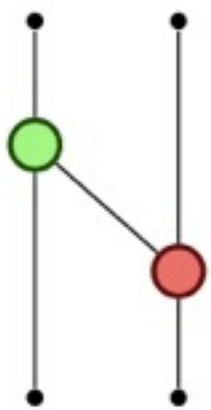
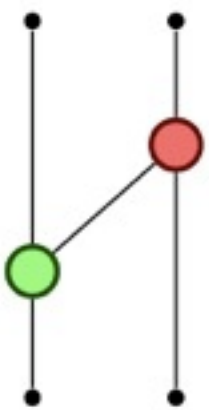
$$= \begin{pmatrix} \cos\frac{\beta}{2} & -i\sin\frac{\beta}{2} \\ -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

Representing Paulis



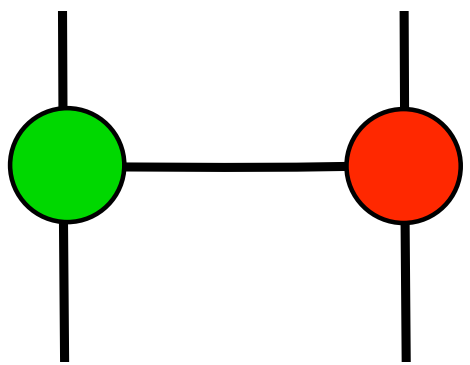


Representing CNot

$$\wedge X = [\![$$  $$]\!] = [\![$$  $$]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

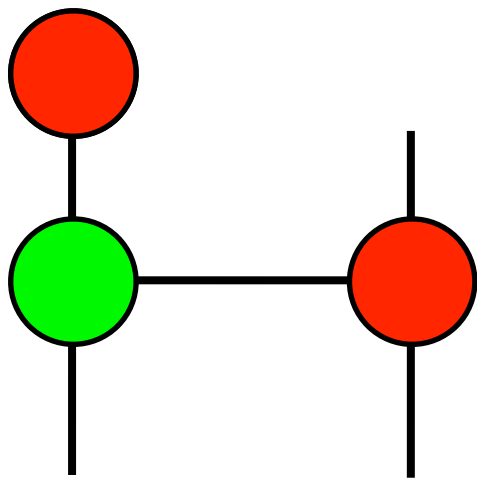
Representing Logic Gates

$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



Representing Logic Gates

$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



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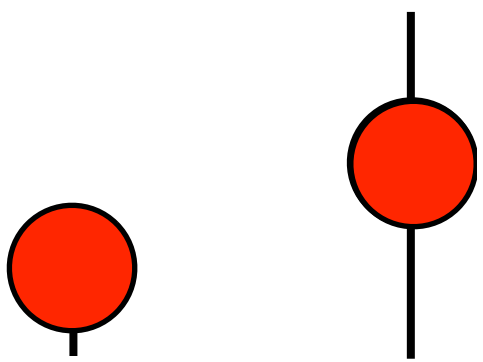

Representing Logic Gates

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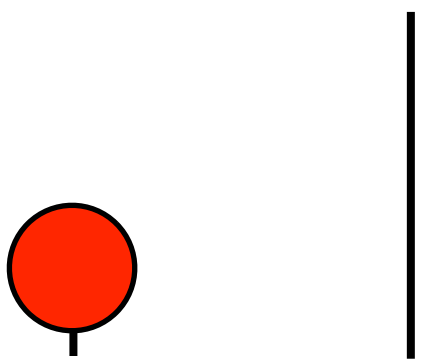

Representing Logic Gates

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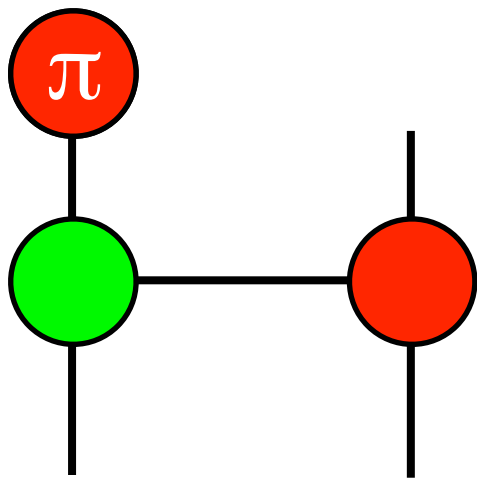
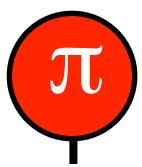
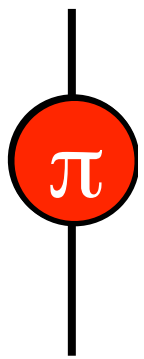


Representing Logic Gates

$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



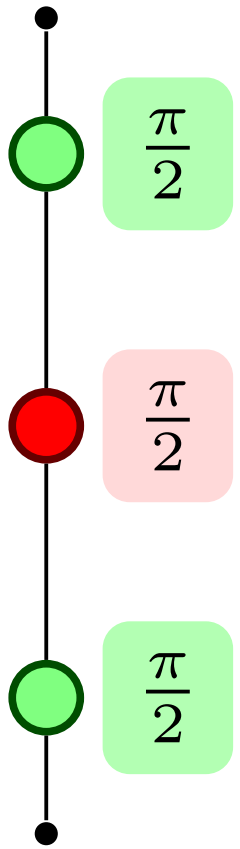


The ZX-calculus is universal

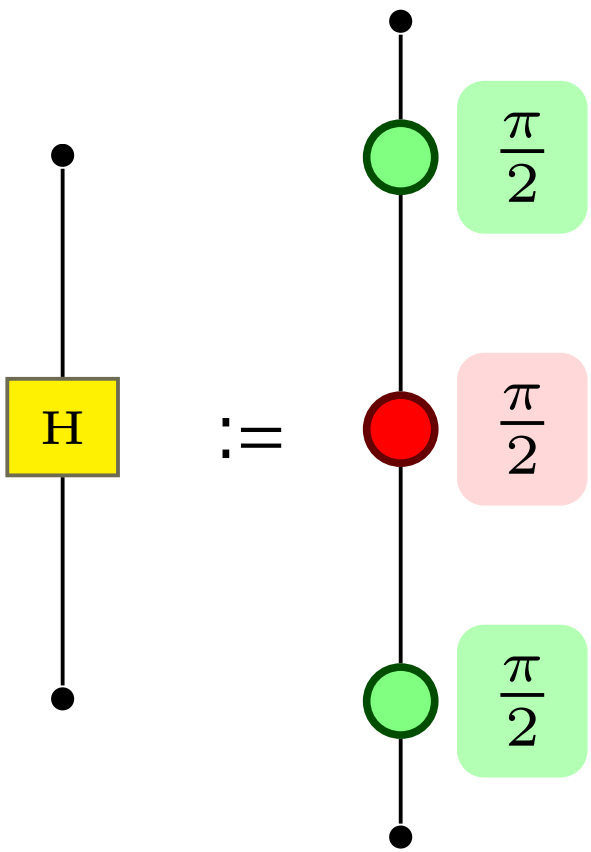
Theorem: Let U be a unitary map on n qubits; then there exists a ZX-calculus term D such that:

$$[\![D]\!] = U$$

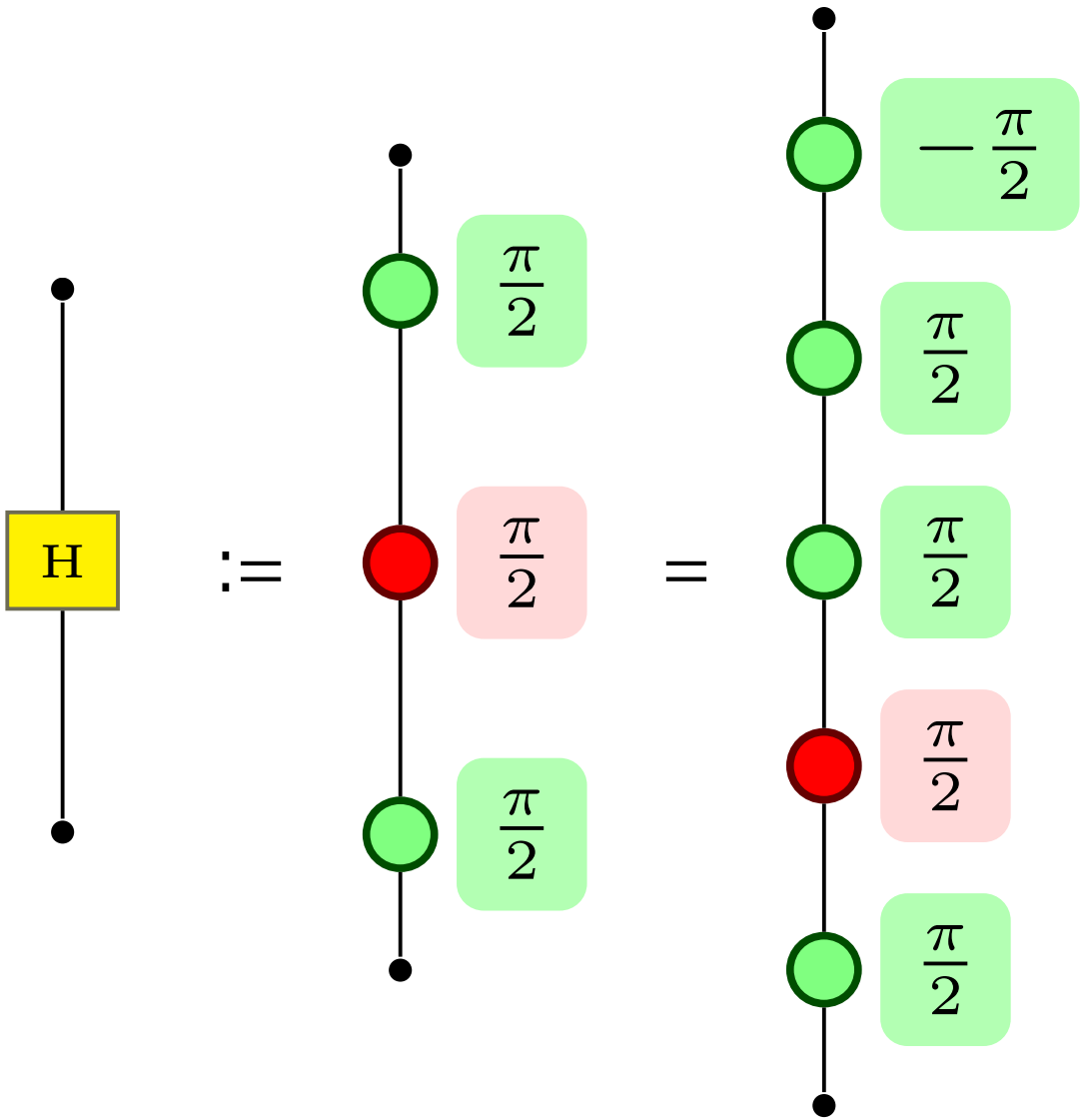
Representing Hadamard

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = [\![$$  $$]\!]$$

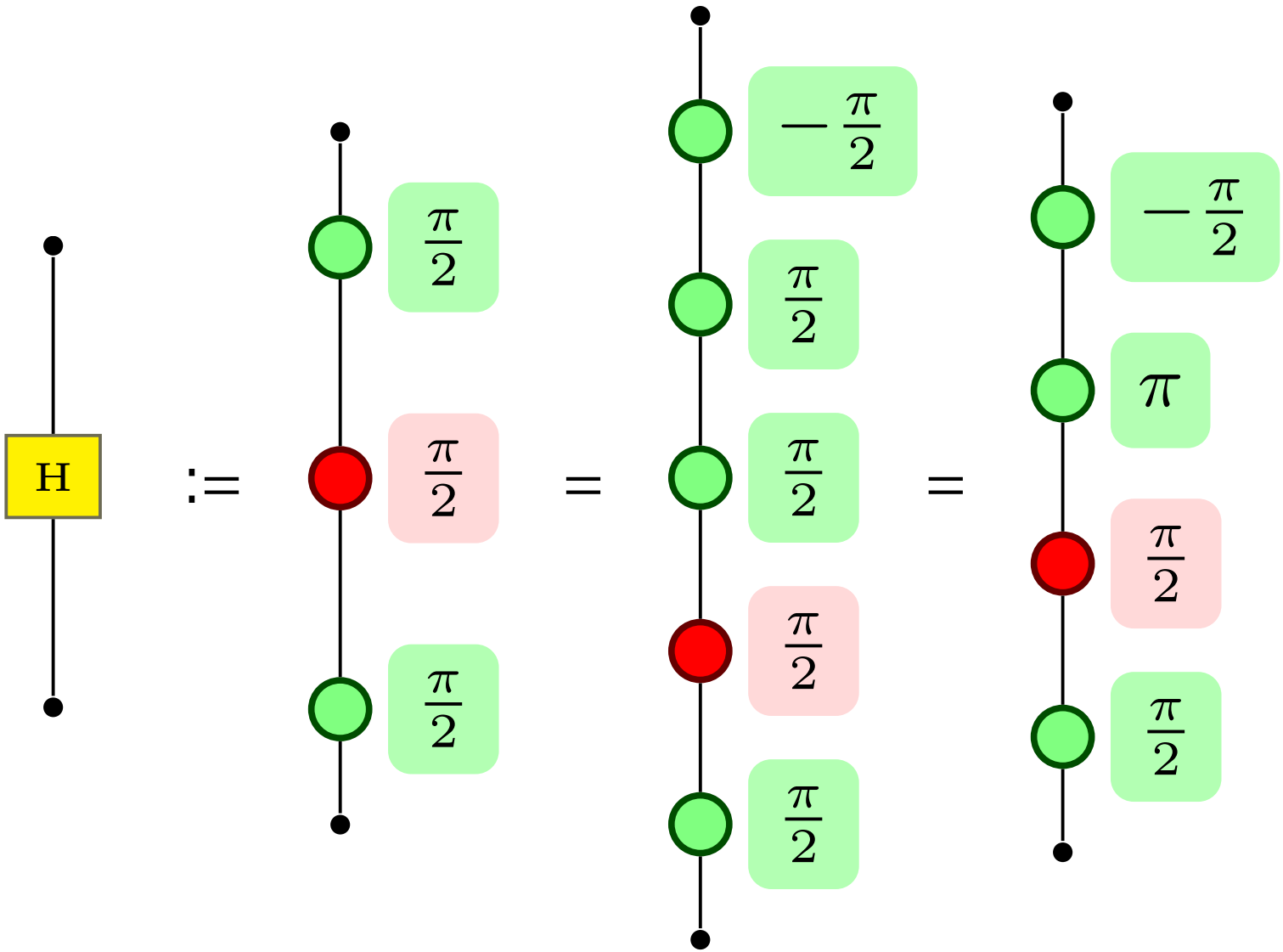
More on the Hadamard



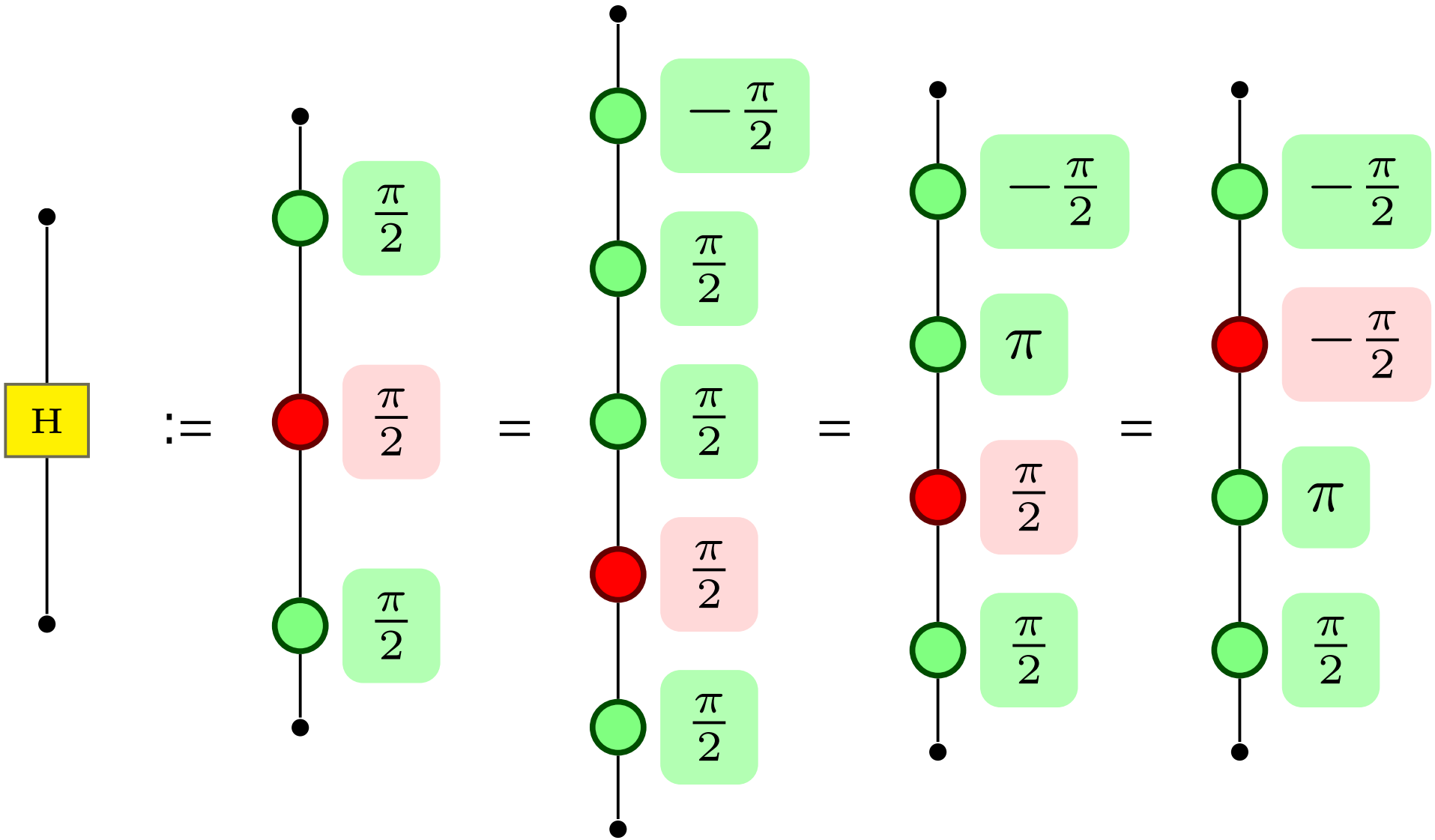
More on the Hadamard



More on the Hadamard

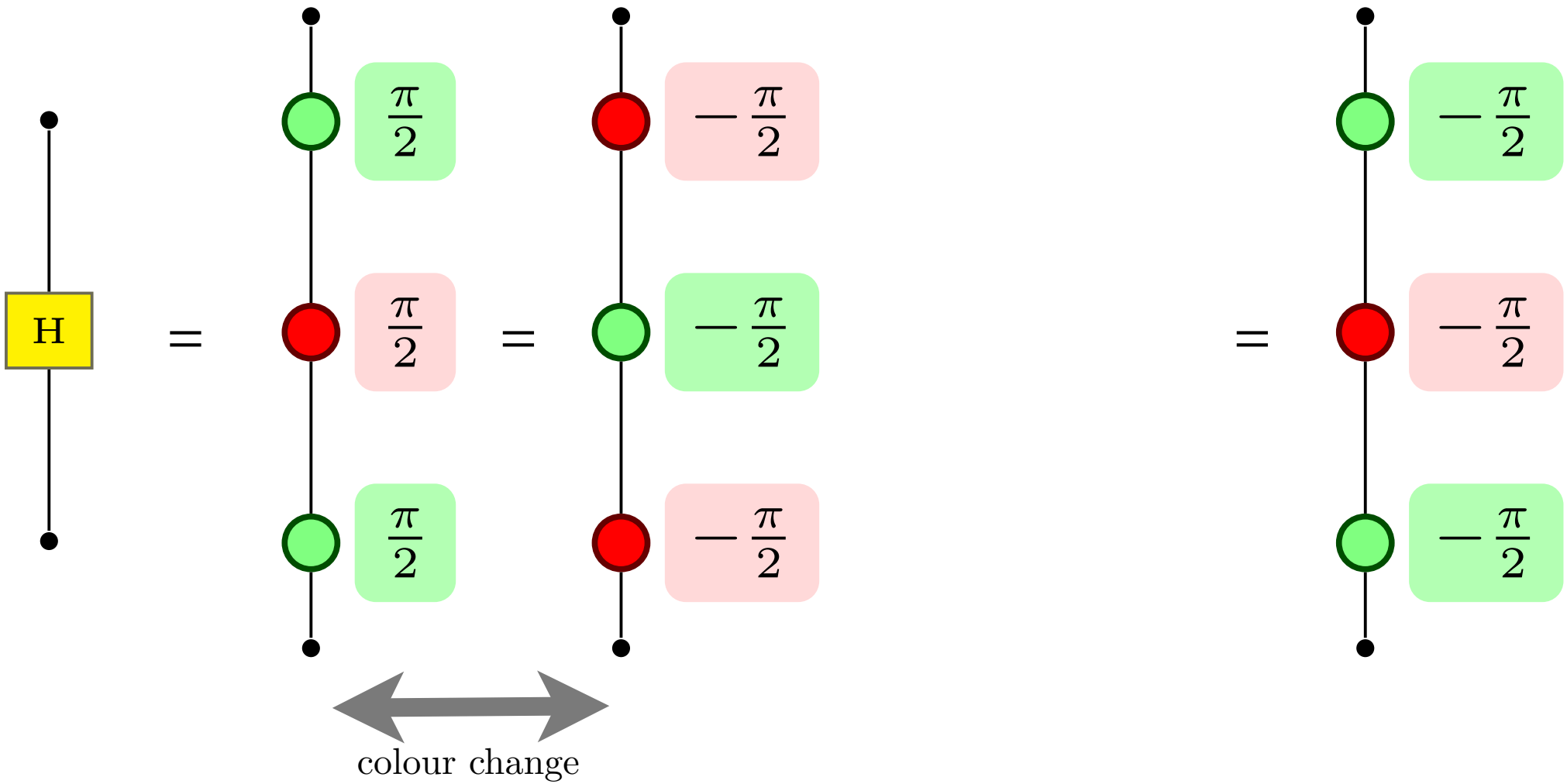


More on the Hadamard

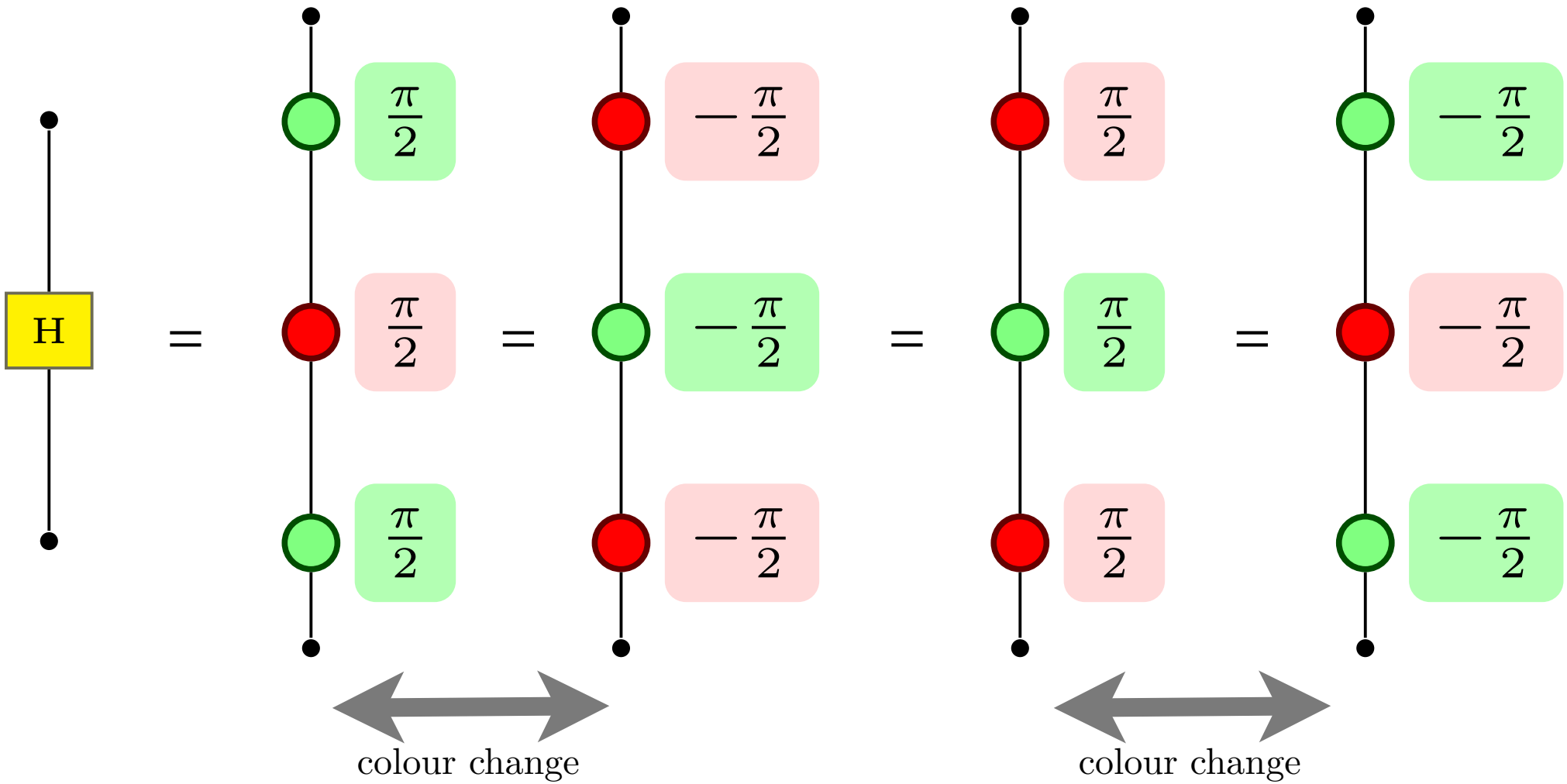


More on the Hadamard

More on the Hadamard

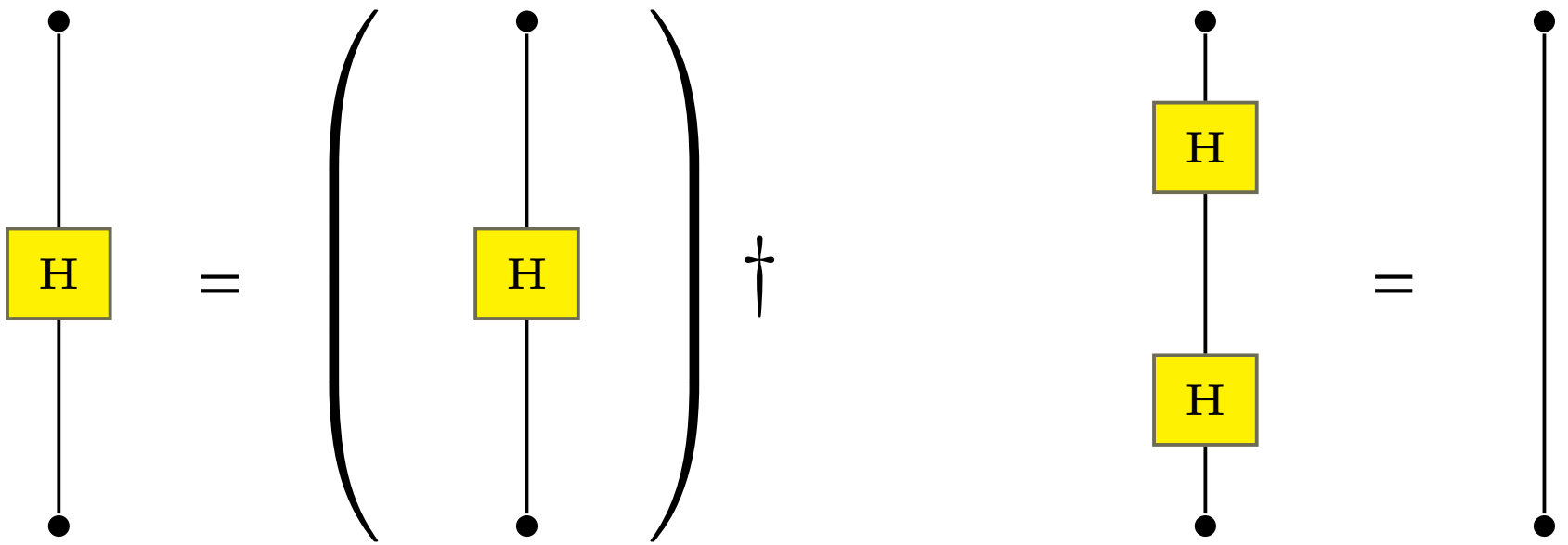


More on the Hadamard



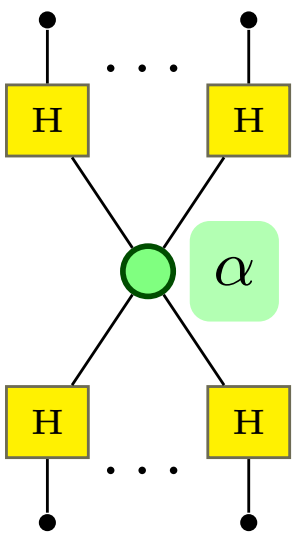
More on the Hadamard

Corollary 1:



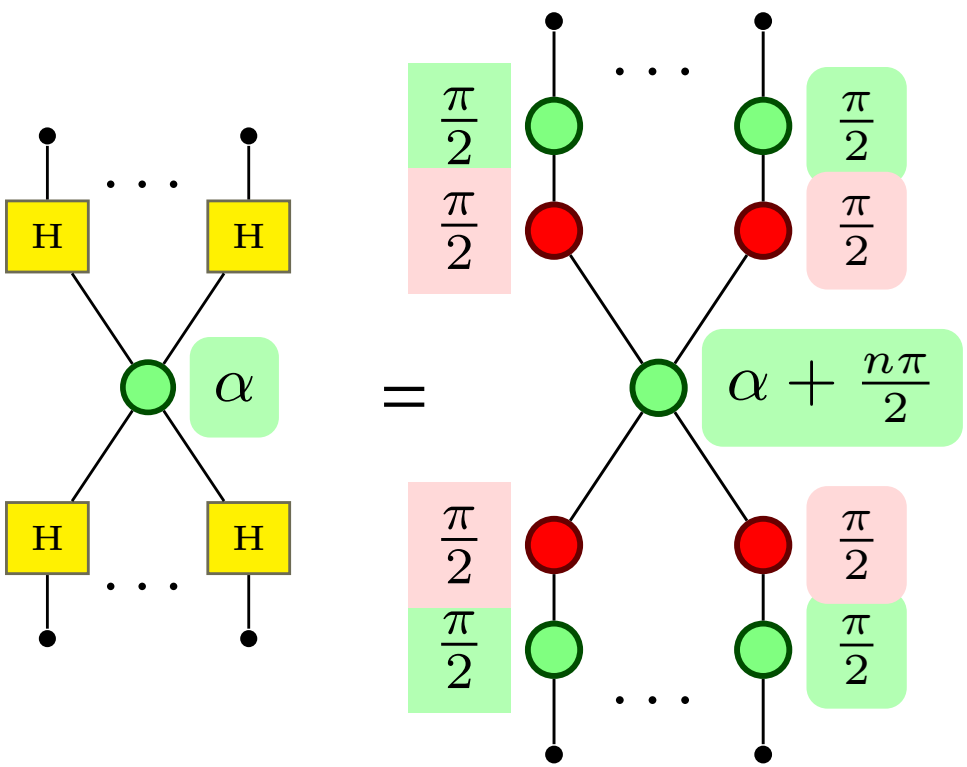
More on the Hadamard

Corollary 2:



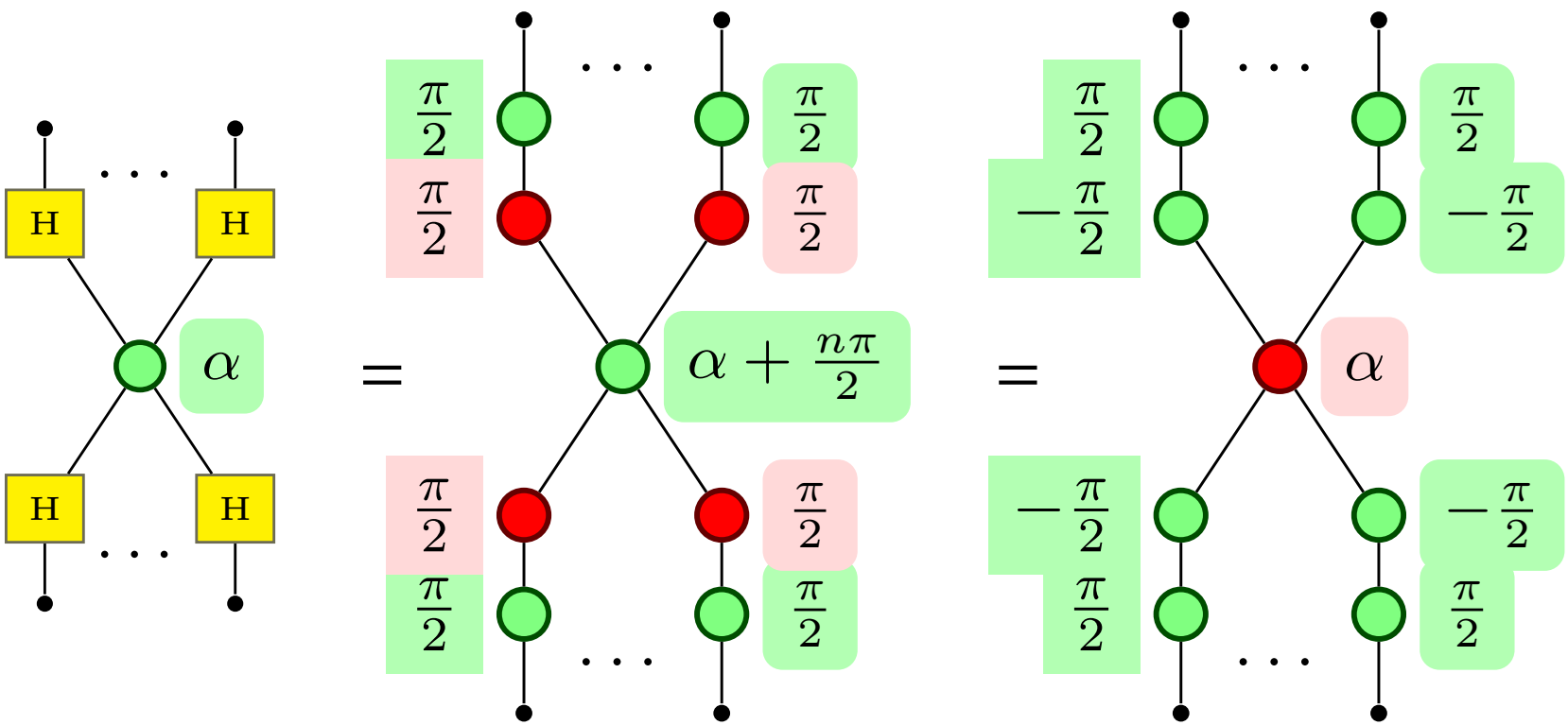
More on the Hadamard

Corollary 2:



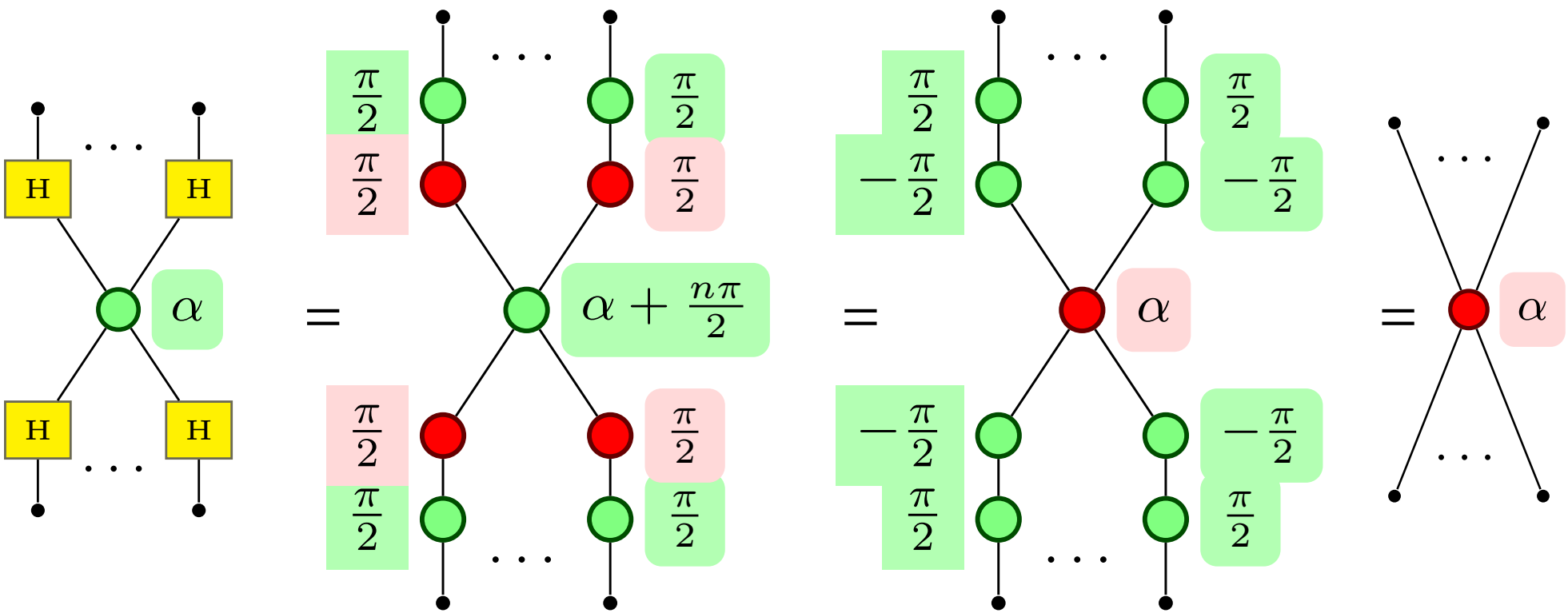
More on the Hadamard

Corollary 2:



More on the Hadamard

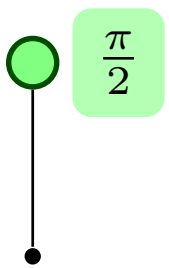
Corollary 2:



Corollary 2.1 :total symmetry between red and green

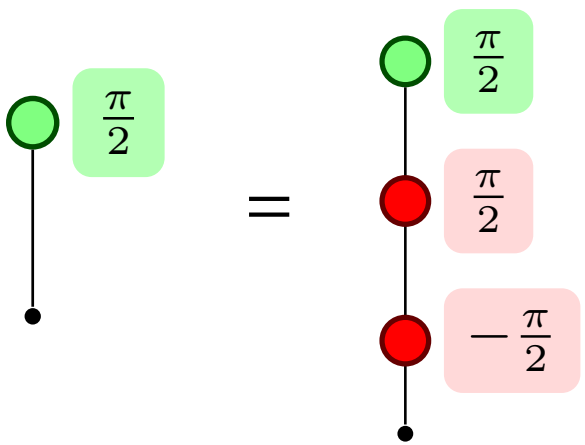
More on the Hadamard

Corollary 3:



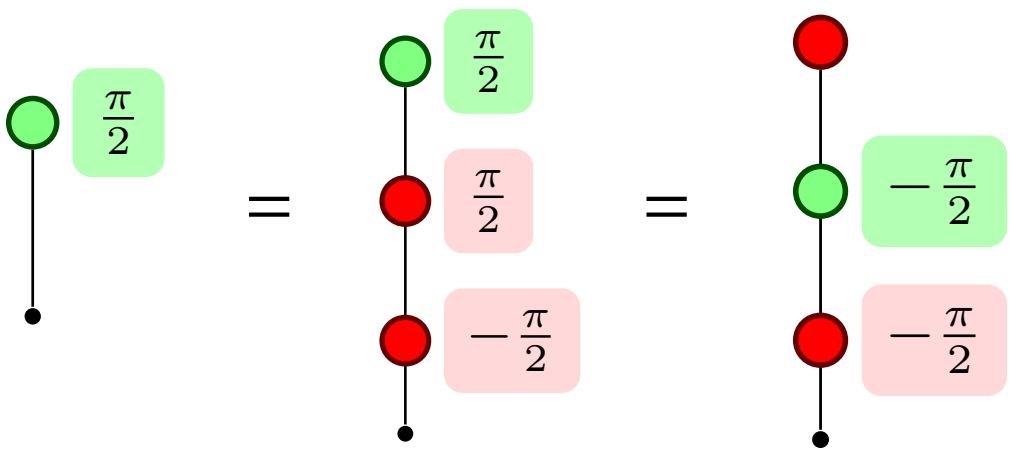
More on the Hadamard

Corollary 3:



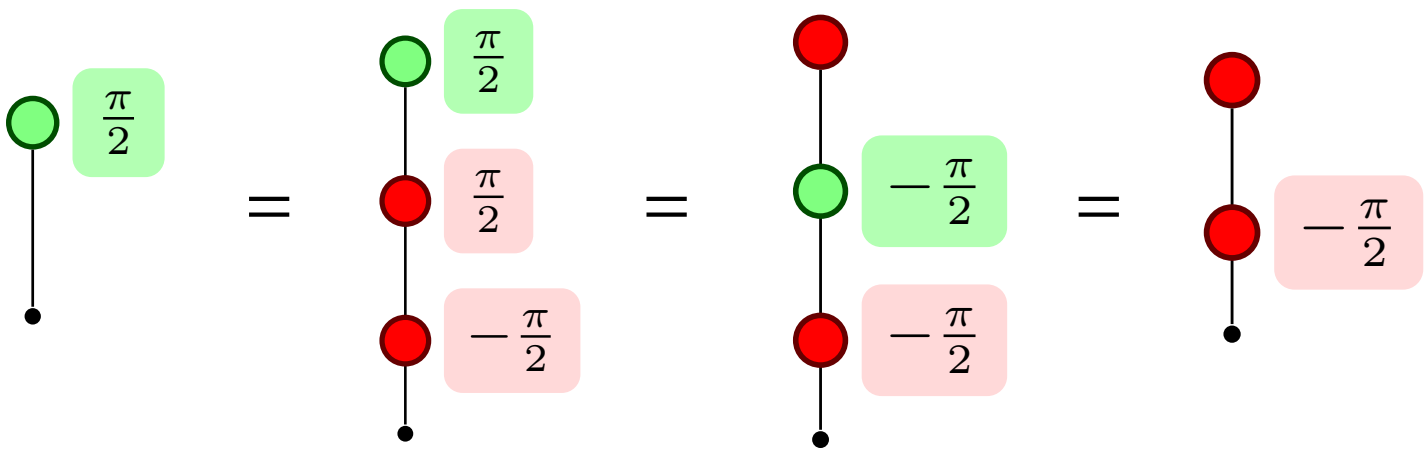
More on the Hadamard

Corollary 3:



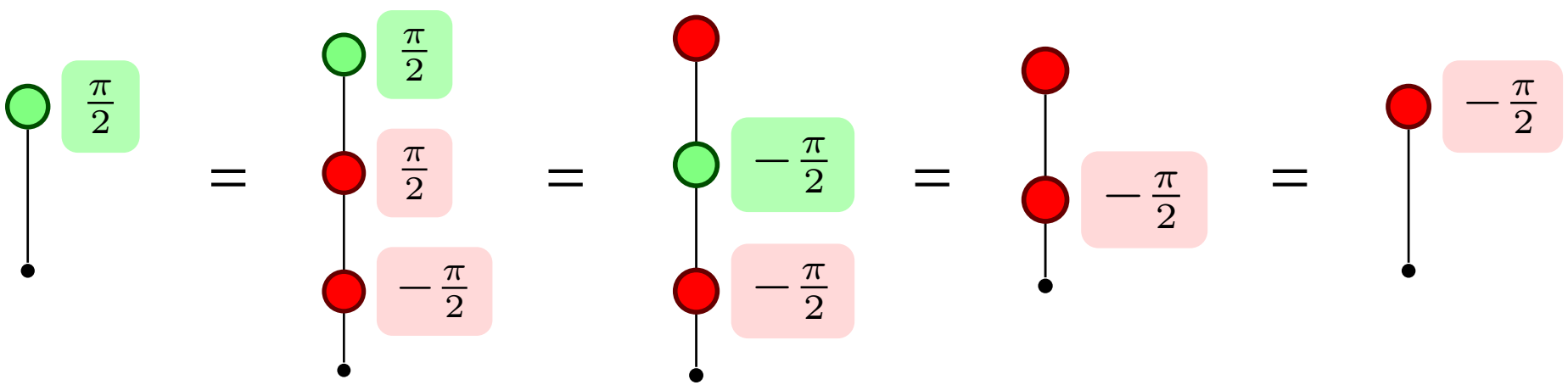
More on the Hadamard

Corollary 3:



More on the Hadamard

Corollary 3:



More on the Hadamard

Corollary 3:

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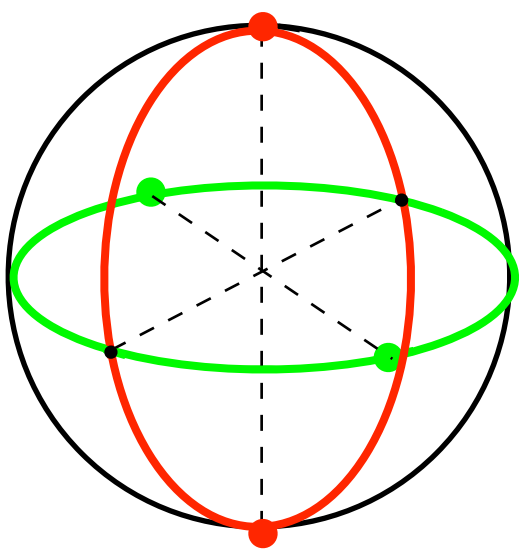

More on the Hadamard

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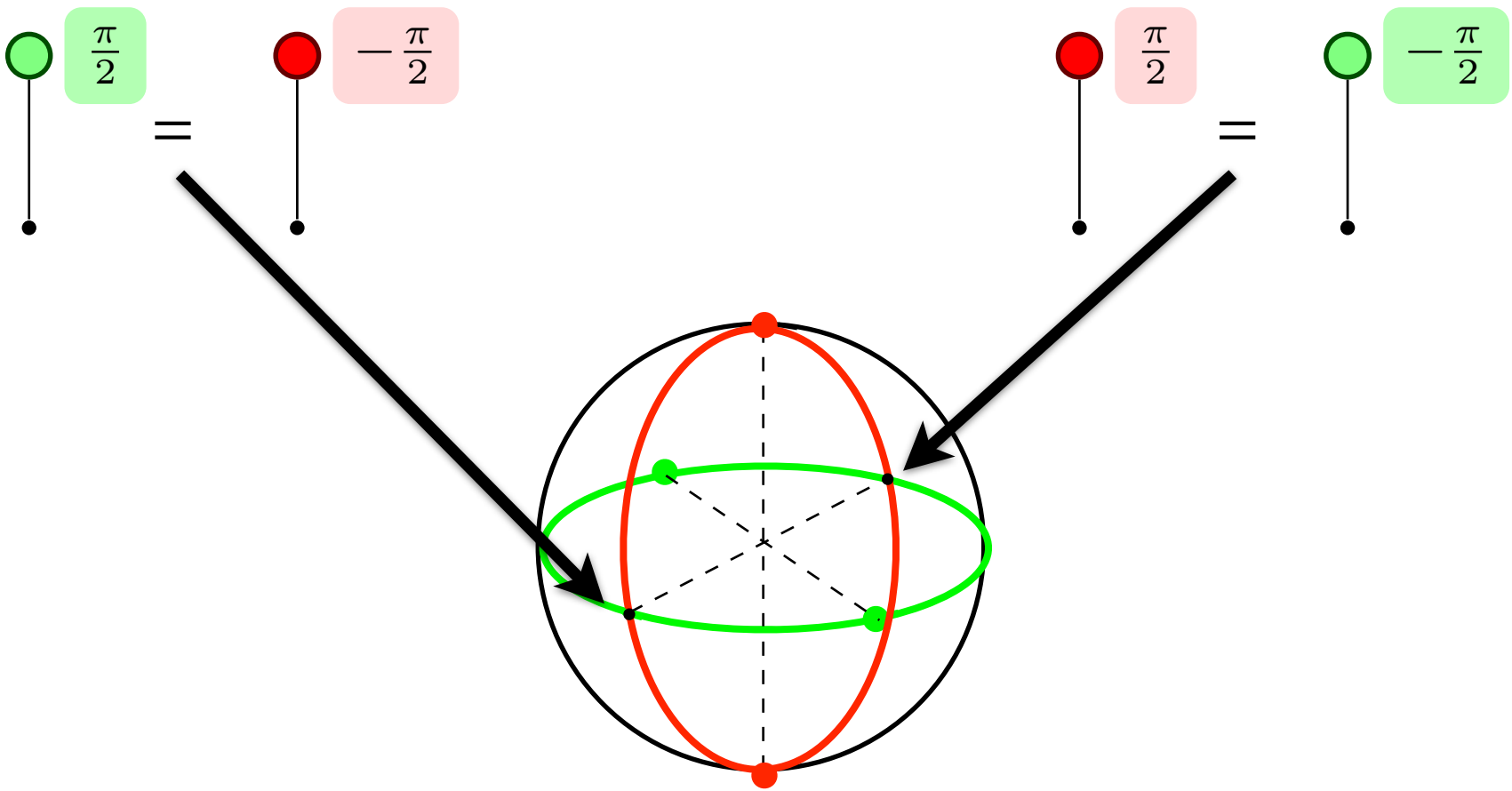
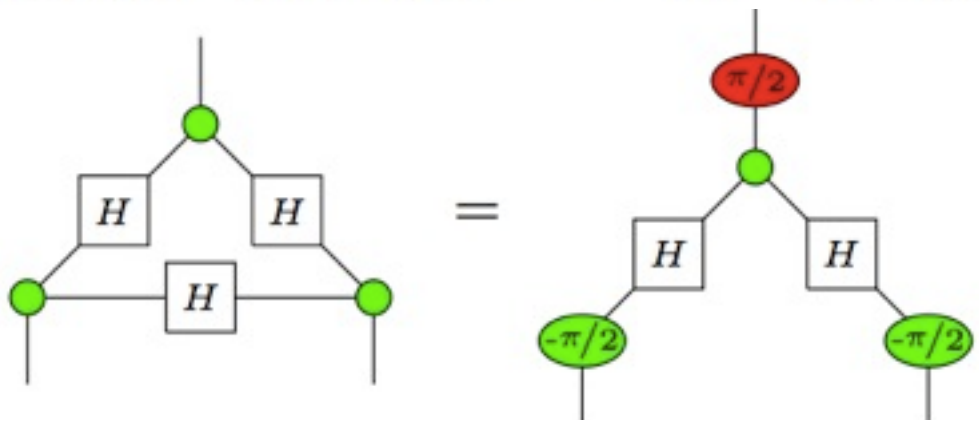


More on the Hadamard

Corollary 4: Van den Nest's theorem

$$R_x(\pi/2)^{(u)} R_z(-\pi/2)^{(N_G(u))} |G\rangle = |G * u\rangle$$

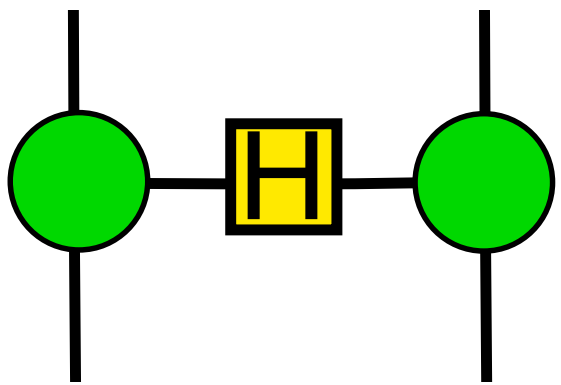


Van den Nest, M., Dehaene, J. and De Moor, B. "*Graphical description of the action of local clifford transformations on graph states*", Phys. Rev. A 69 (2004) 022316. arXiv:quant-ph/0411115

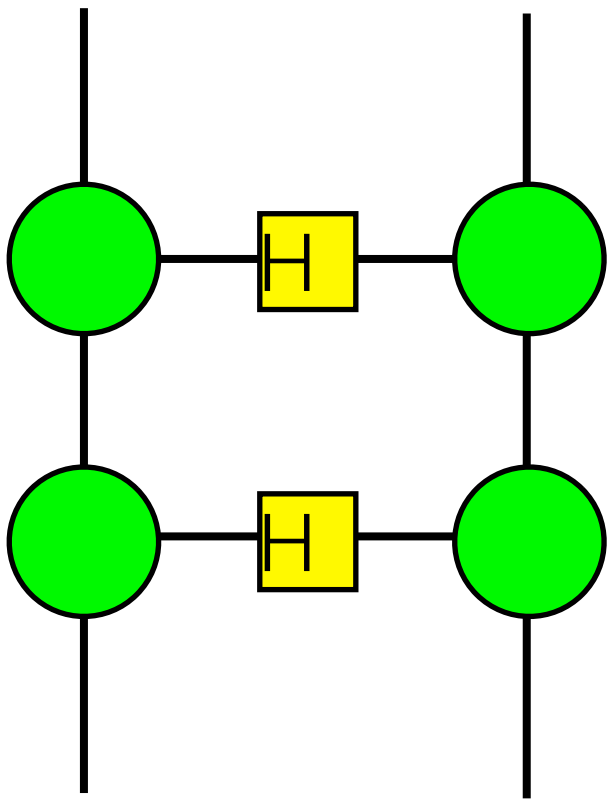
Duncan, R. and Perdrix, S. "*Graph States and the necessity of Euler Decomposition*", Proc. CiE09 (2009), LNCS 5635. arXiv:0902.0500

Representing Logic Gates

$$\wedge Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



Example: 2 CZs



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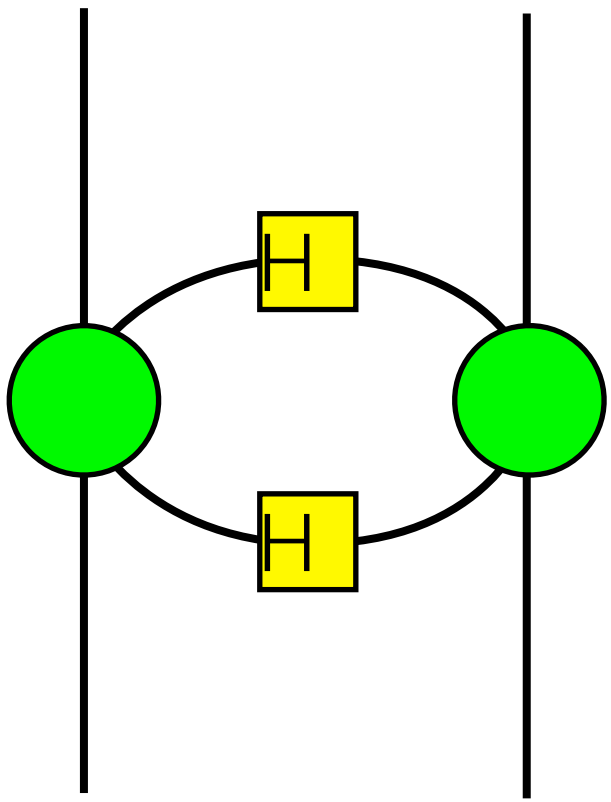

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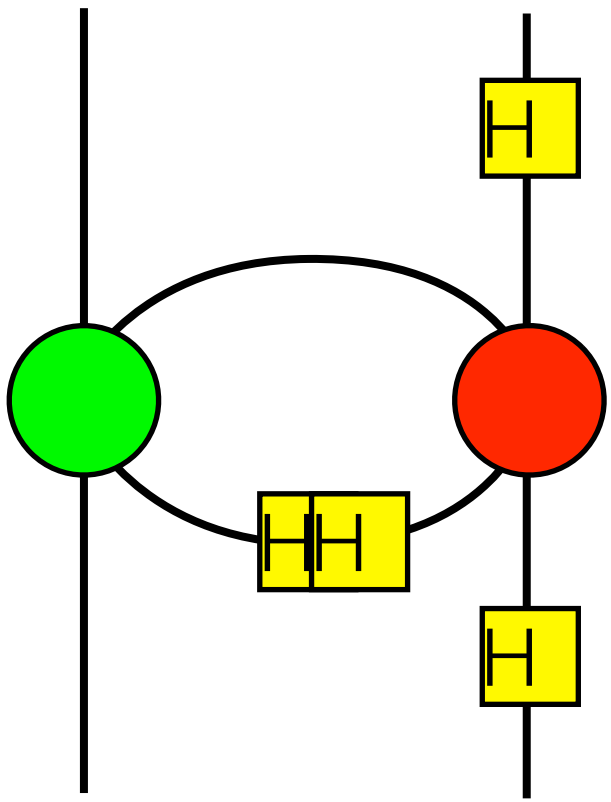

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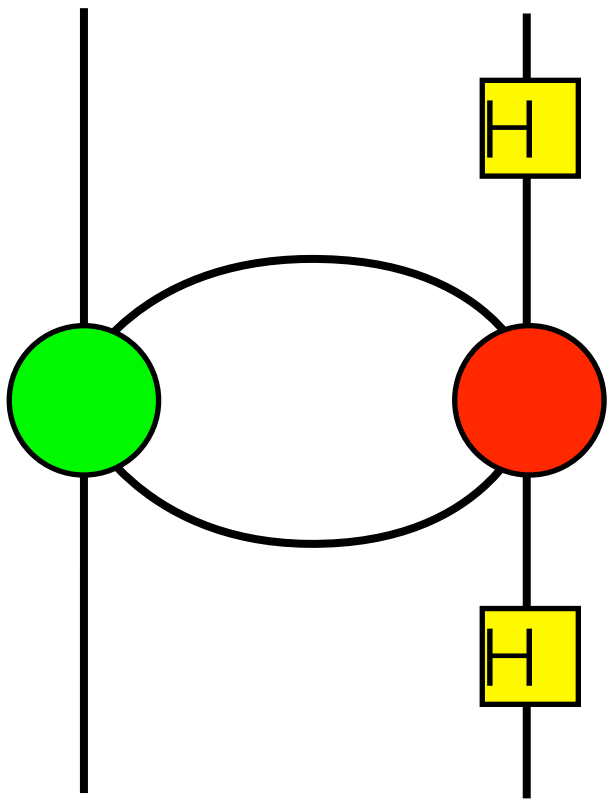

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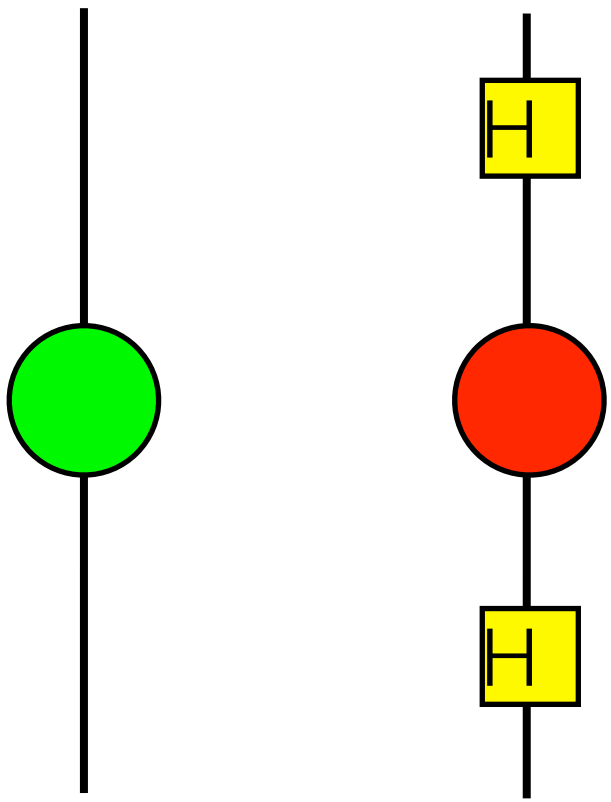

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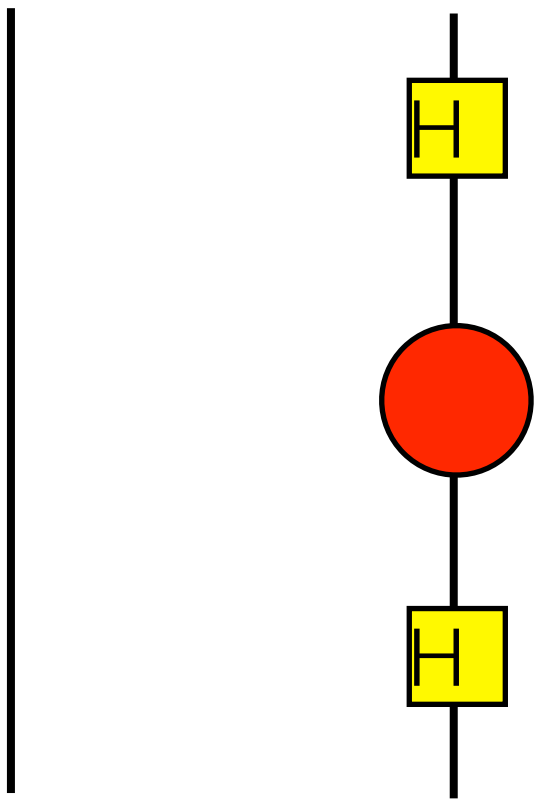

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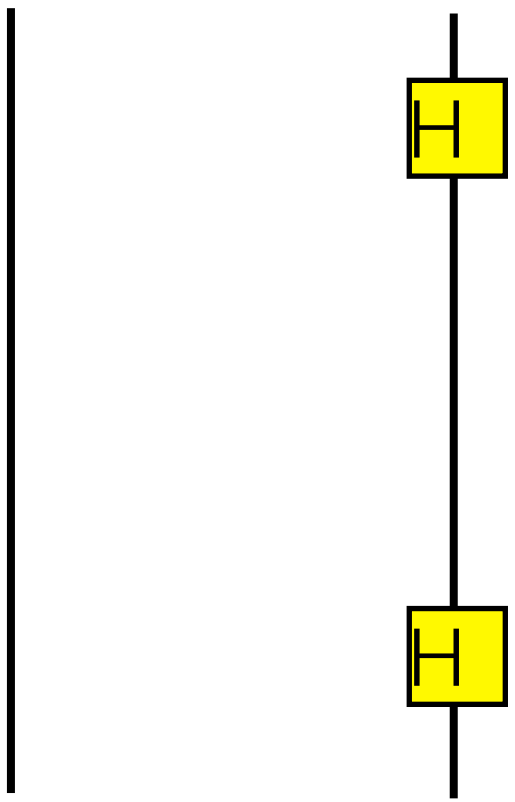

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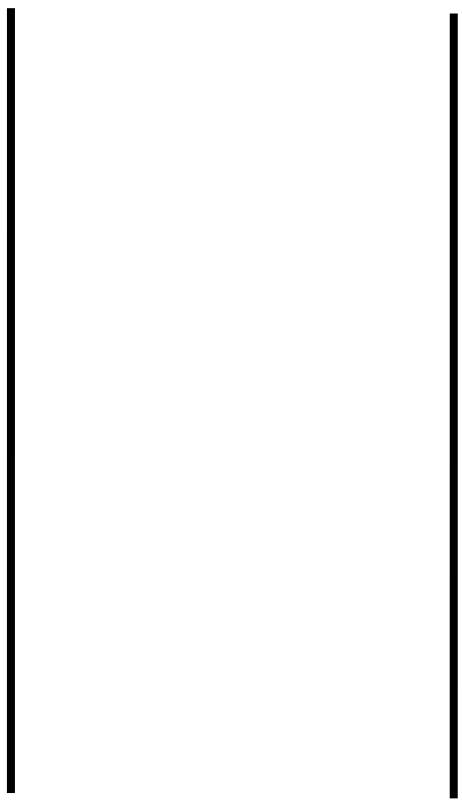
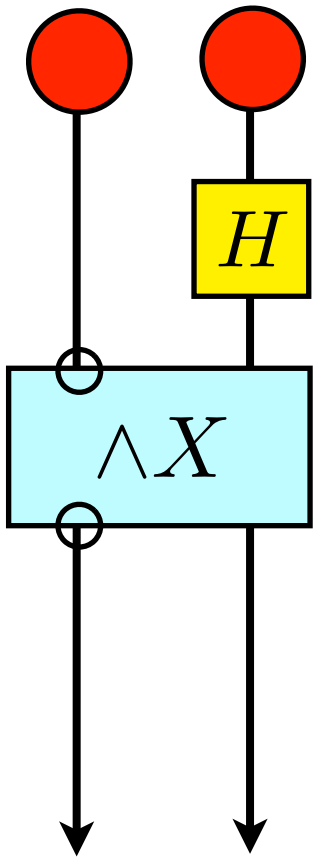


Example: Preparing a Bell state



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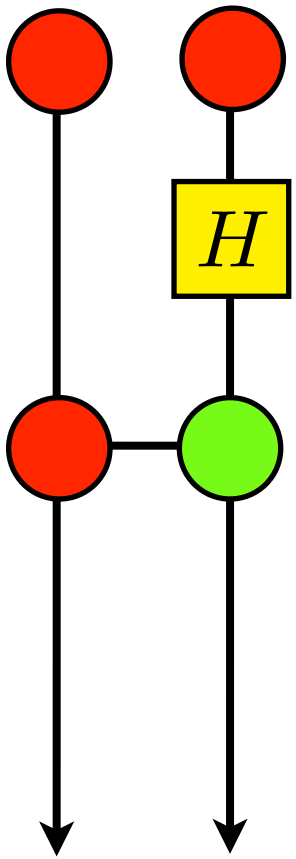

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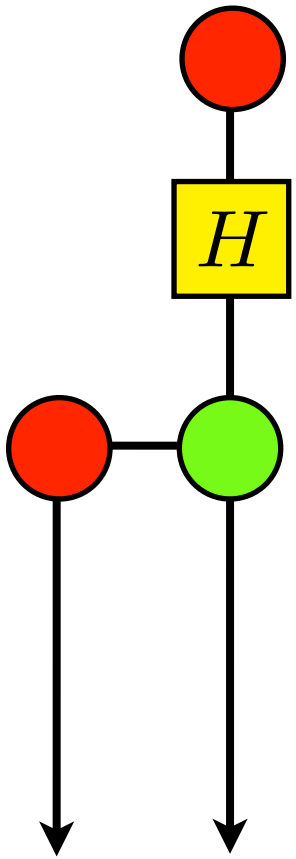

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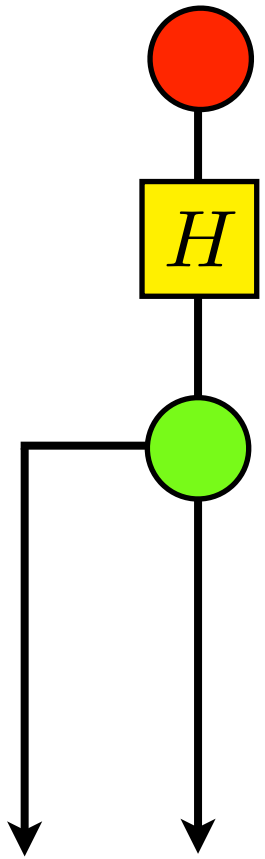

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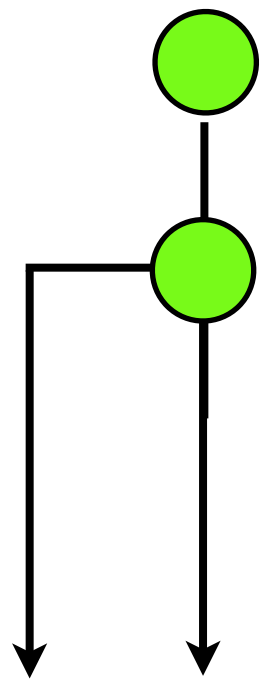

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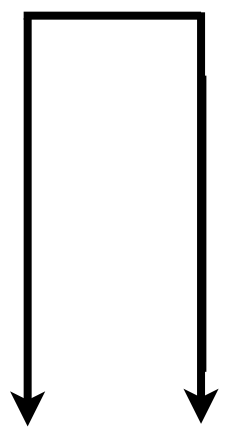
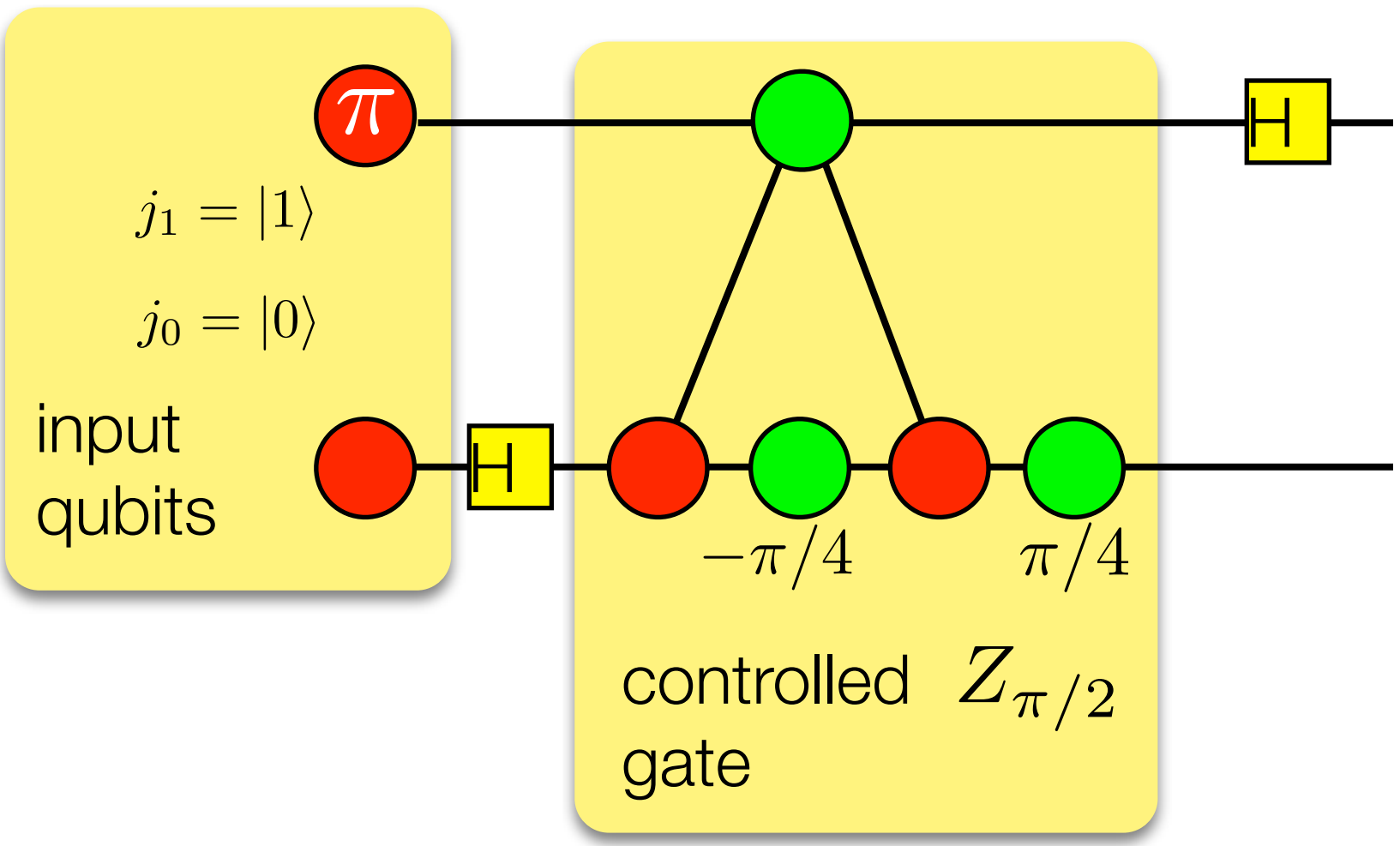


Example: 2-Qubit Quantum Fourier Transform



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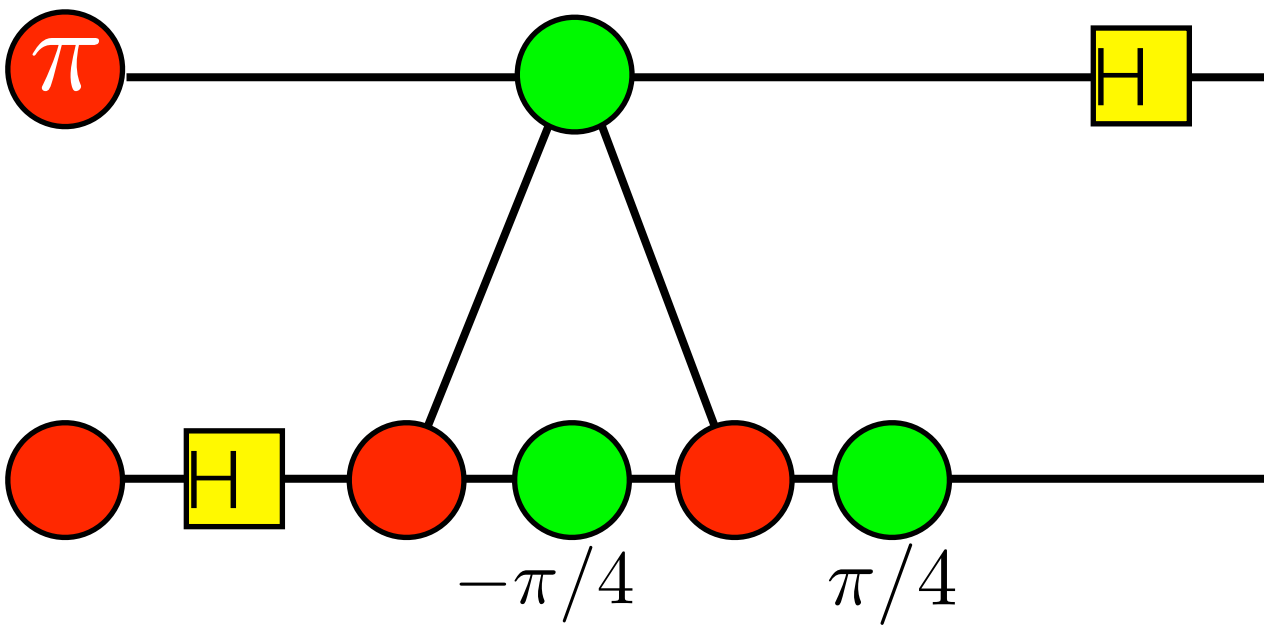

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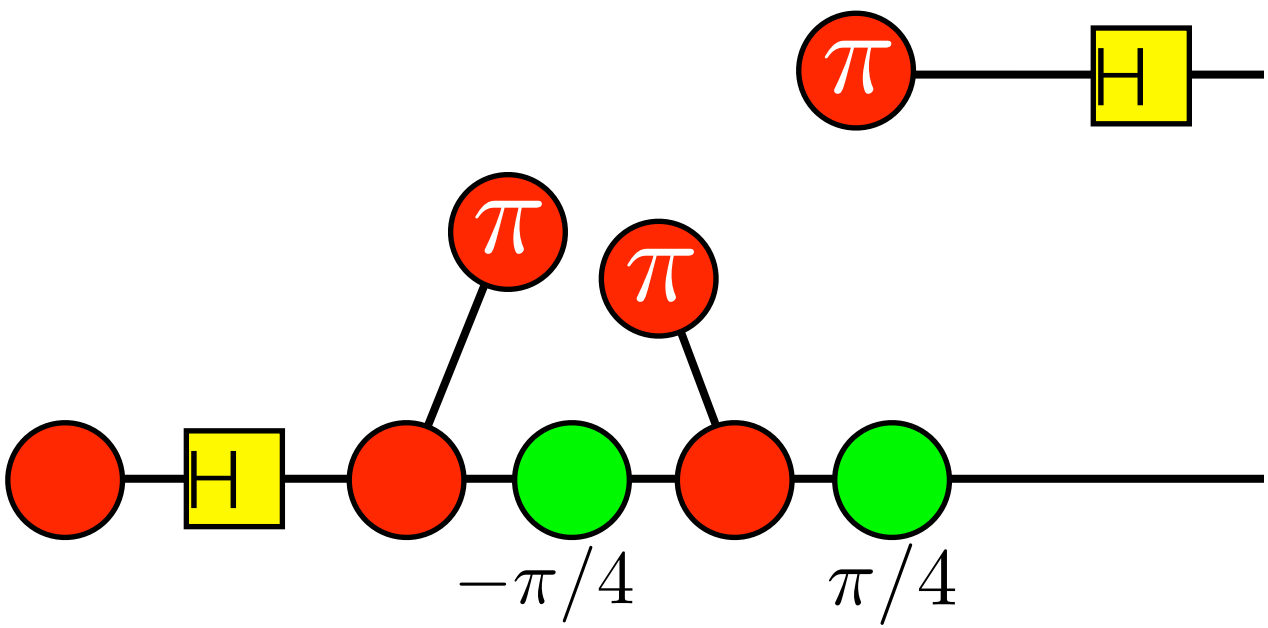

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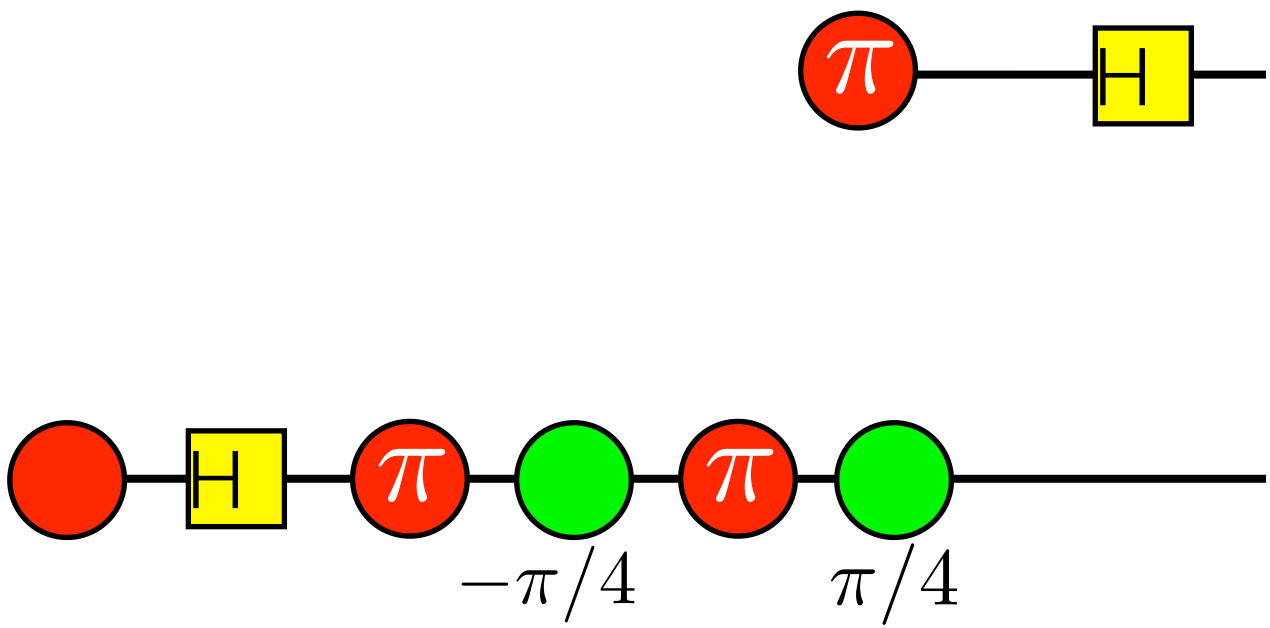

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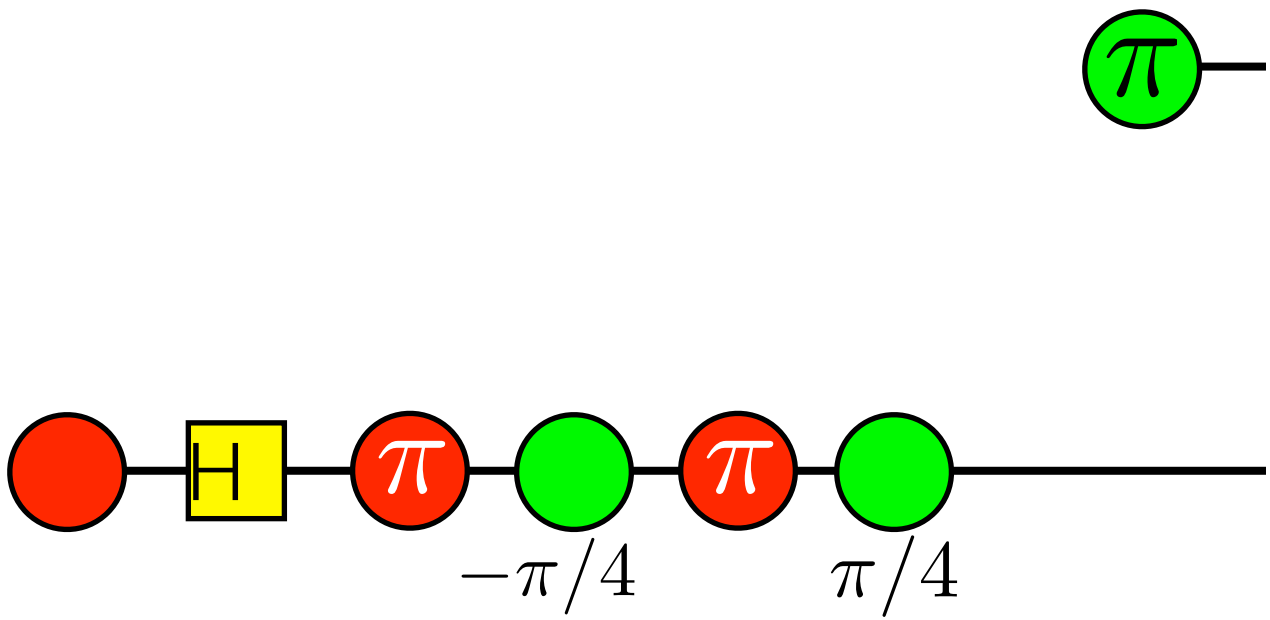

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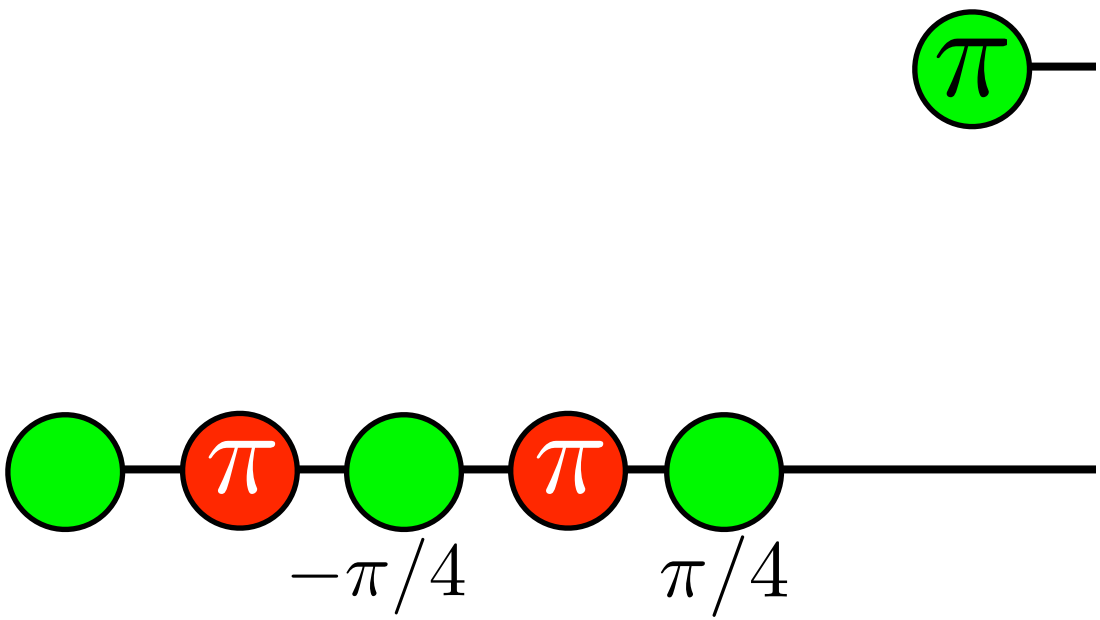

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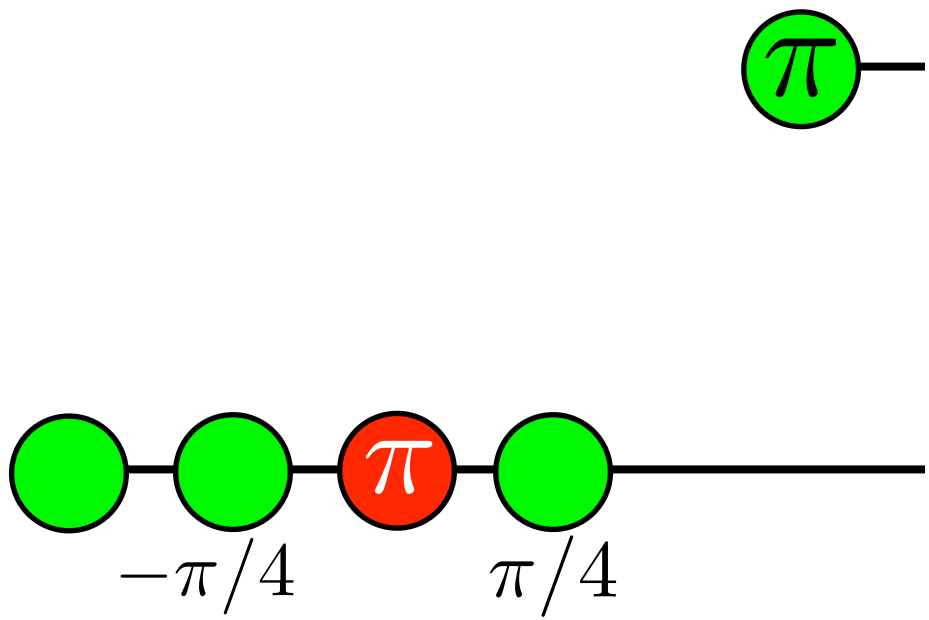

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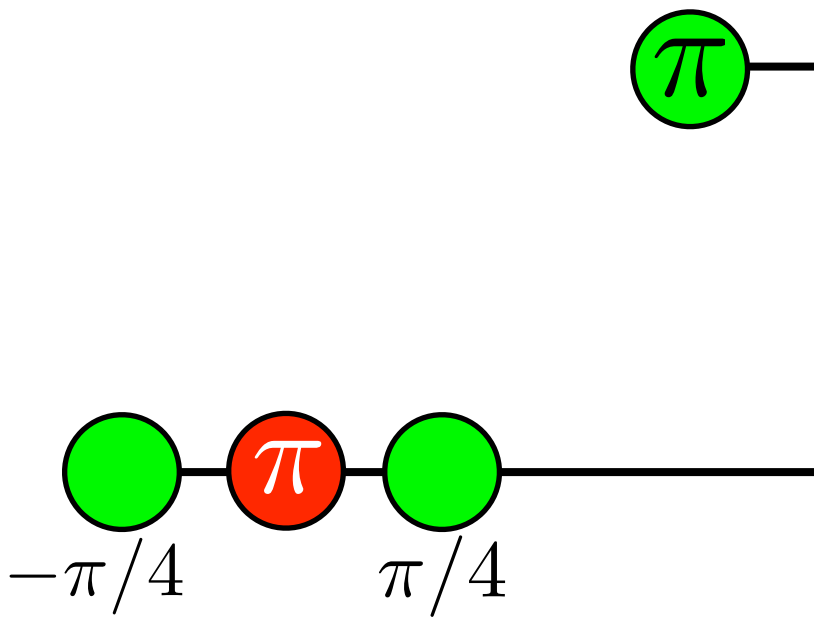

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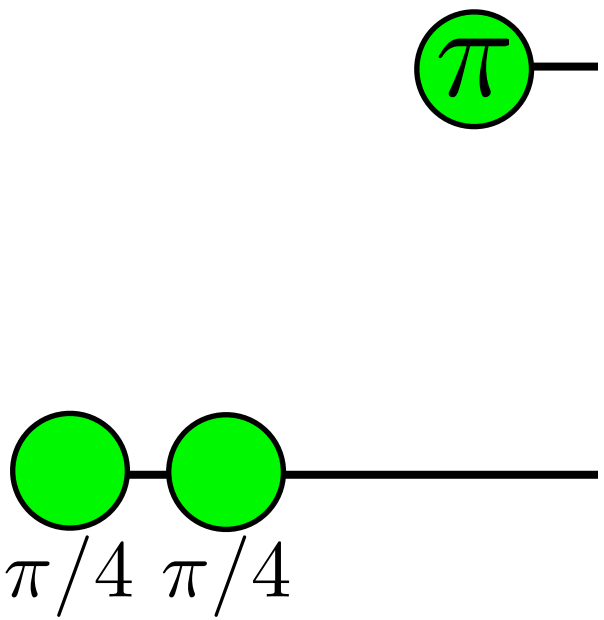

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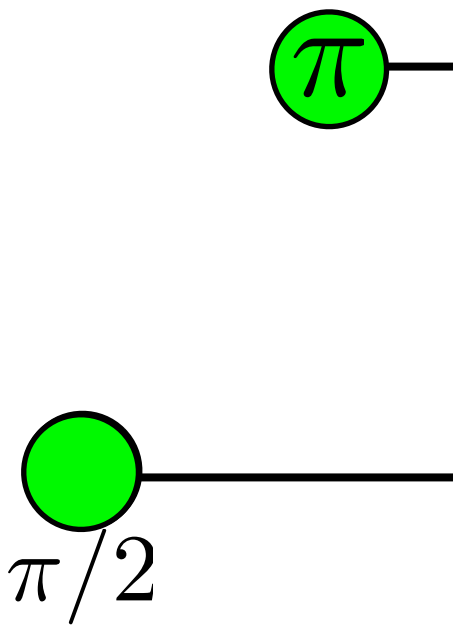

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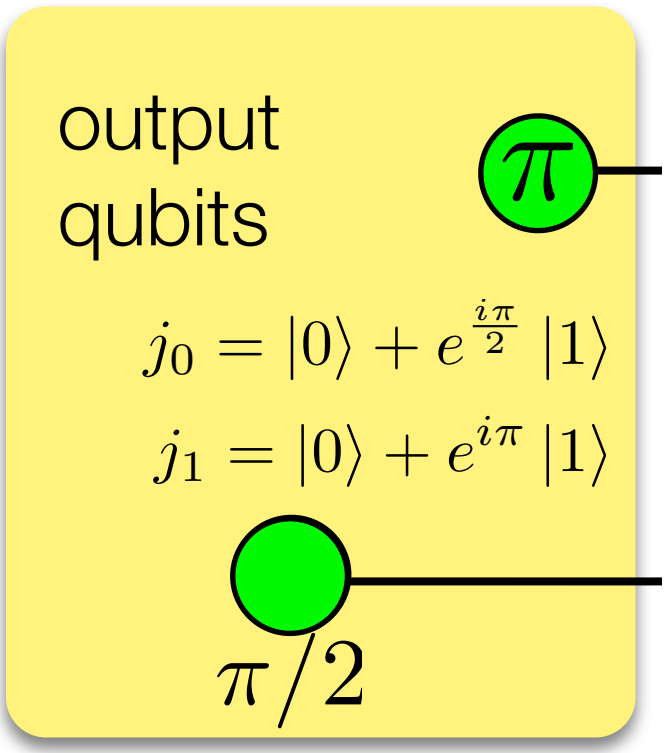
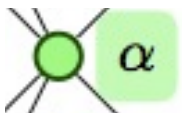
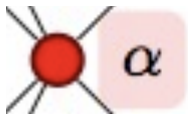


Which diagrams are circuits?

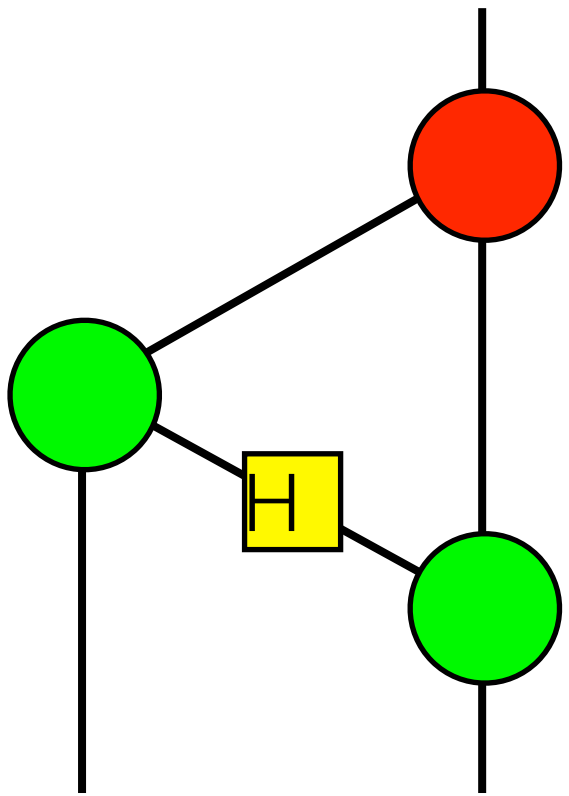
Defn: a diagram is called **circuit-like** if:

- All of its 

, 

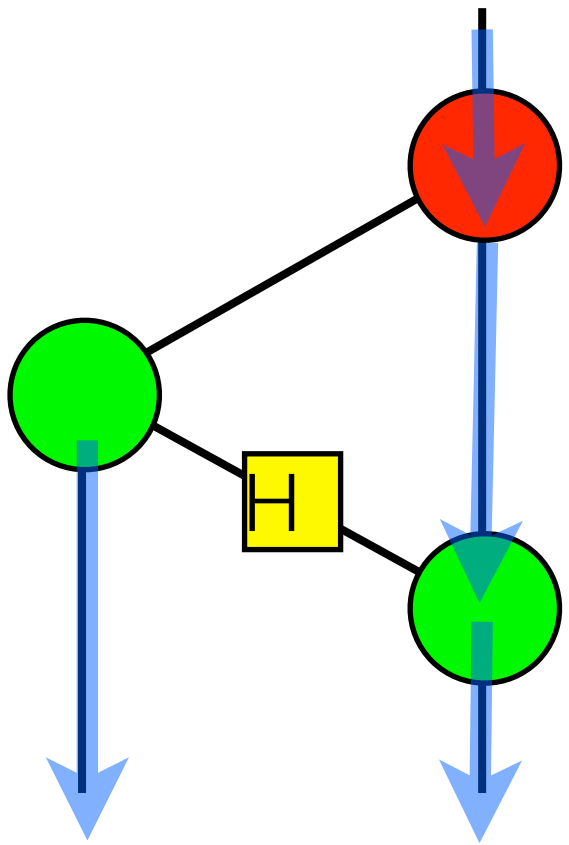
, and boundary vertices can be covered by set of disjoint paths P, each of which ends in an output.
- Every cycle in the diagram which overlaps with 2 paths in P traverses an edge in the opposite direction to P.
- It is 3-coloured.

Which diagrams are circuits?



Which diagrams are circuits?

YES!



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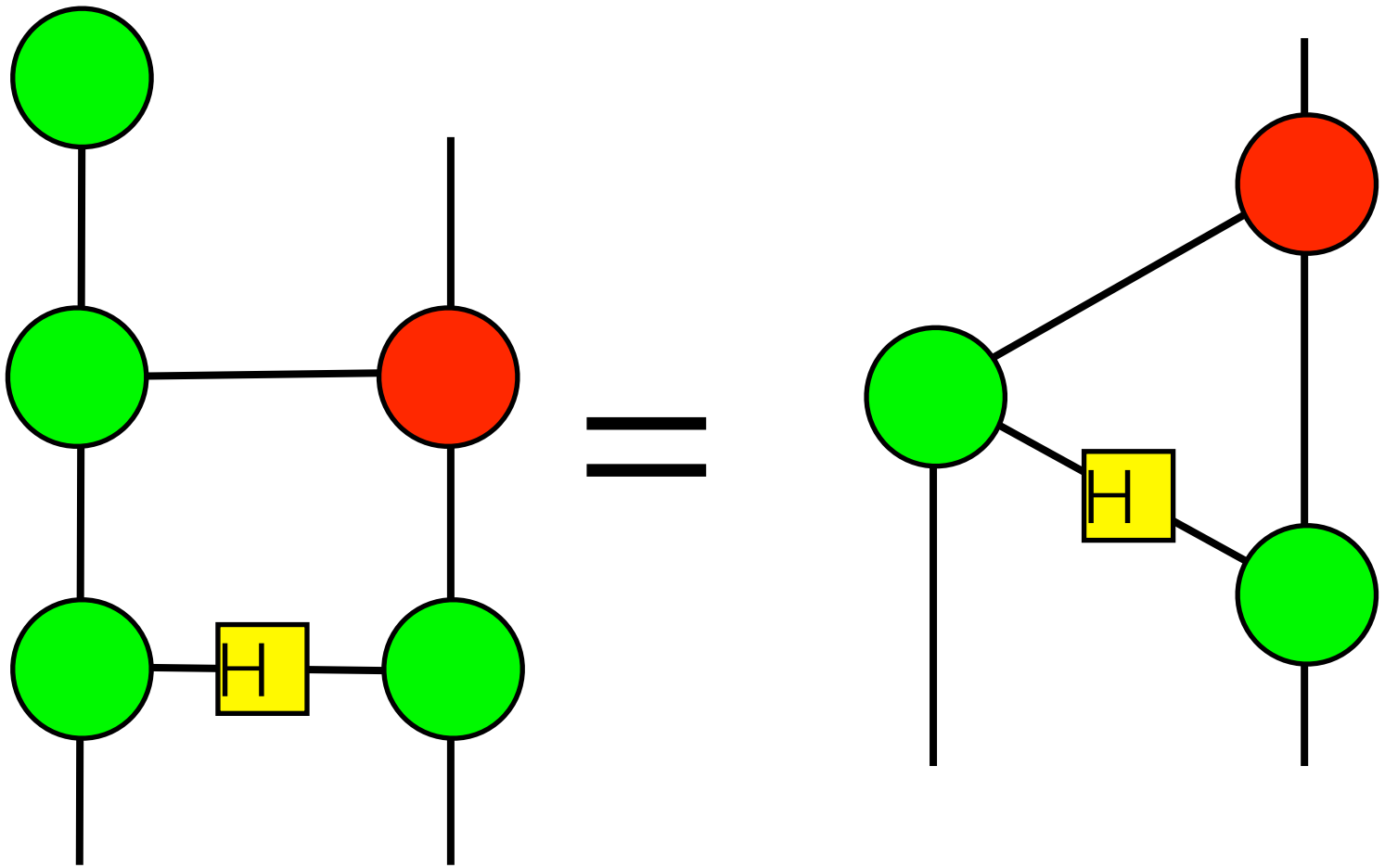

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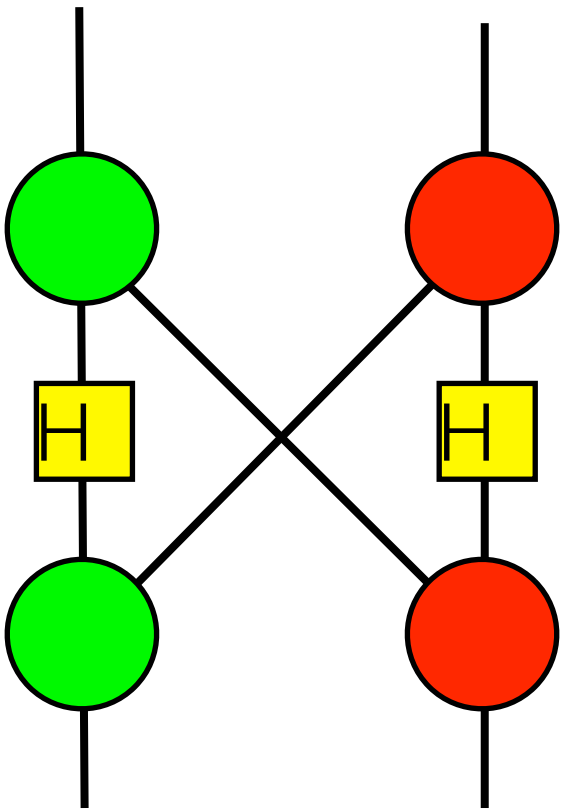
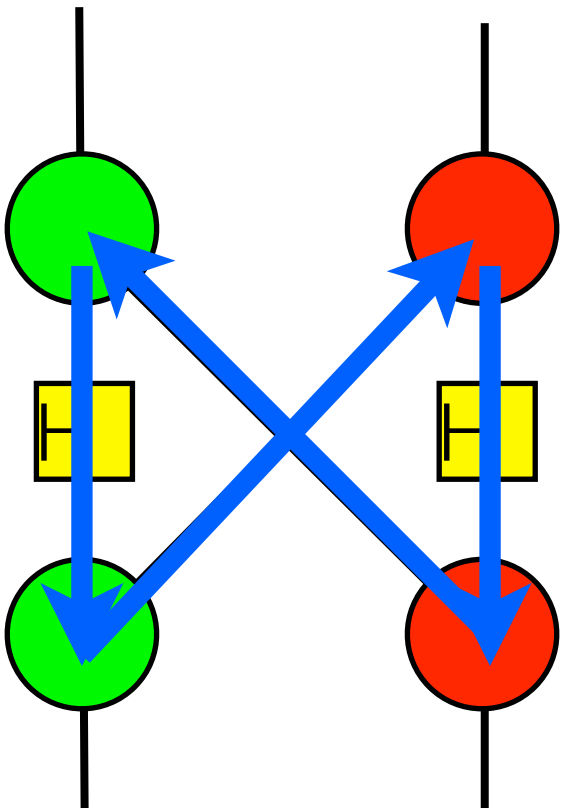


Which diagrams are circuits?

NO :(

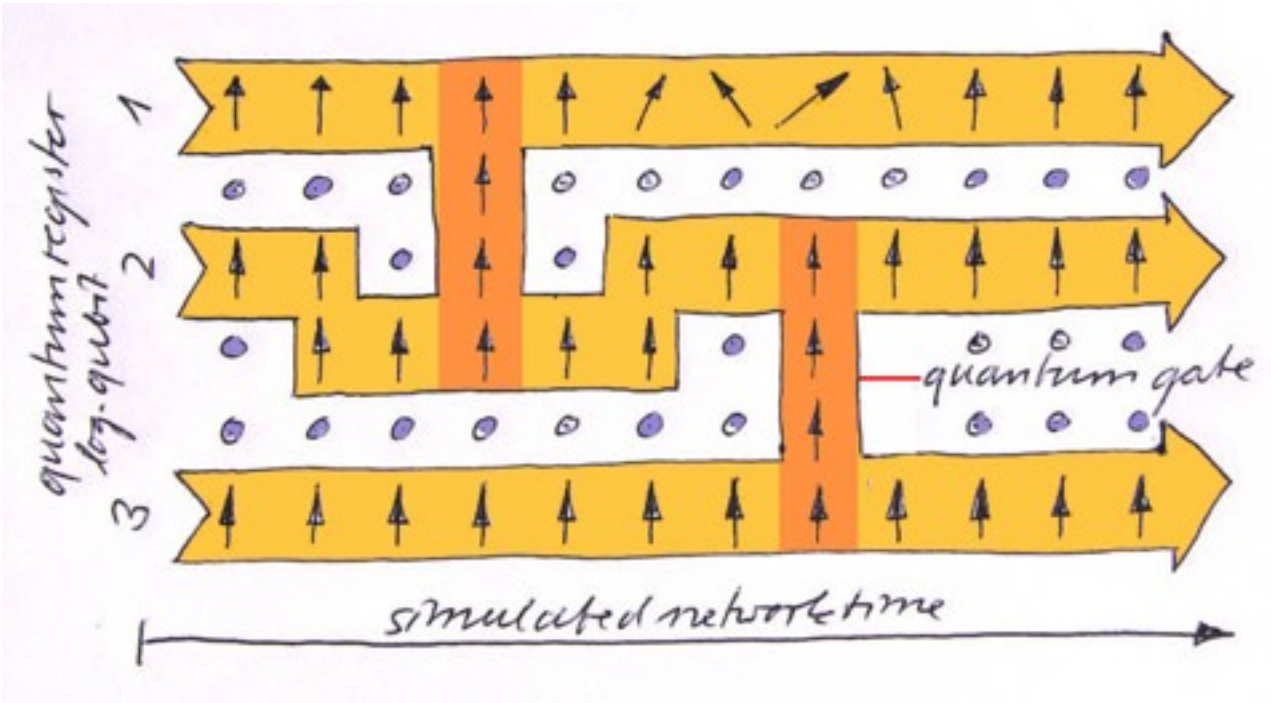


Which diagrams are circuits?

Thm: if a diagram D is circuit-like then $[\![D]\!]$ is a unitary embedding.

Being circuit-like is a little stronger than being a unitary embedding: we require a particular gate set, and a particular encoding of the gate set as diagrams, and the minimality w.r.t. to spider. Up to these conditions, the theorem is *if and only if*.

Information Flow in the 1-Way Quantum Computer



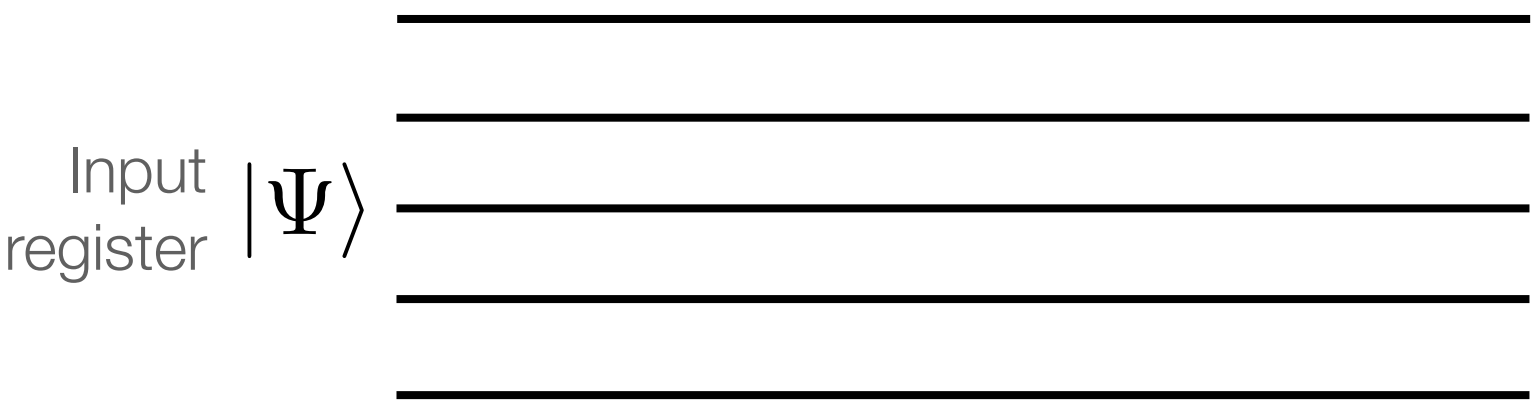
Using measurements to compute

Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:

Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:



Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:

Quantum Circuits

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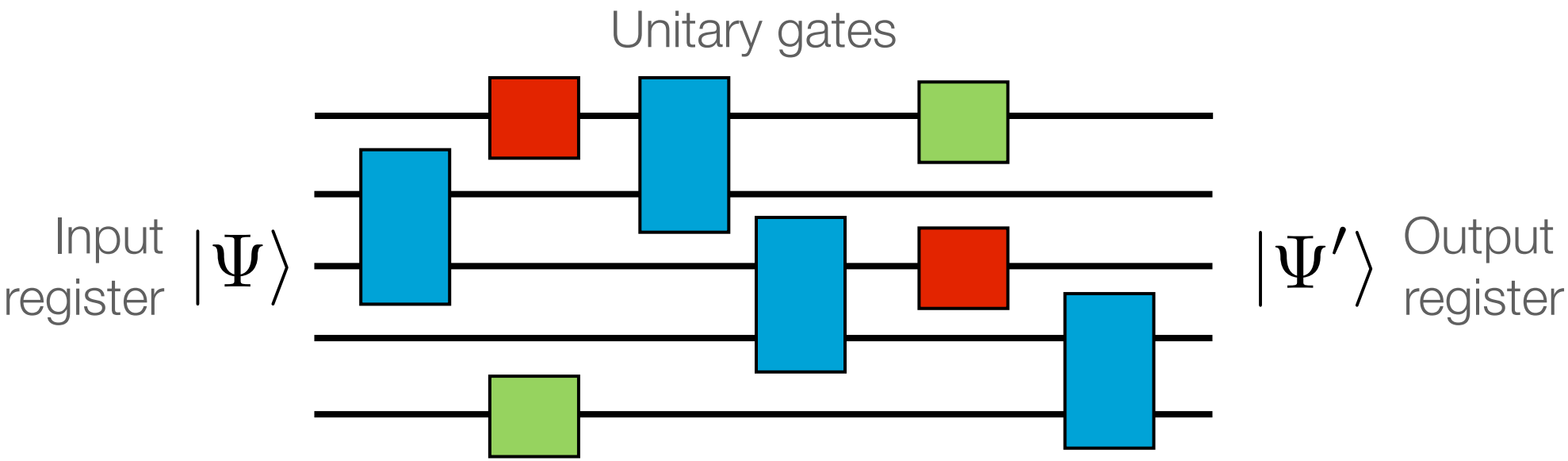
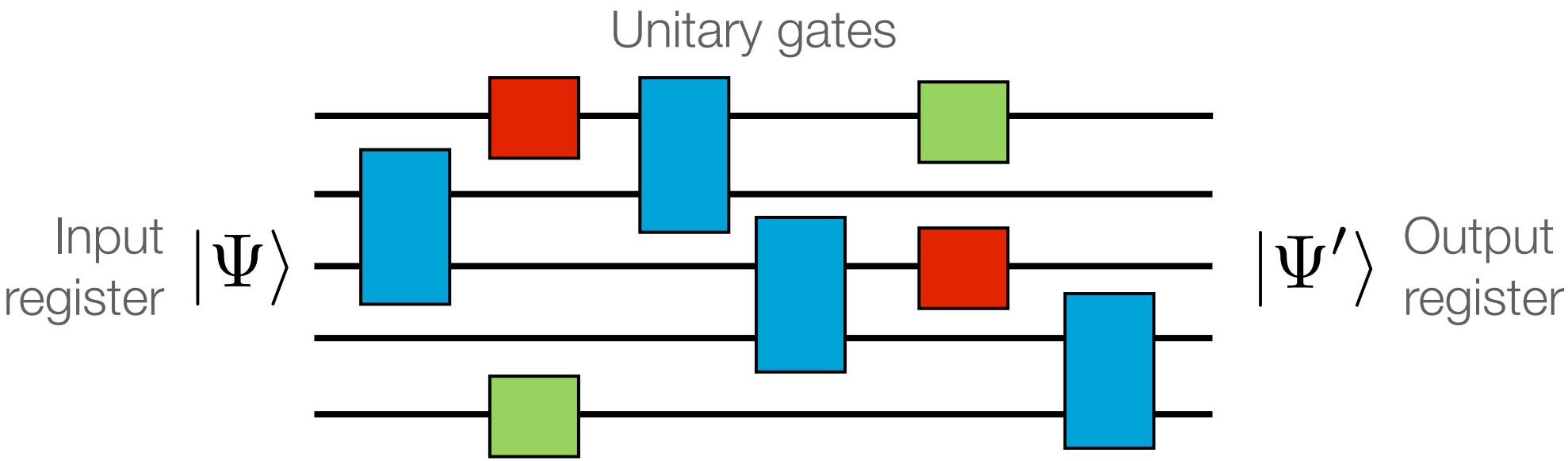


Quantum Circuits

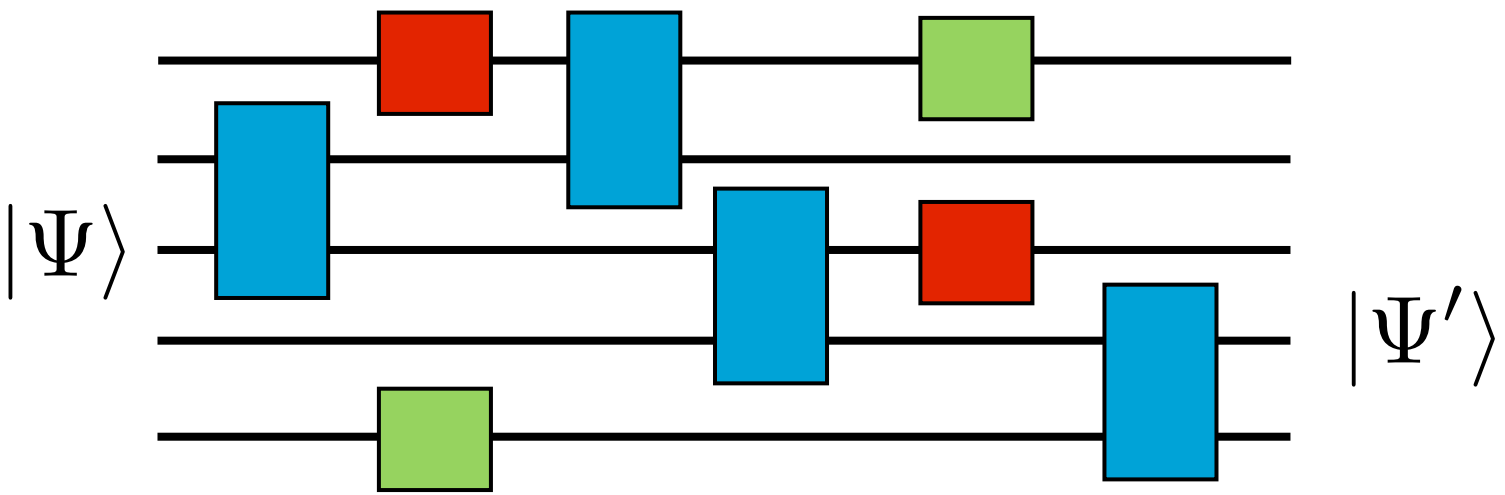
Model of quantum computation analogous to classical Boolean circuits:



$$U : \mathbb{C}^{2^n} \longrightarrow \mathbb{C}^{2^n}$$

Quantum Circuits

Circuits can be generalised:



Quantum Circuits

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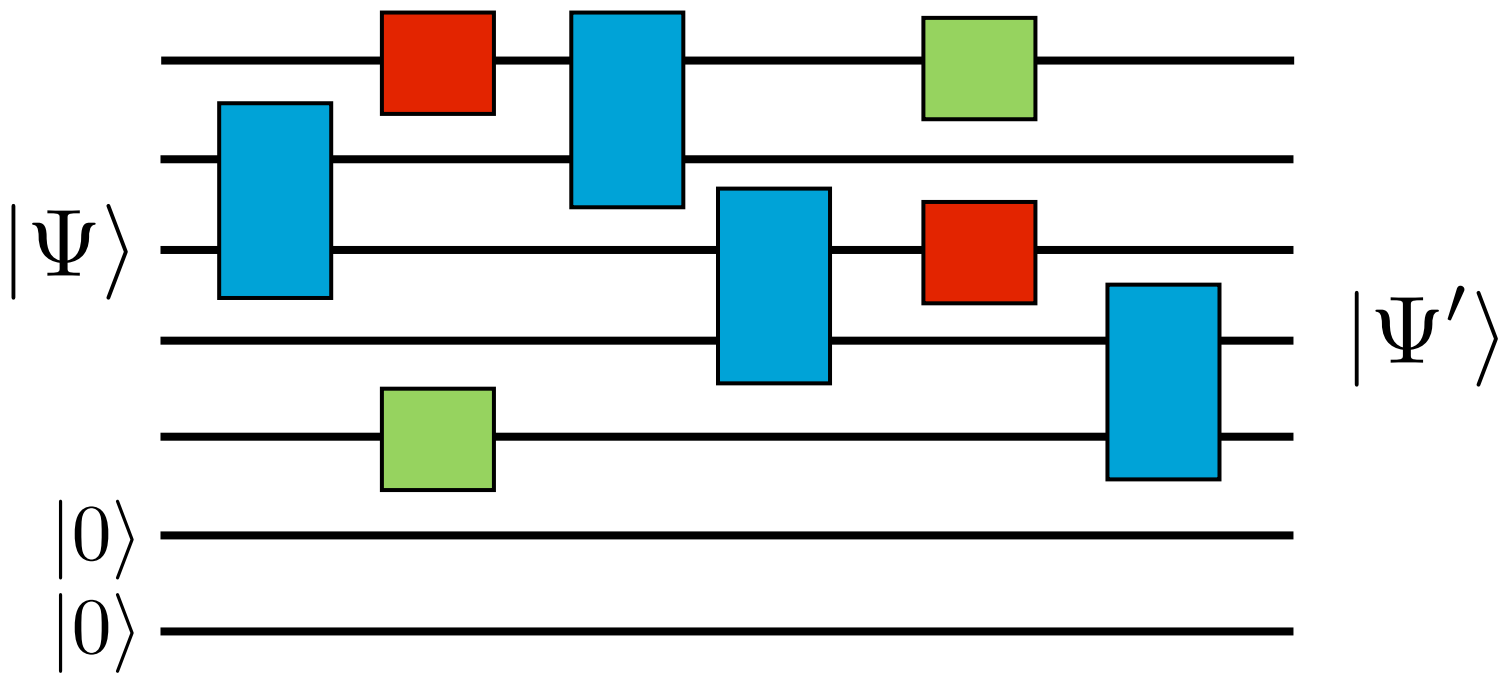

Quantum Circuits

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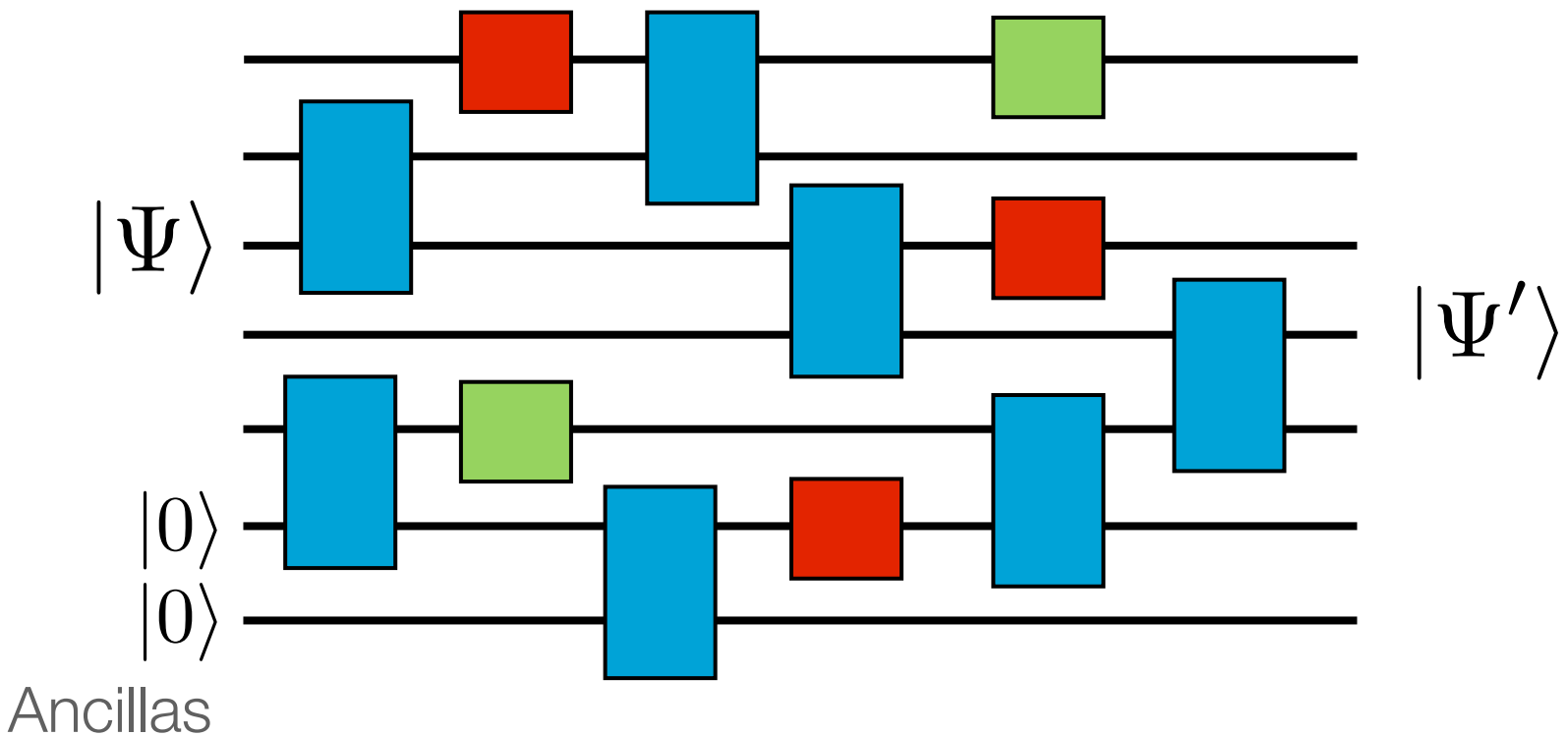

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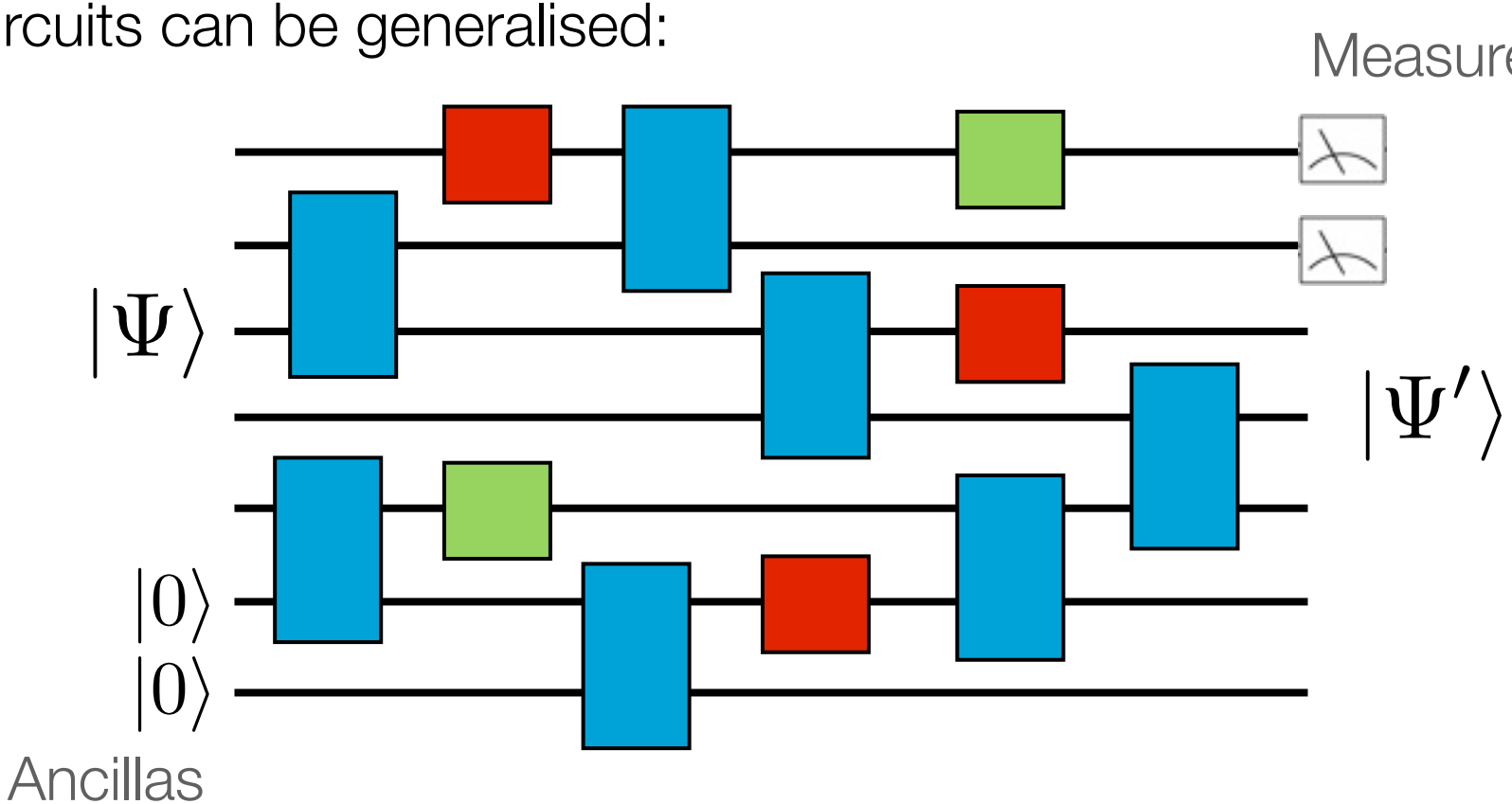

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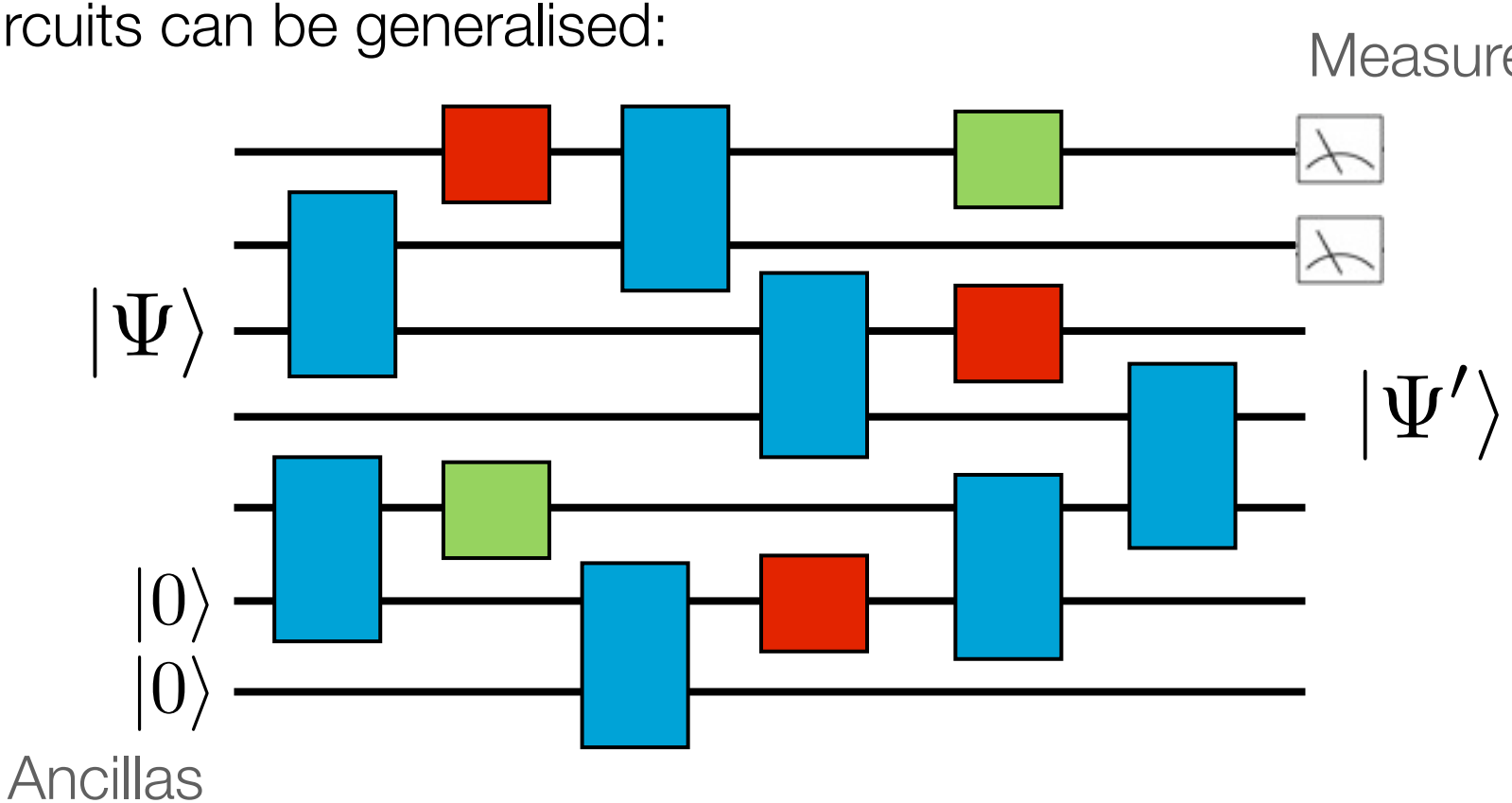
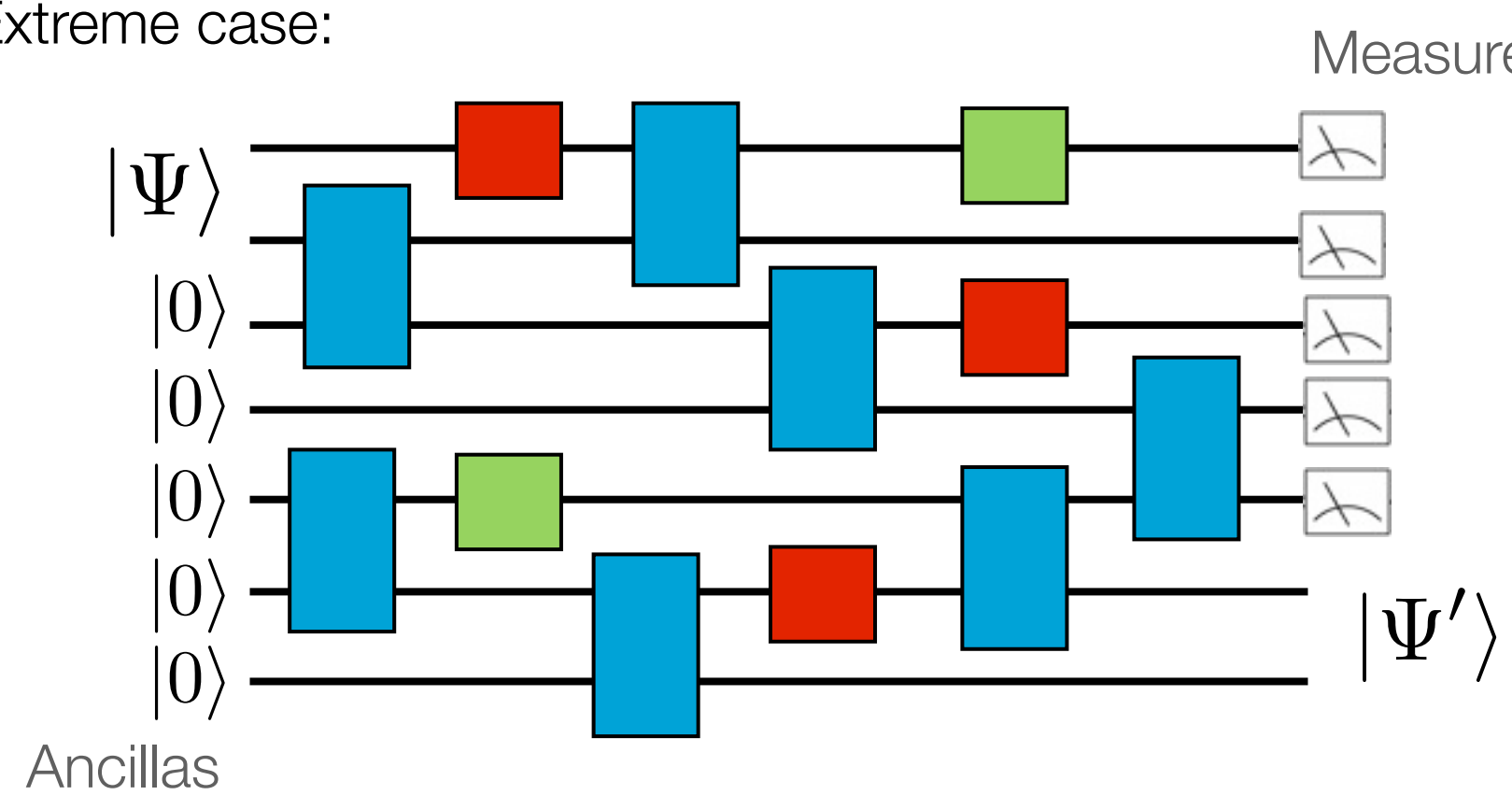


$$C : \mathbb{C}^{2^n \times 2^n} \longrightarrow \mathbb{C}^{2^m \times 2^m}$$

Quantum Circuits

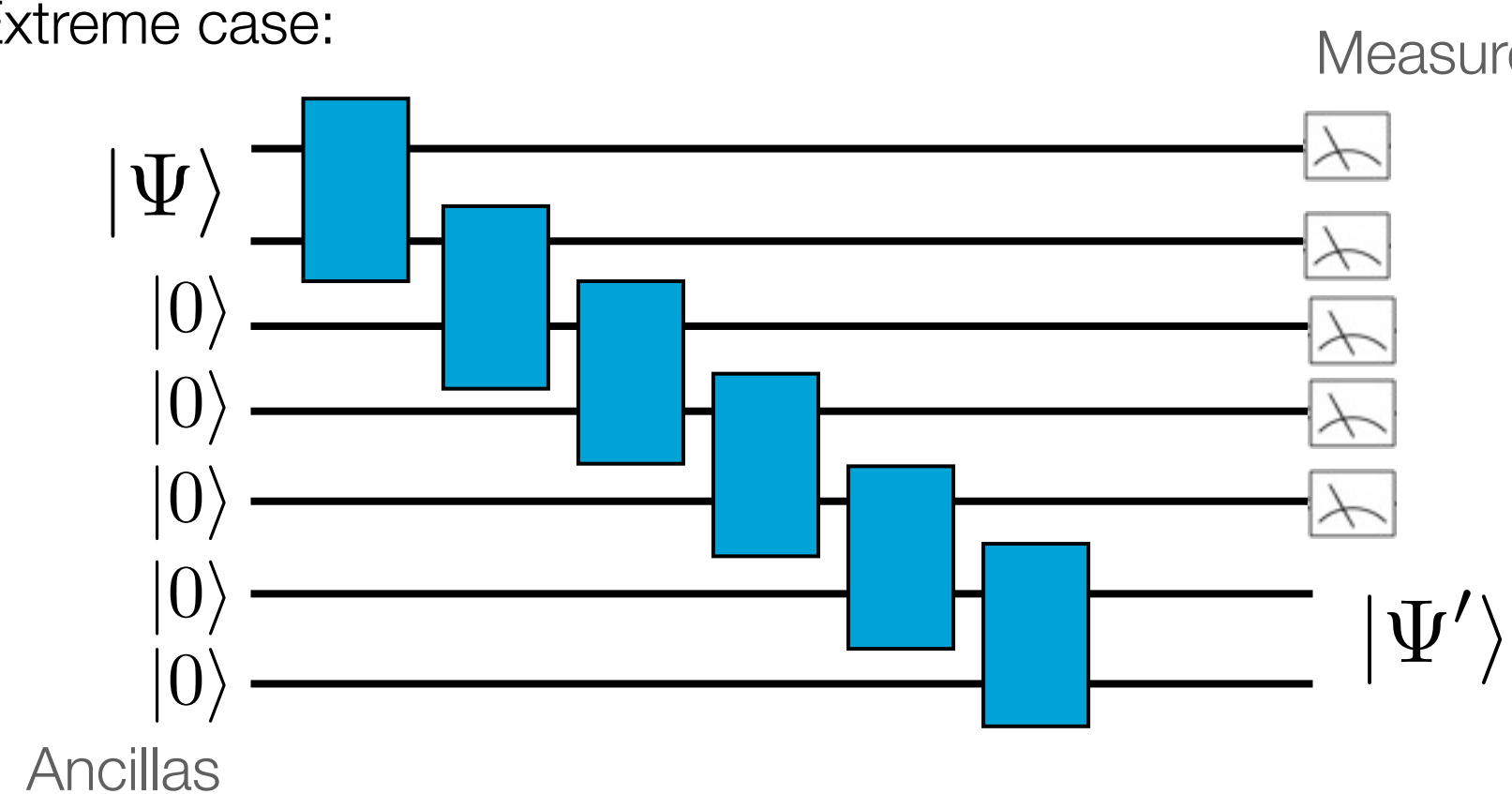
Extreme case:



$$C : \mathbb{C}^{2^n \times 2^n} \longrightarrow \mathbb{C}^{2^m \times 2^m}$$

Quantum Circuits

Extreme case:

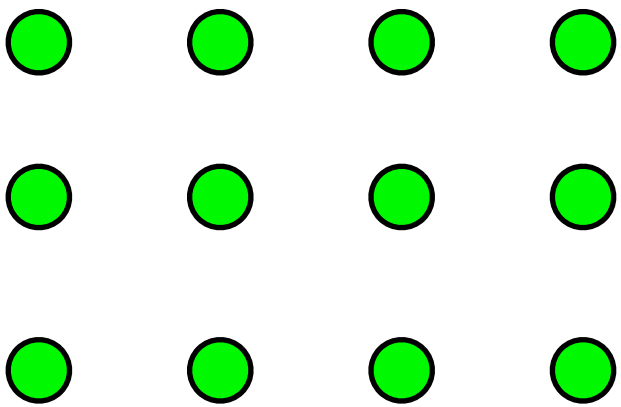


$$C : \mathbb{C}^{2^n \times 2^n} \longrightarrow \mathbb{C}^{2^m \times 2^m}$$

Raussendorf and Briegel, PRL (86) 2001
Raussendorf, Browne and Briegel, J. Mod. Opt. (46) 2003

The One-Way Model

Some qubits, each initialised in the state $|+\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



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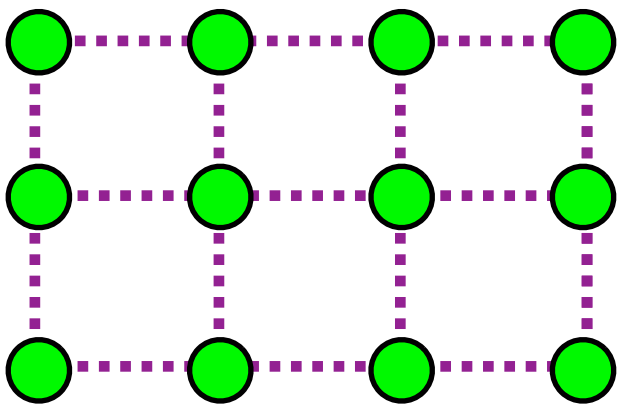


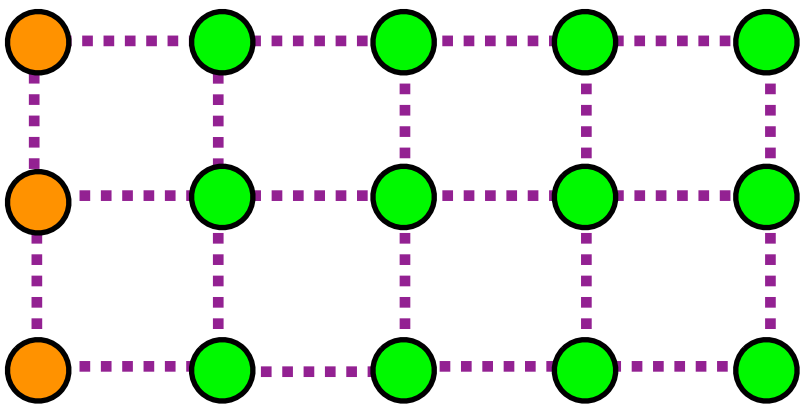
Entangling operation on pairs of qubits to make a **graph state.**

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Raussendorf and Briegel. PRL (86) 2001
Raussendorf, Browne, Briegel J. Mod. Opt. (46) 2003

The One-Way Model

A graph state, coupled to some input qubits:



The only operation is to measure single qubits in the basis:

$$|+_\alpha\rangle := \frac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle)$$

$$|-_\alpha\rangle := \frac{1}{\sqrt{2}}(|0\rangle - e^{i\alpha}|1\rangle)$$

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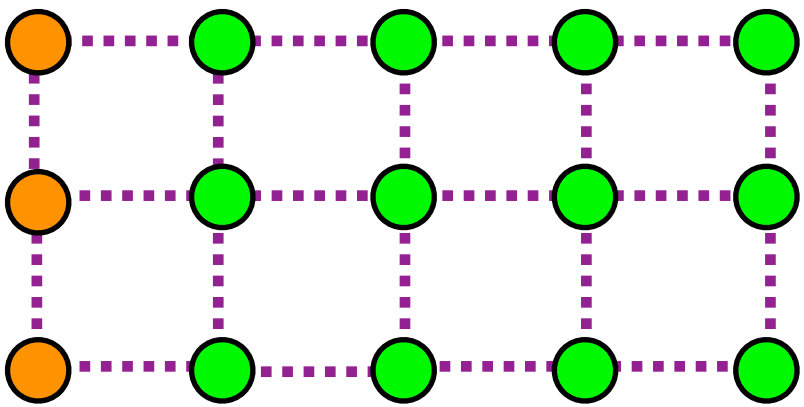

* Measured qubits are removed from the cluster;
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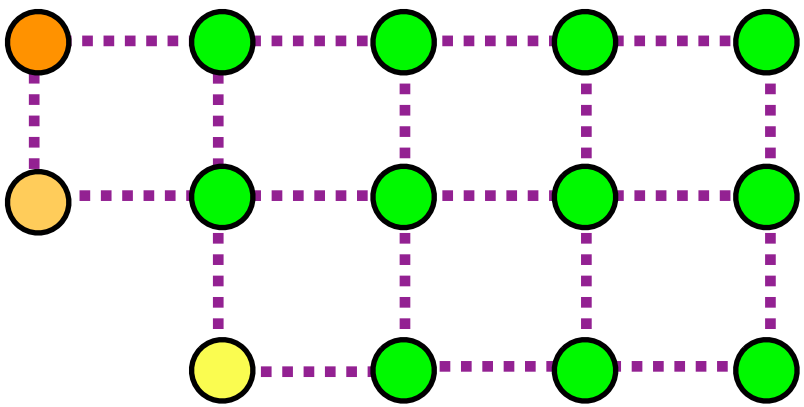

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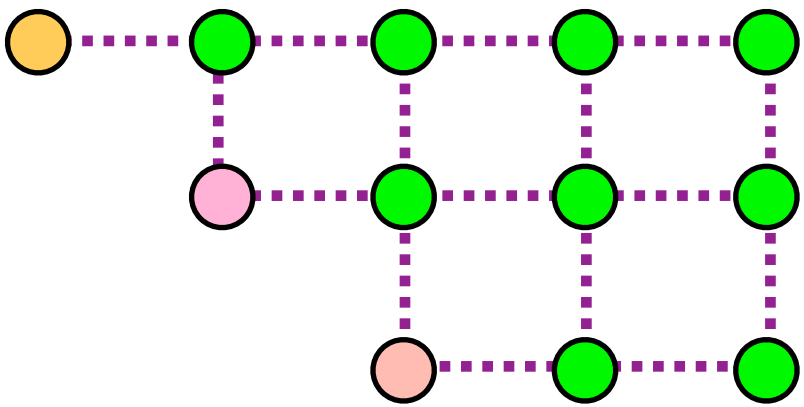

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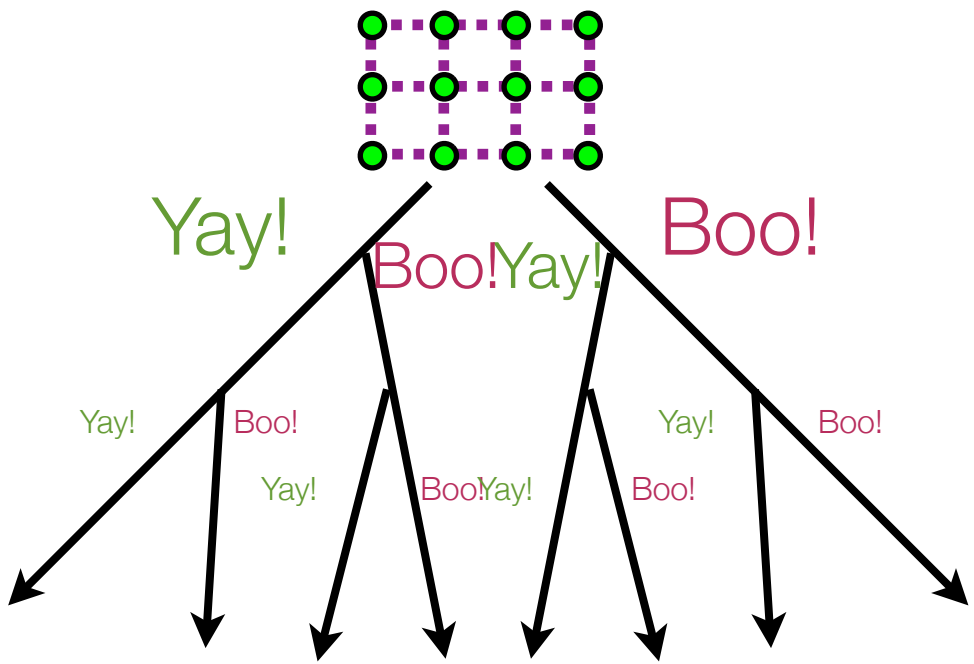
Non-determinism

Non-determinism of measurements leads to probabilistic



Attempt to control branching by using *adaptive measurements*:

(choice of measurements depend on the outcome of earlier ones)

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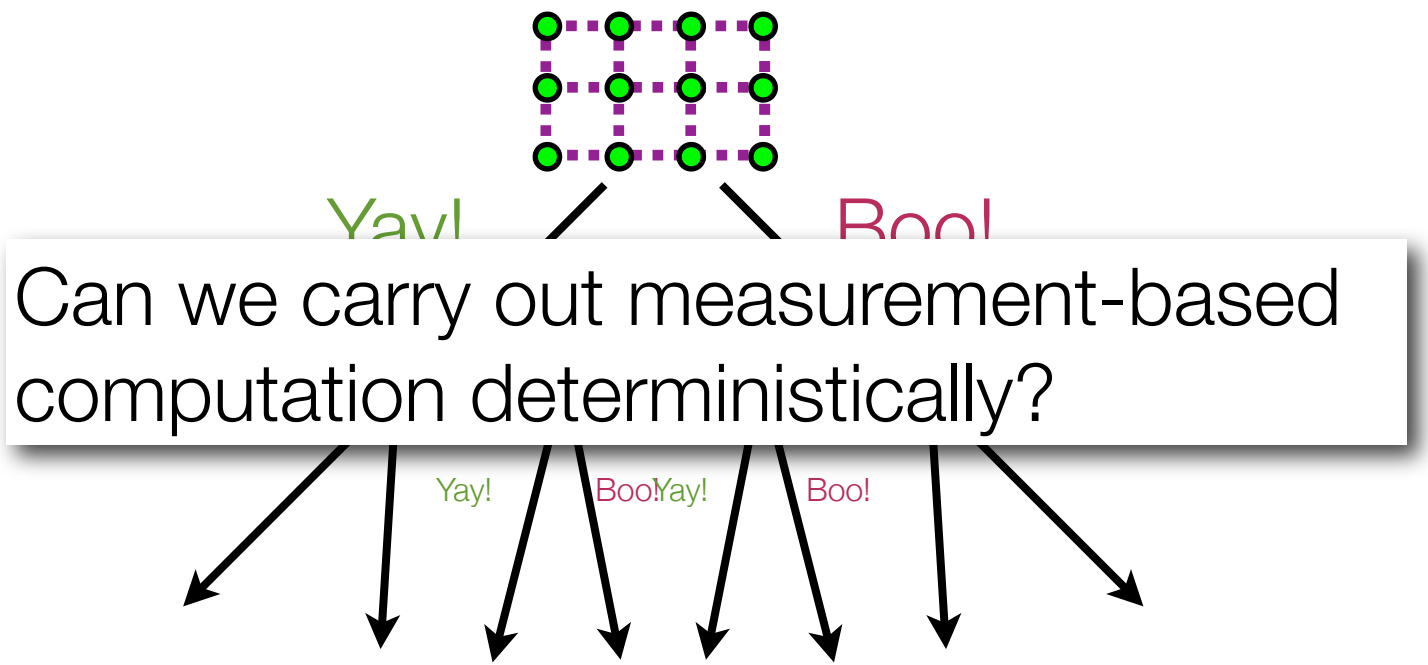

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Measurement Calculus

The measurement calculus is a formal syntax for programming the one-way model.

- N_i : initialise qubit i in the $|+\rangle$ state
- E_{ij} : entangle qubits i and j using a CZ operation
- ${}^{s}[M_i^{\alpha}]^{t}$: measure qubit i in the basis $|0\rangle \pm e^{(-1)^s \alpha + t\pi} |1\rangle$
- X_i^s, Z_i^s: apply a Pauli X or Z operator to qubit i

The boolean variables s, t are *signals*; their values are determined by the results of measuring the corresponding qubit.

- Standard form: operations occur in the order above
  - Every pattern can be standardised

How to Measure

Suppose we have a measuring device for the standard 1-qubit basis:

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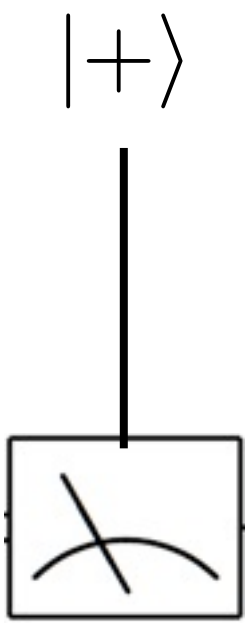
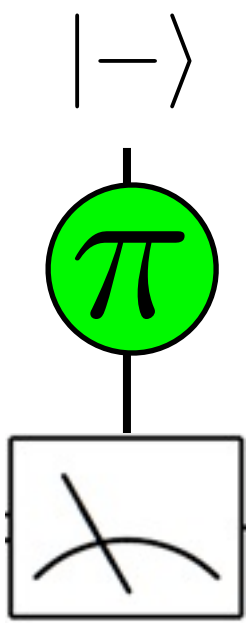
Should be this:

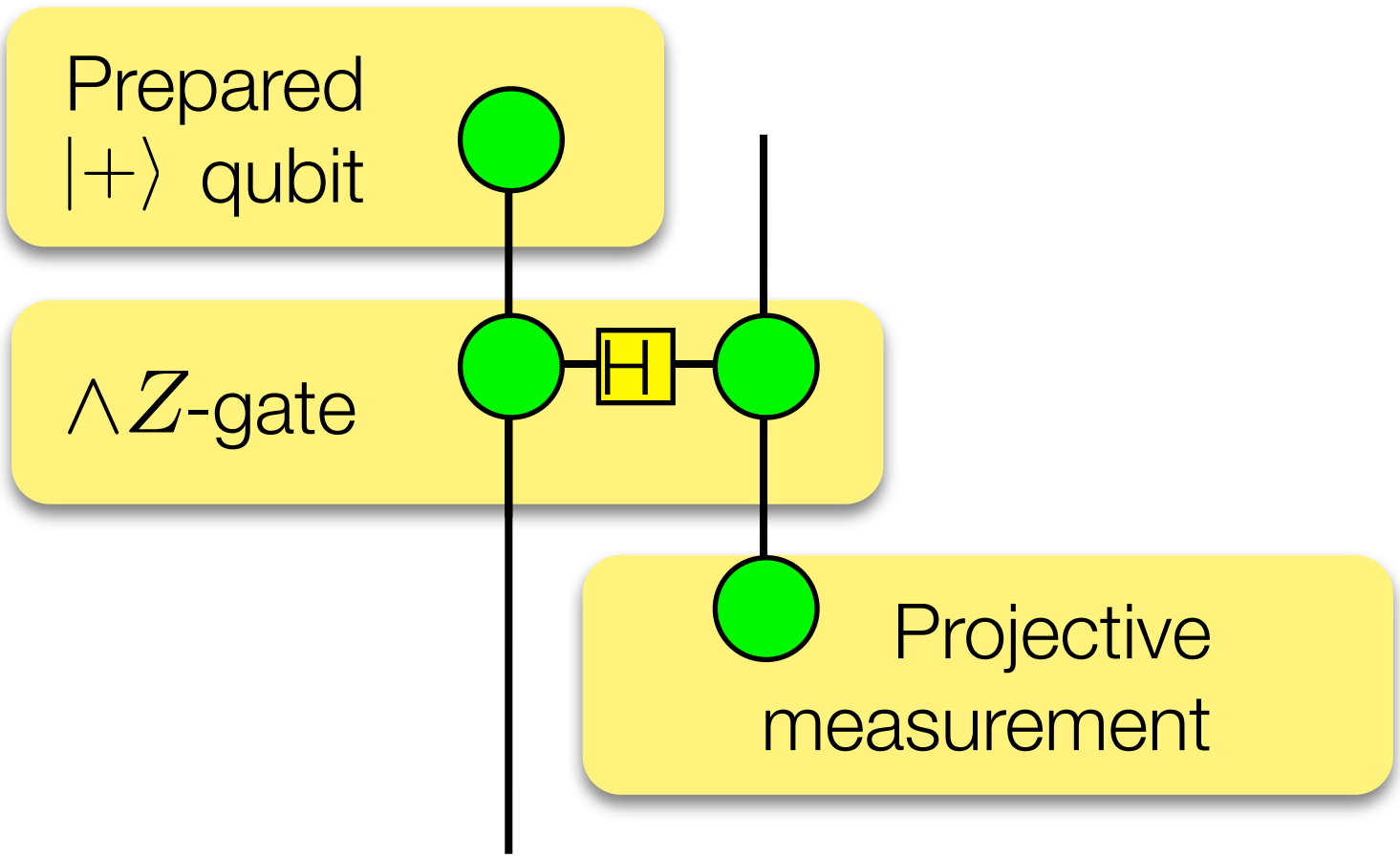
Yay!





Yay!

Example: Hadamard



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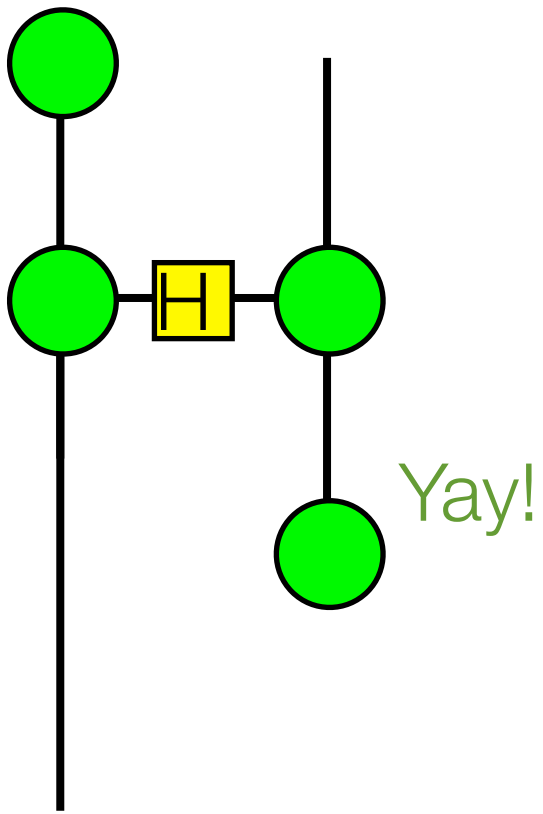

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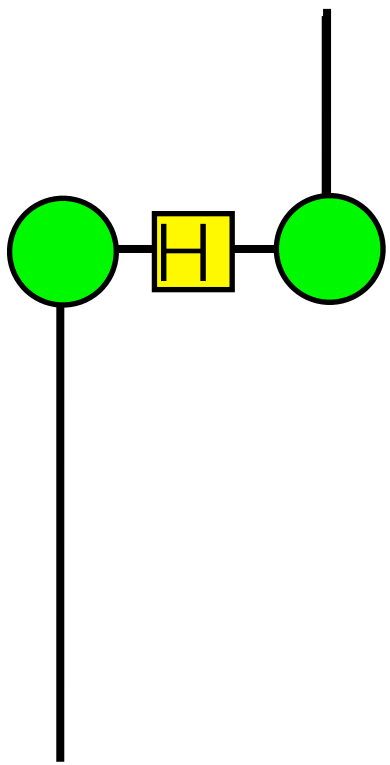

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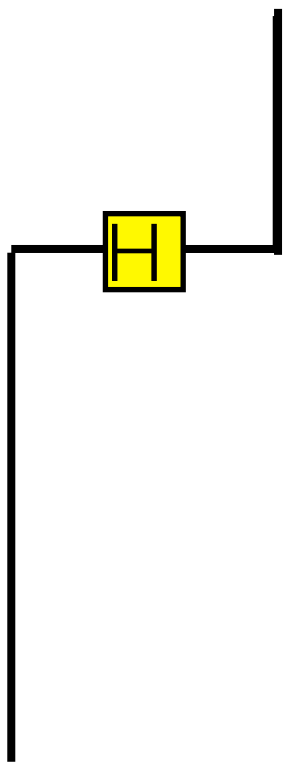

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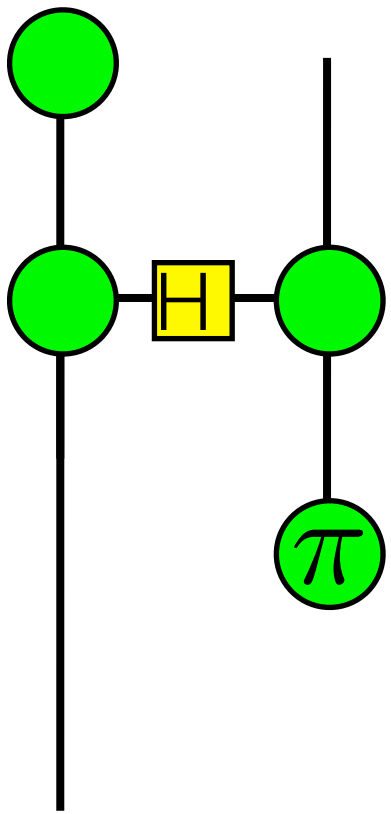

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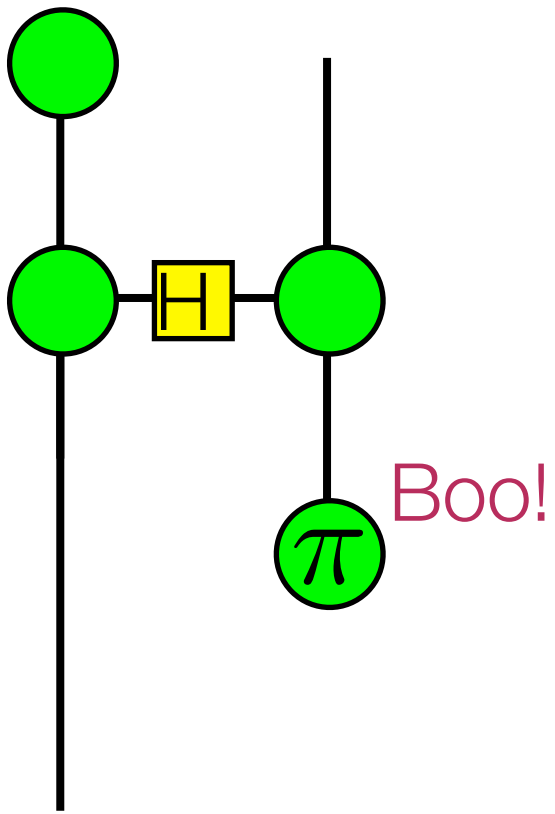

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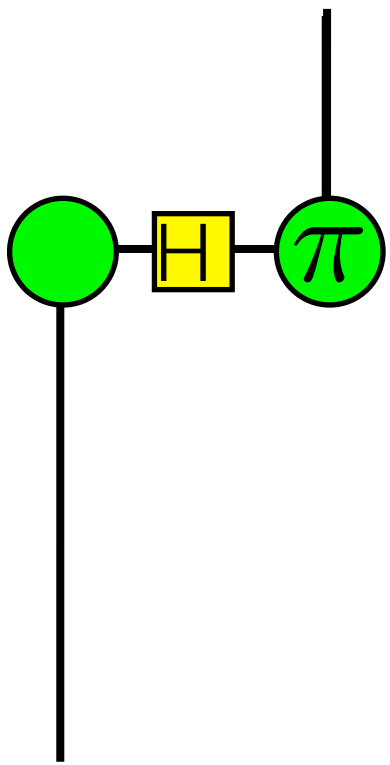

Boo!

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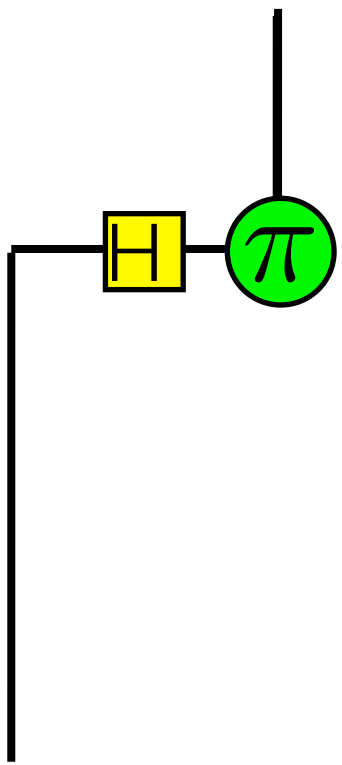

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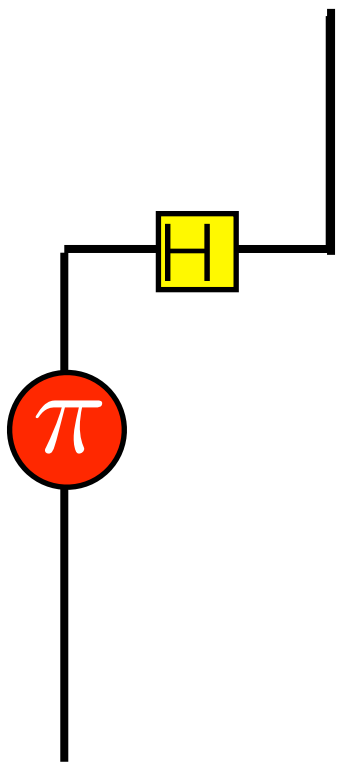
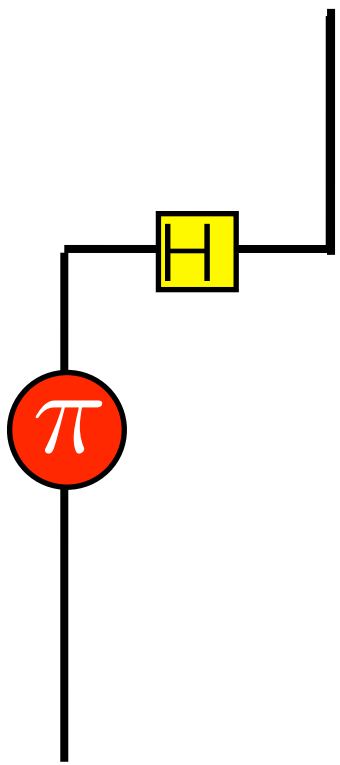


Boo!

Example: Hadamard

Add correction here:



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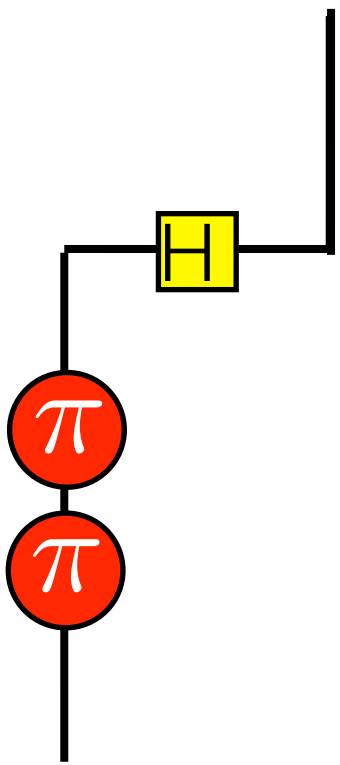

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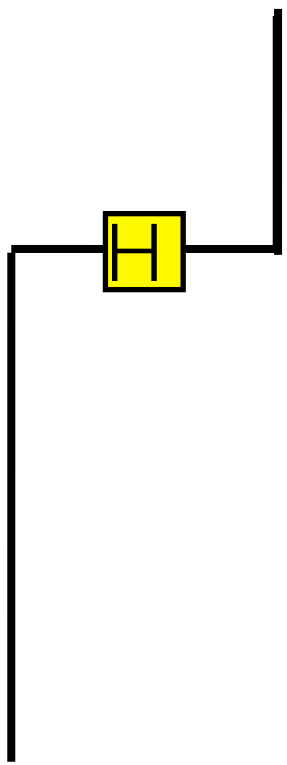
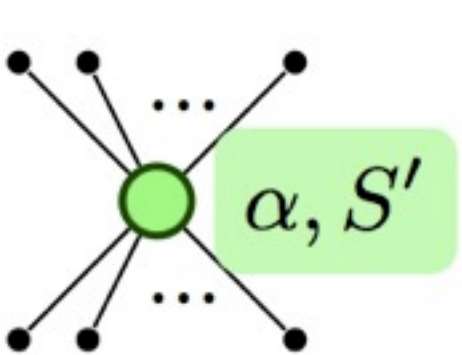


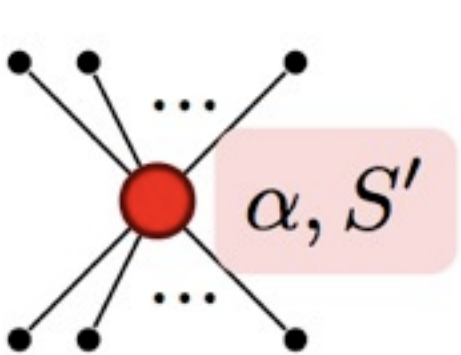
Boo!

Add correction here:

Conditional Operations

Defn: Let S be a set of variables ("signals"); define $\mathbb{D}(S)$ to be the category of diagrams generated by:





where $S' \subseteq S$

Semantics

Defn: a *valuation* of S is a function $v : S \to \{0, 1\}$. For each valuation we can define a map $\hat{v} : \mathbb{D}(S) \to \mathbb{D}$ by replacing the angle alpha with 0 if

$$\prod_{s \in S'} v(s) = 0$$

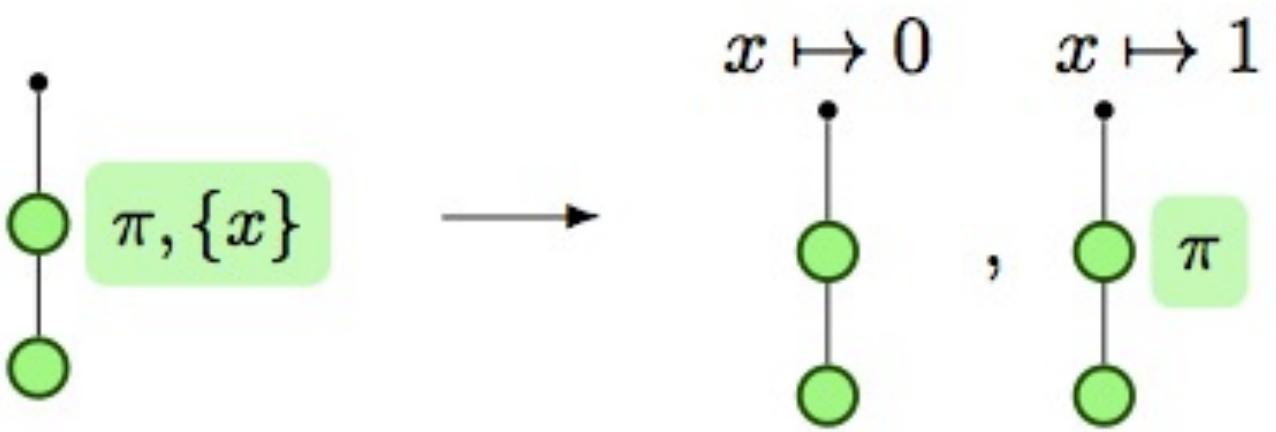
and leaving if unchanged otherwise.

Defn: The **denotation** of a diagram D in $\mathbb{D}(S)$ is a superoperator:

$$\rho \mapsto \sum_{v \in 2^S} [\![\hat{v}(D)]\!] \rho [\![\hat{v}(D)]\!]^\dagger$$

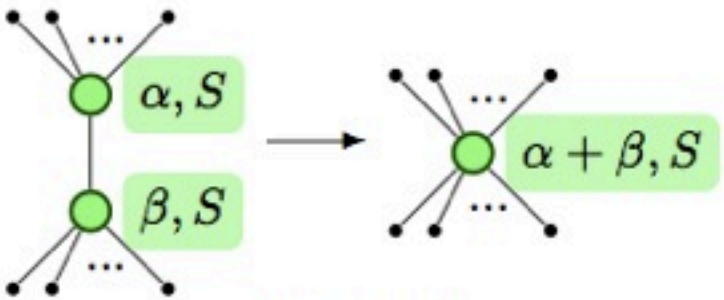
Example

A measurement in the basis: $|+\rangle, |-\rangle$



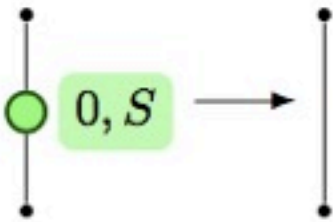
$$\rho \mapsto \langle +|\rho|+\rangle + \langle +|Z\rho Z|+\rangle$$
$$= \langle +|\rho|+\rangle + \langle -|\rho|-\rangle$$

Equations for conditional diagrams

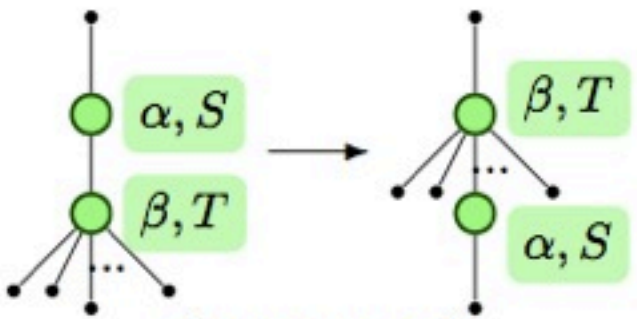


(spider)

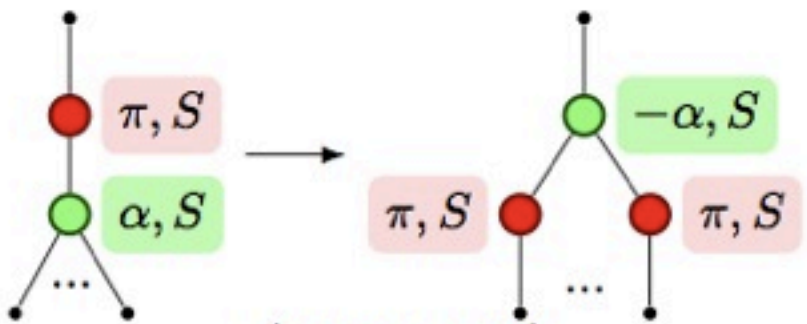
(anti-loop)



(identity)



(α-commute)



(π-commute)

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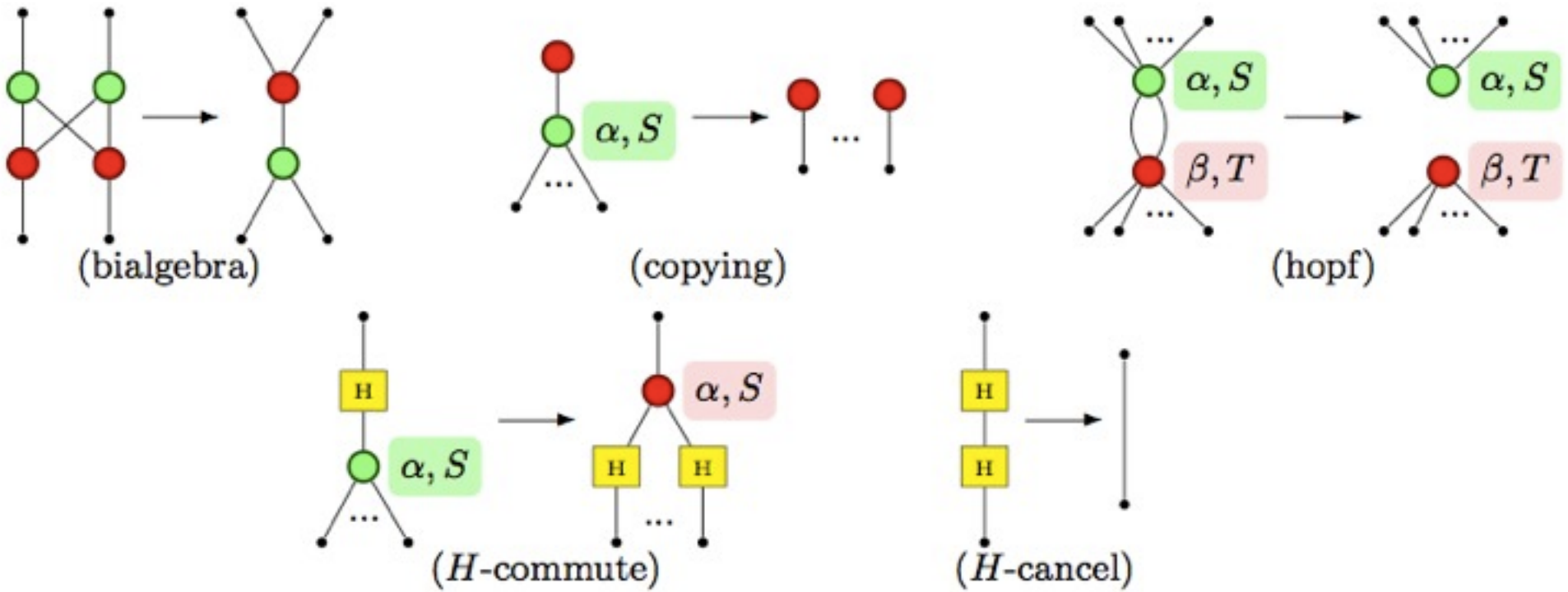

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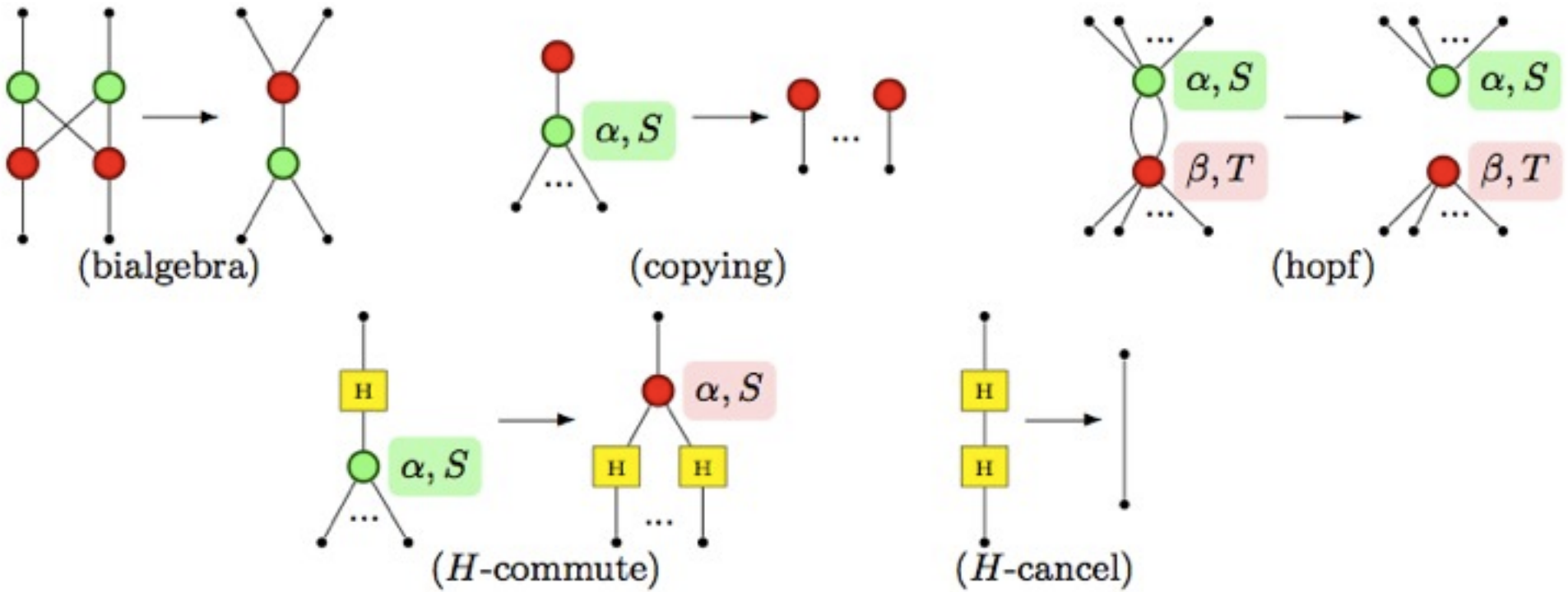
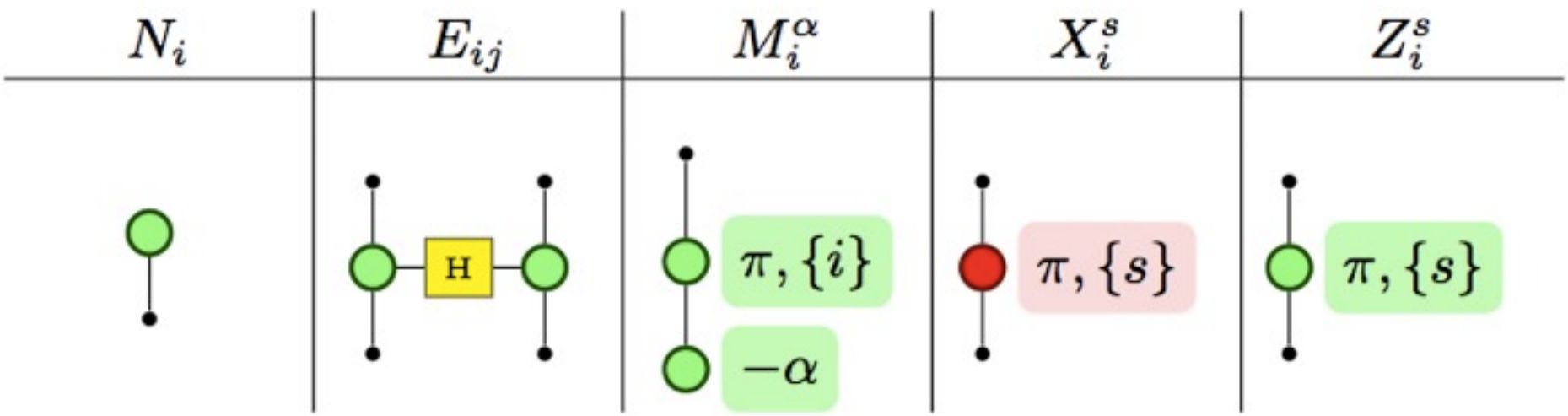


.... plus the same again with the colours exchanged

Translation from MC to ZX-calculus

We can translate any measurement pattern to diagram using the table below:



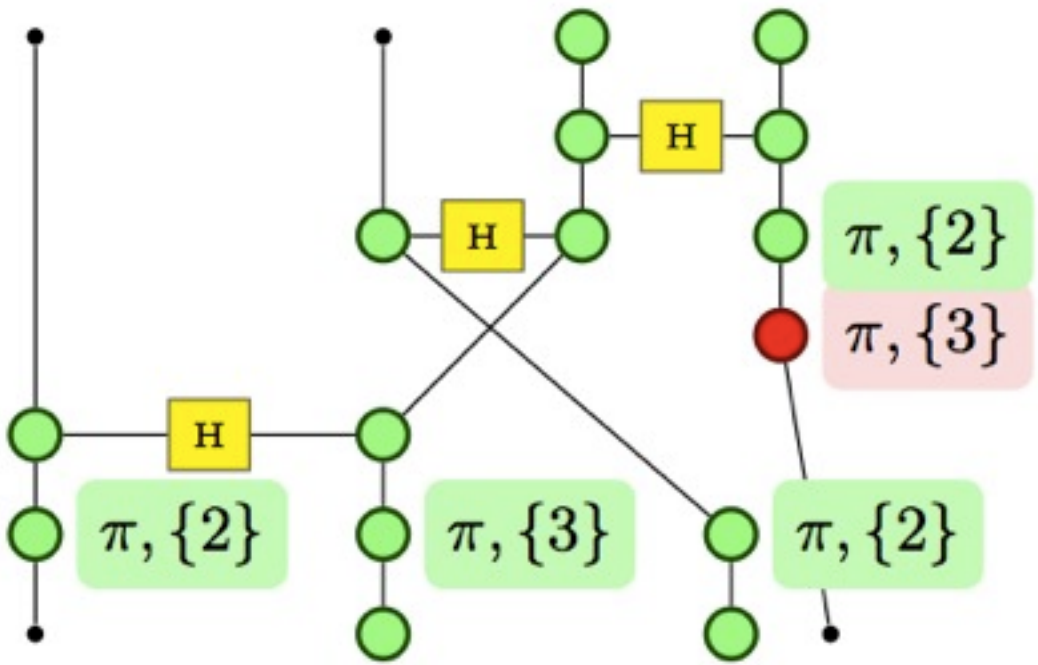
Example

The CNOT is computed by the pattern:

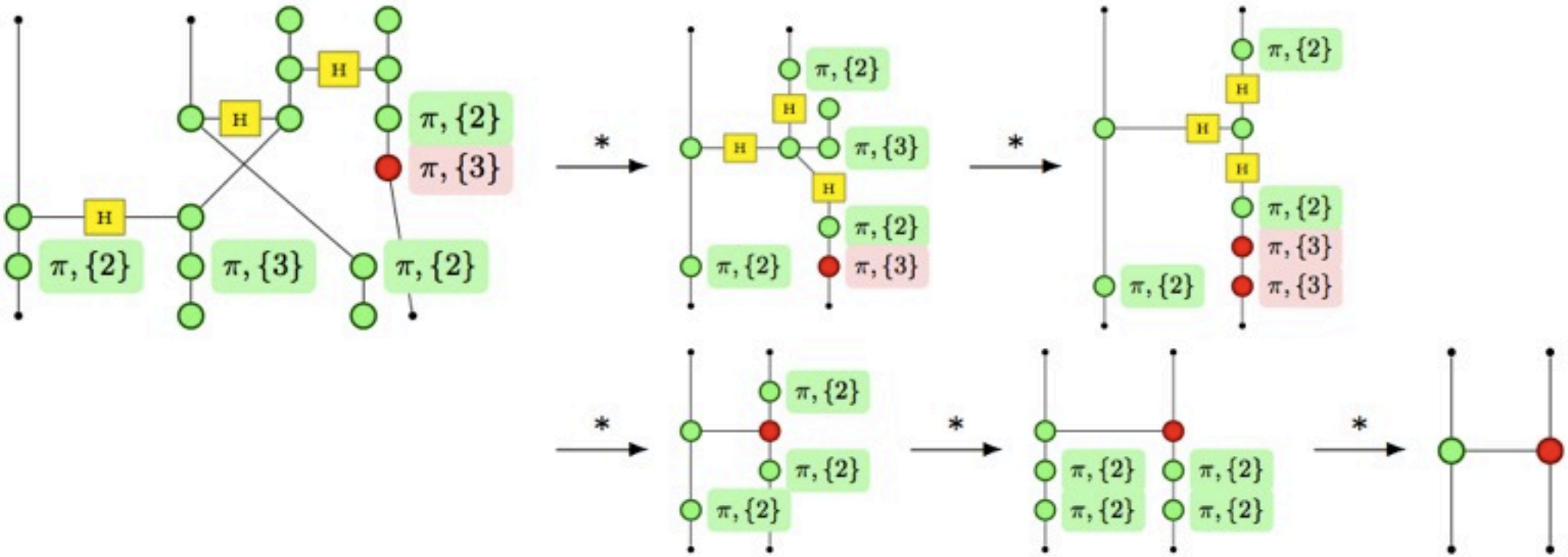
$$\mathfrak{P} = X_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 E_{13} E_{23} E_{34} N_3 N_4$$

Which yields the diagram:

$D_{\mathfrak{P}} =$



Example Rewrite



We removed all the conditional operations, therefore this **pattern** is deterministic.

Can we do this in general?

Geometry of a pattern

The following pattern computes the CNOT gate:

$$\mathfrak{P} = \bar{X}_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 \bar{E}_{13} E_{23} E_{34} N_3 N_4$$

Each pattern implicitly defines a *geometry* (G, I, O)

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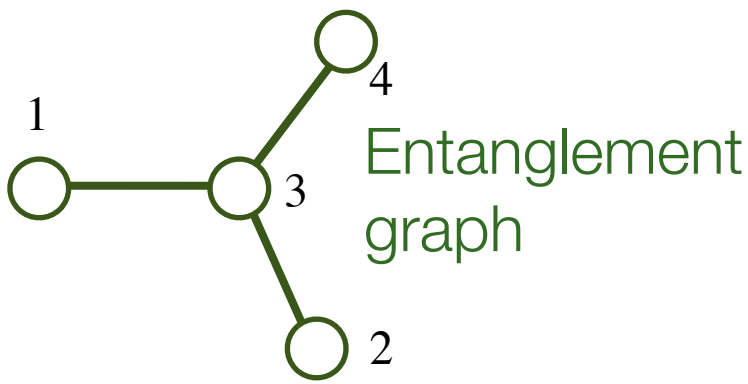

Entanglement graph

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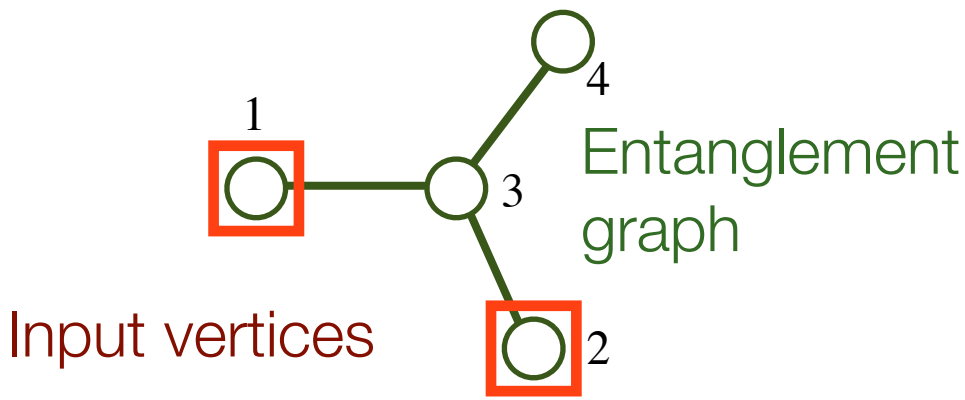

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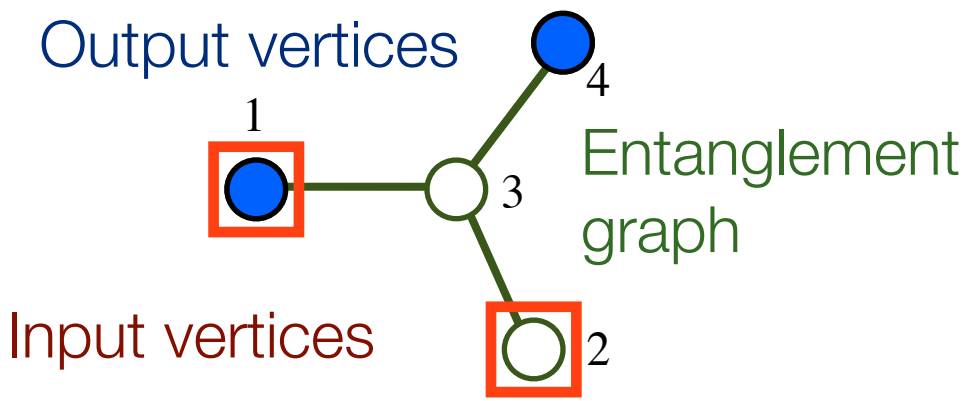


Flow and determinism

Defn: let (G, I, O) be a geometry; a *flow* on G is a pair $(f, <)$ where $f : O^C \to I^C$ is a function and $<$ is a partial order s.t.

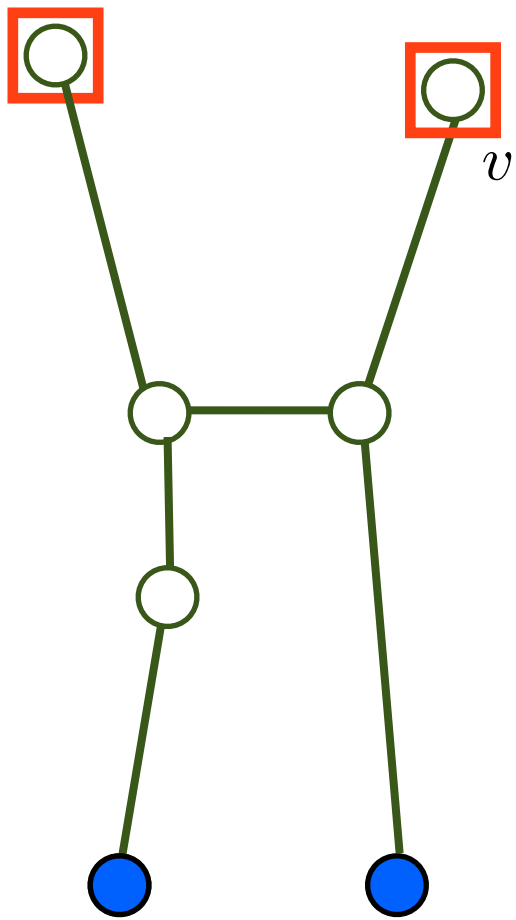
- $f(u) \sim u$
- $u < f(u)$
- If $f(u) \sim v, v \neq u$ then $u < v$

Intuitively $f(u)$ is the qubit which must correct the error produced by measuring u. The partial order guarantees that this is consistent with causality.

Thm: if a geometry has a flow there exists a uniformly deterministic pattern on it.

Flow

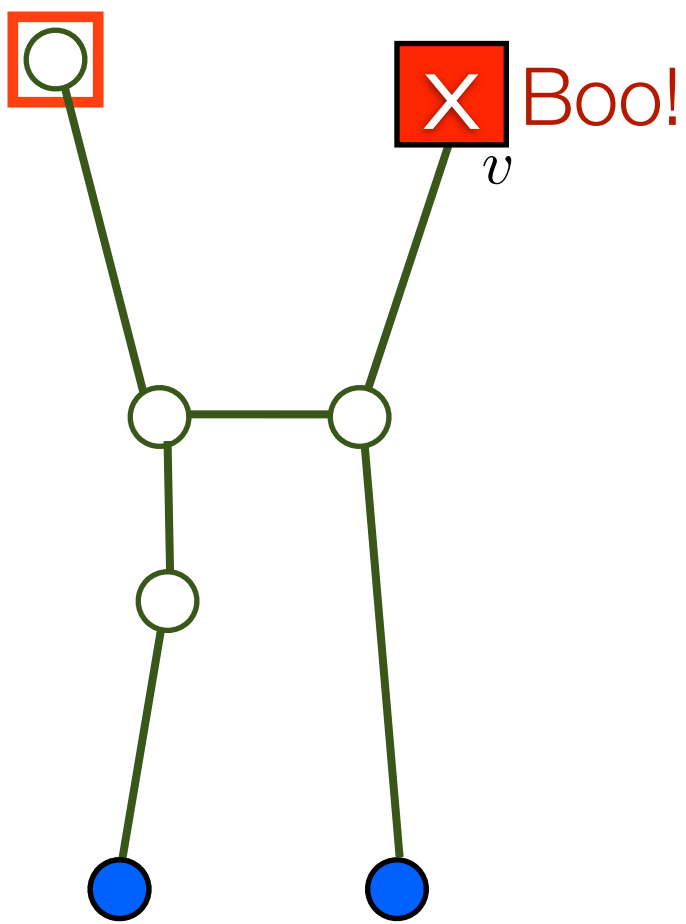
Input vertices



Output vertices

Flow

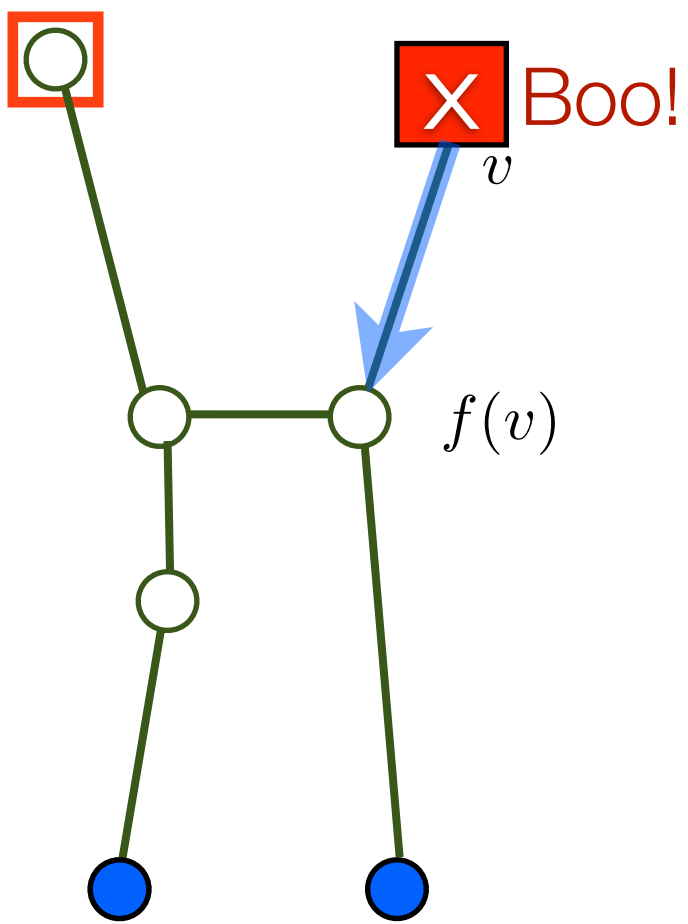
Input vertices



Output vertices

Flow

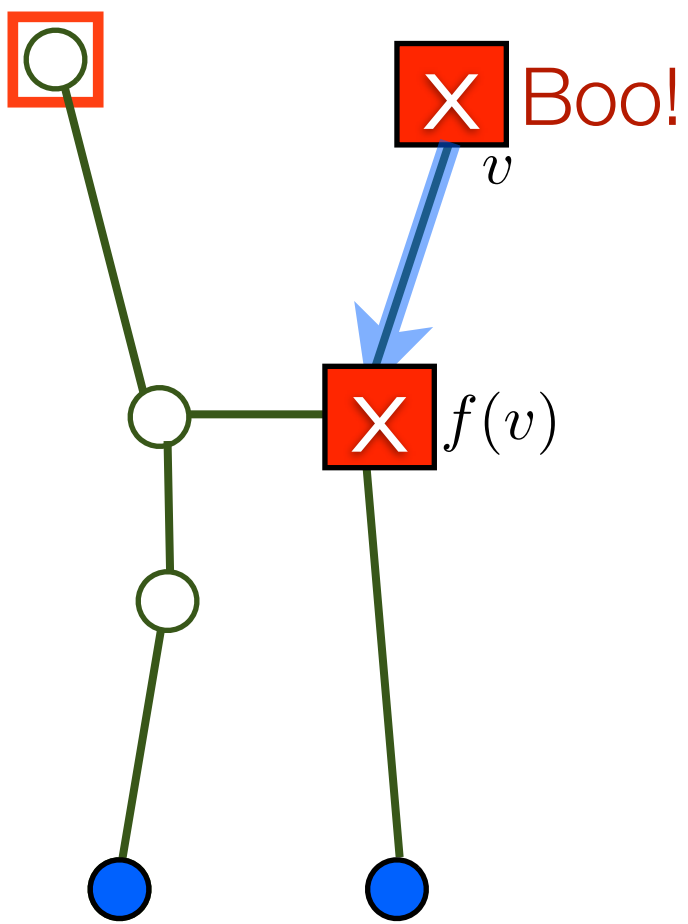
Input vertices



Output vertices

Flow

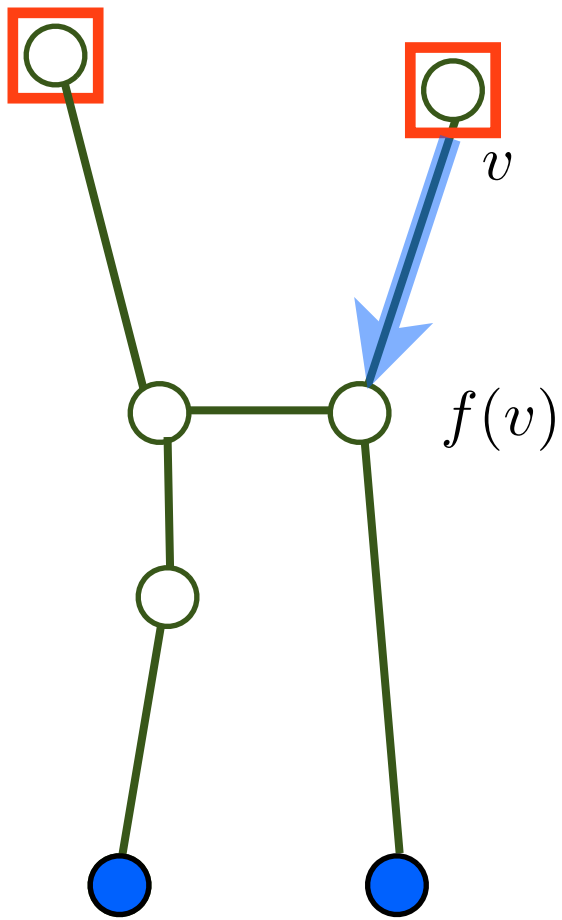
Input vertices



Output vertices

Flow

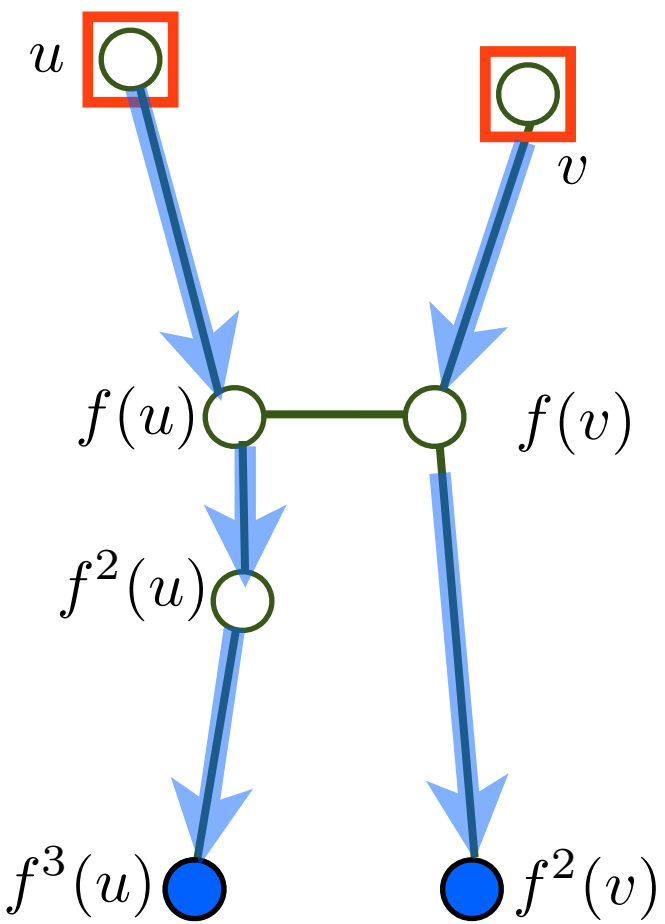
Input vertices



Output vertices

Flow

Input vertices



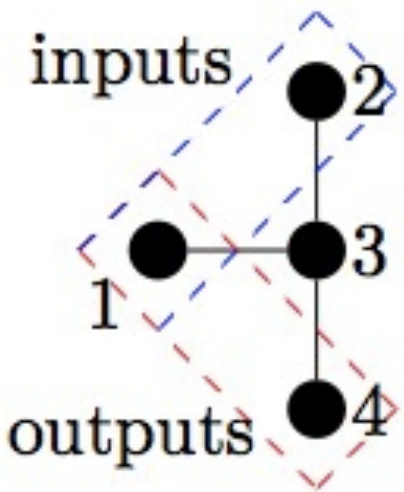
Output vertices

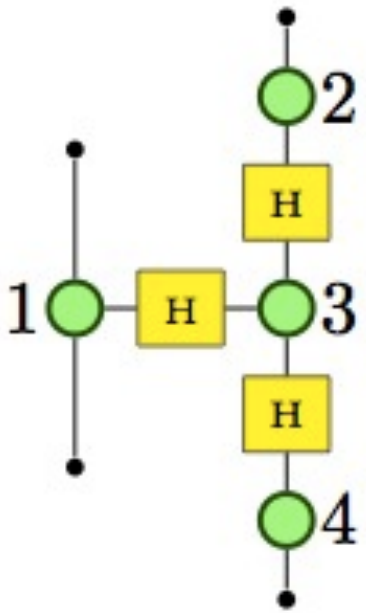
The sequence
$u, f(u), f^2(u), \ldots$
determines a path from inputs to outputs.

Effectively the flow path" is a "logical qubit" in a circuit equivalent to the pattern.

Geometry of a pattern

We can define a diagram directly from the geometry (G, I, O)

$\Gamma_{\mathfrak{P}_\wedge x} =$ 

$D(\Gamma_{\mathfrak{P}_\wedge x}) =$ 

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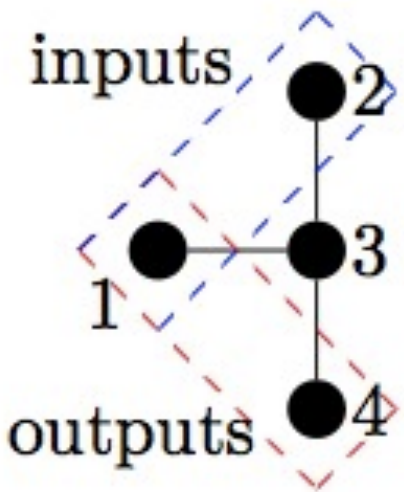
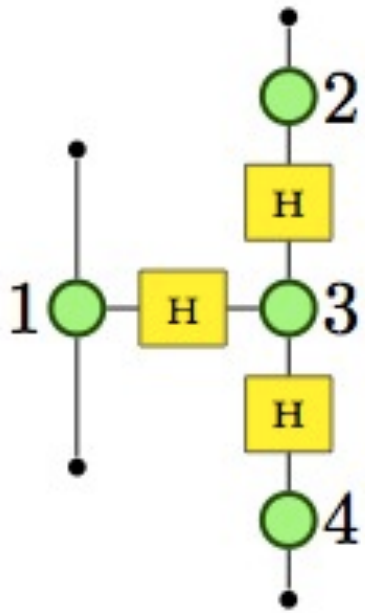
$\Gamma_{\mathfrak{P}_\wedge x} =$ 

$D(\Gamma_{\mathfrak{P}_\wedge x}) =$ 

Theorem: If geometry G has a flow then $D(G)$ is (equivalent to) a circuit-like diagram.

Getting back to the pattern

Getting back to the pattern

We have $D(G(P))$ and $D(P)$. What is the relation?

Getting back to the pattern

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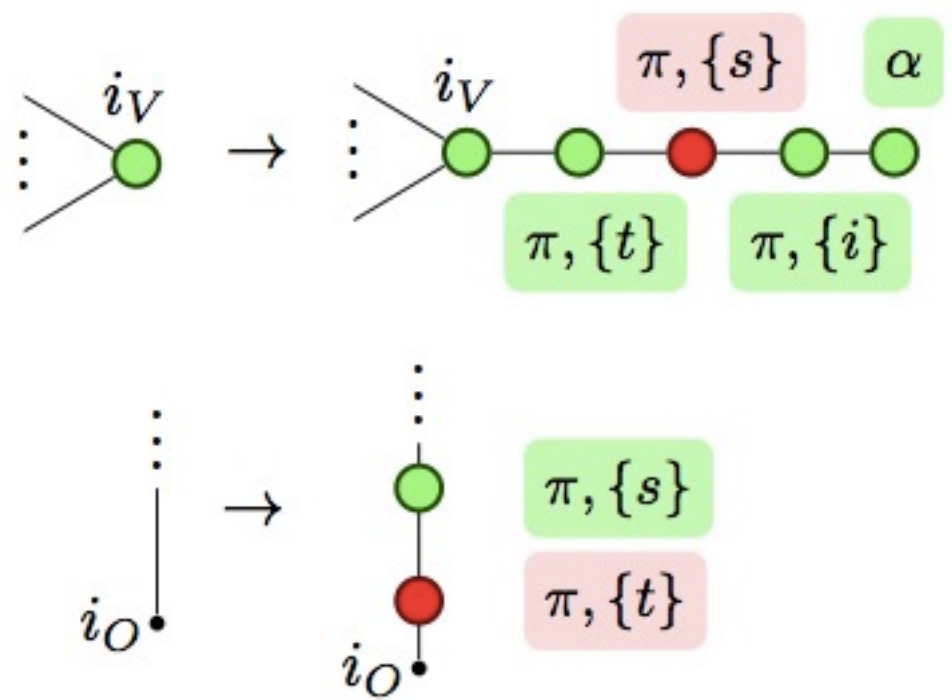
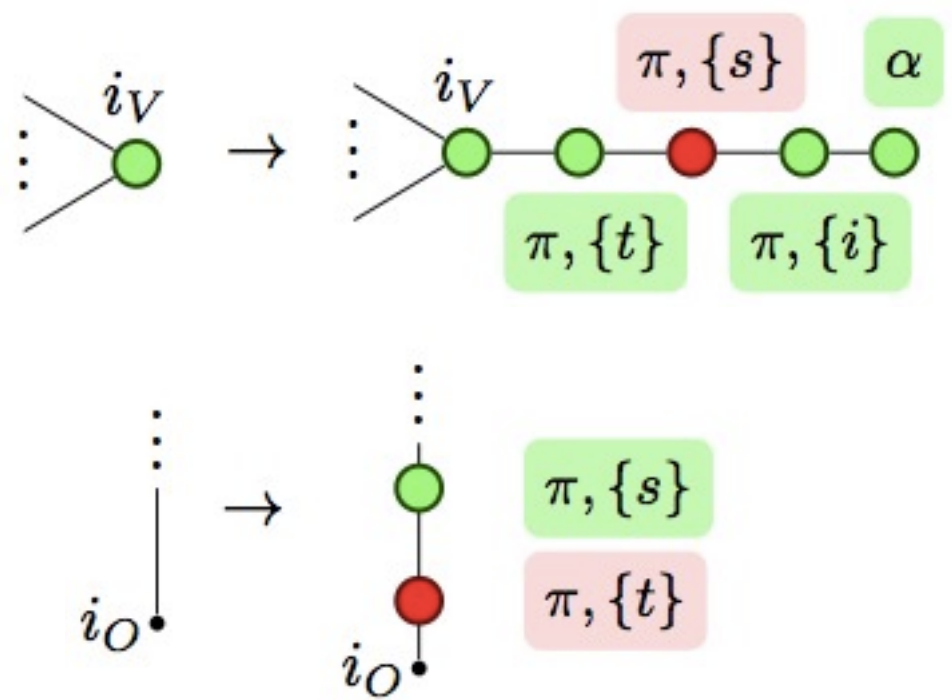
Define $D^*(P)$ by adding pieces to $D(G(P))$:



Getting back to the pattern

We have $D(G(P))$ and $D(P)$. What is the relation?

Define $D^*(P)$ by adding pieces to $D(G(P))$:

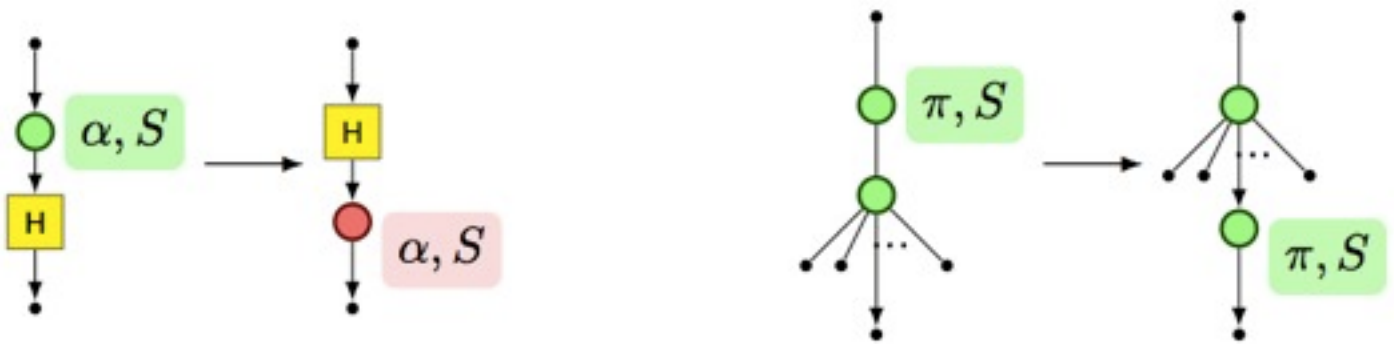


Prop: $D(P)$ rewrites to $D^*(P)$

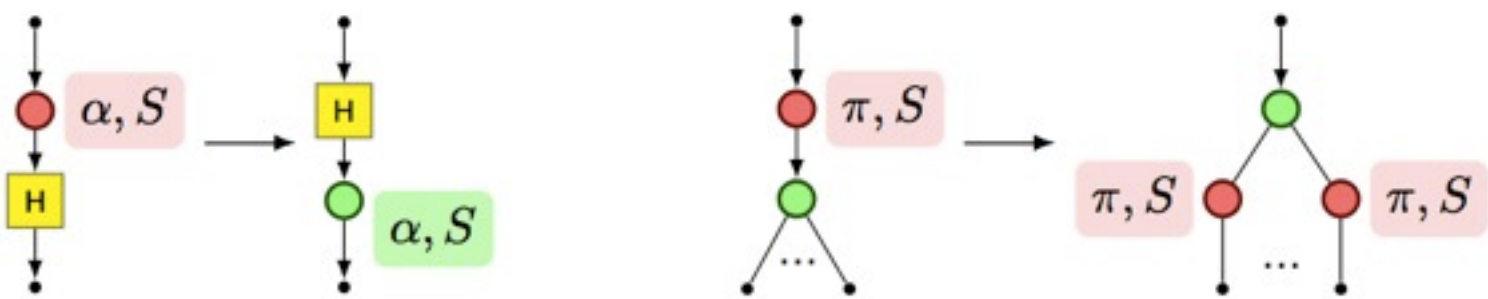
Flow Strategy

If the geometry has a flow we can define a rewrite strategy to propagate the “errors” forward:

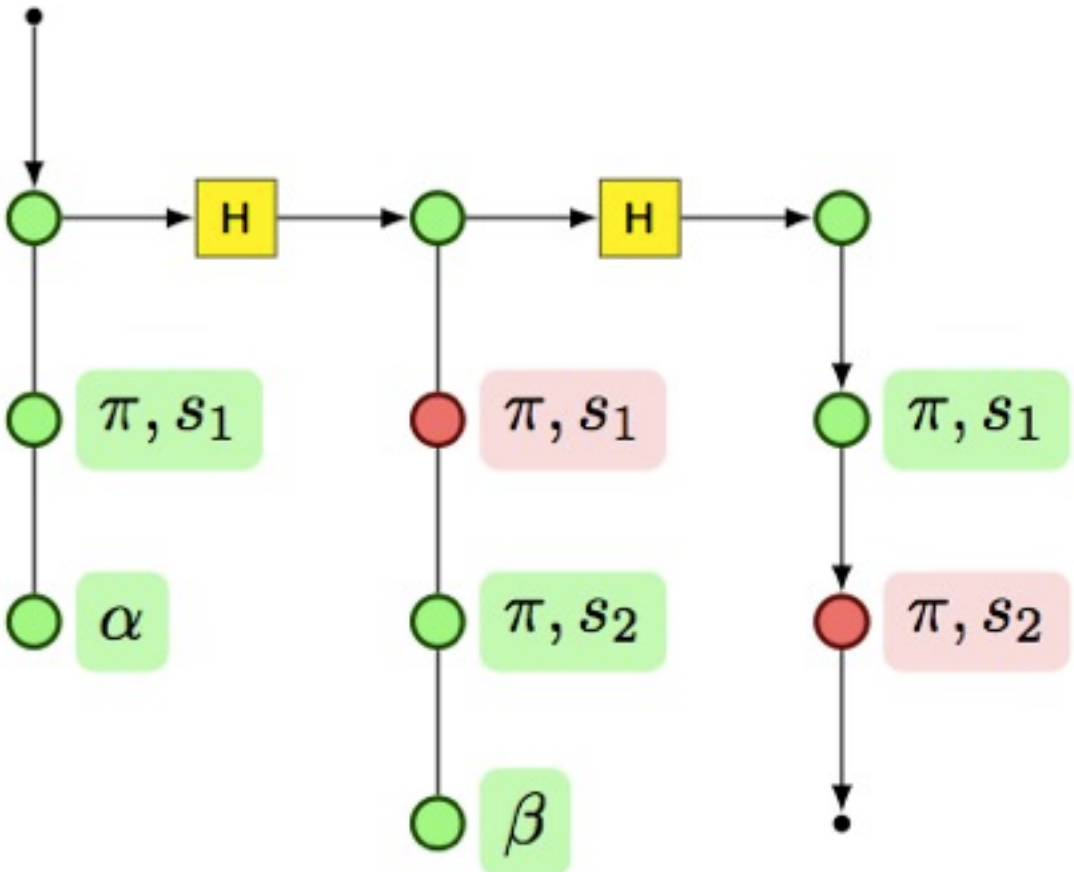
'π-green is attracted by the flow':



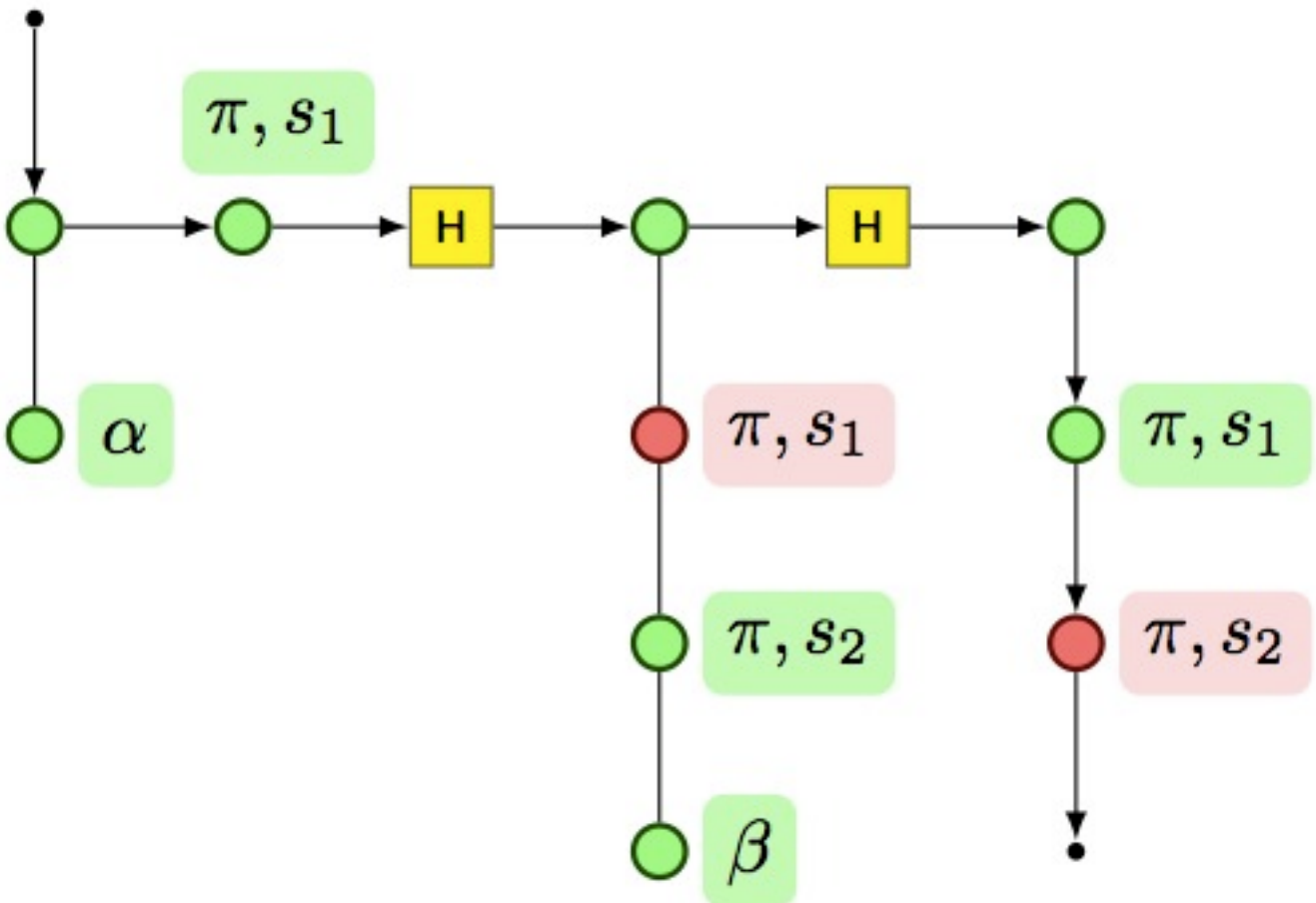
'π-red is pushed by the flow':



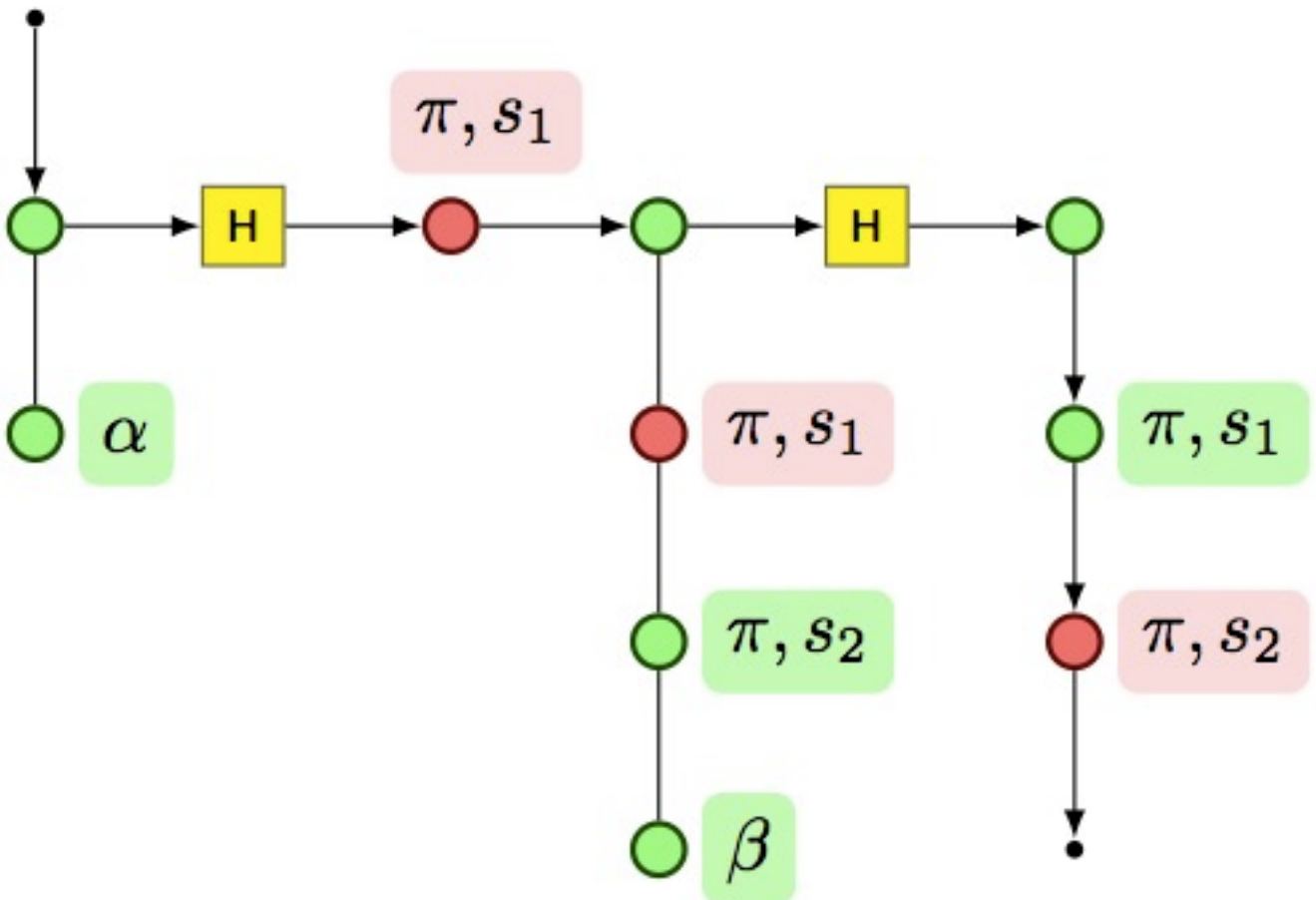
Flow Strategy



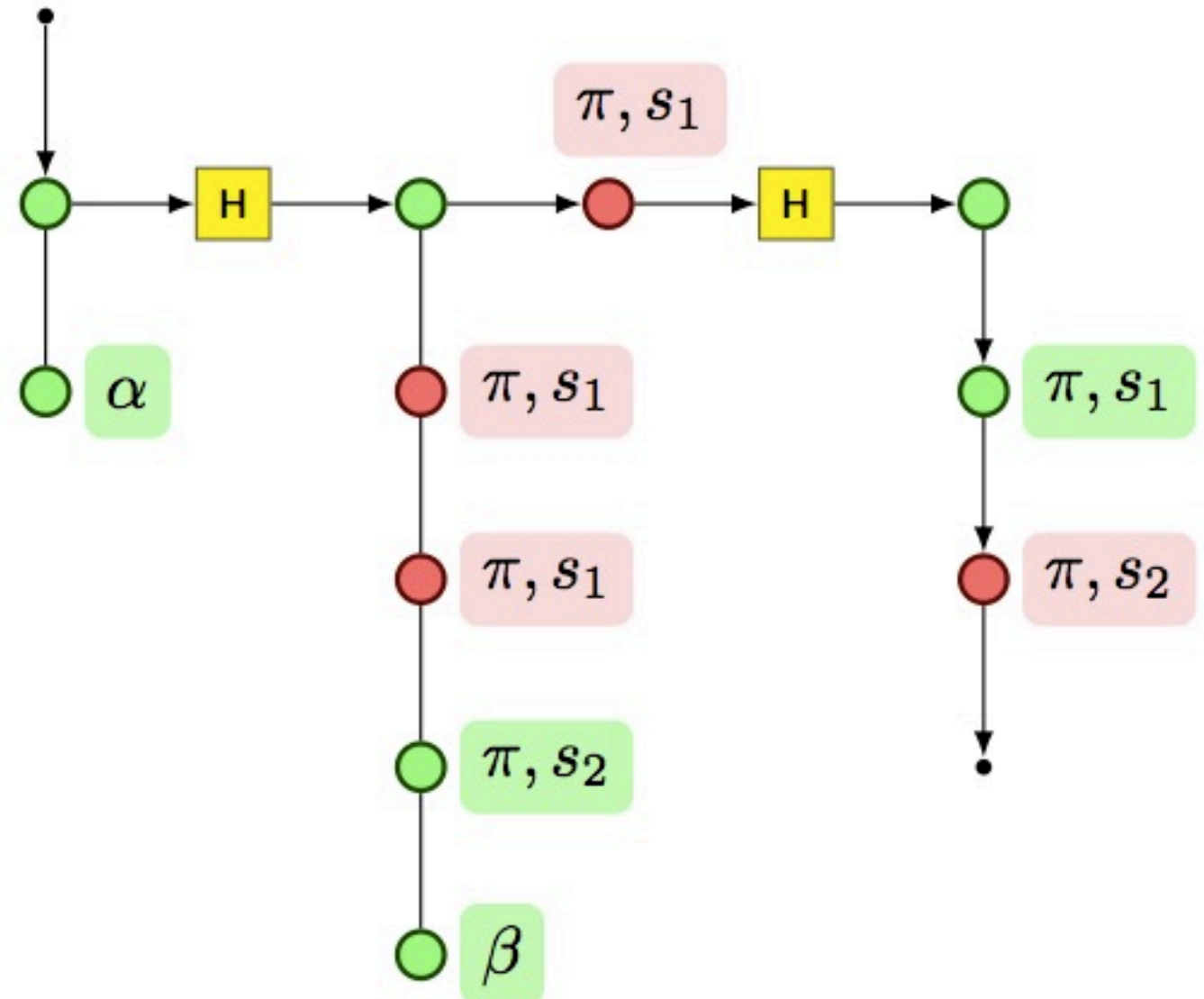
Flow Strategy



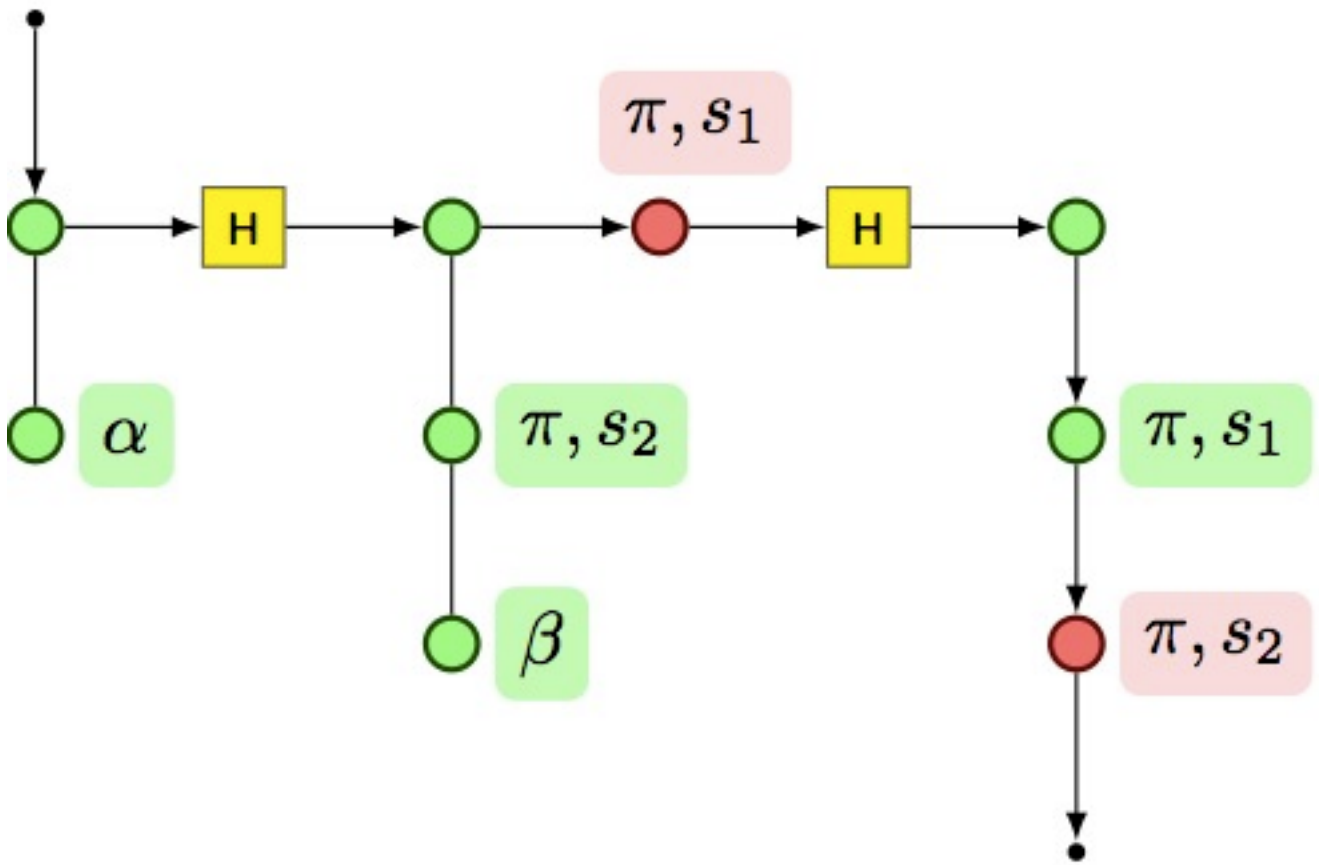
Flow Strategy



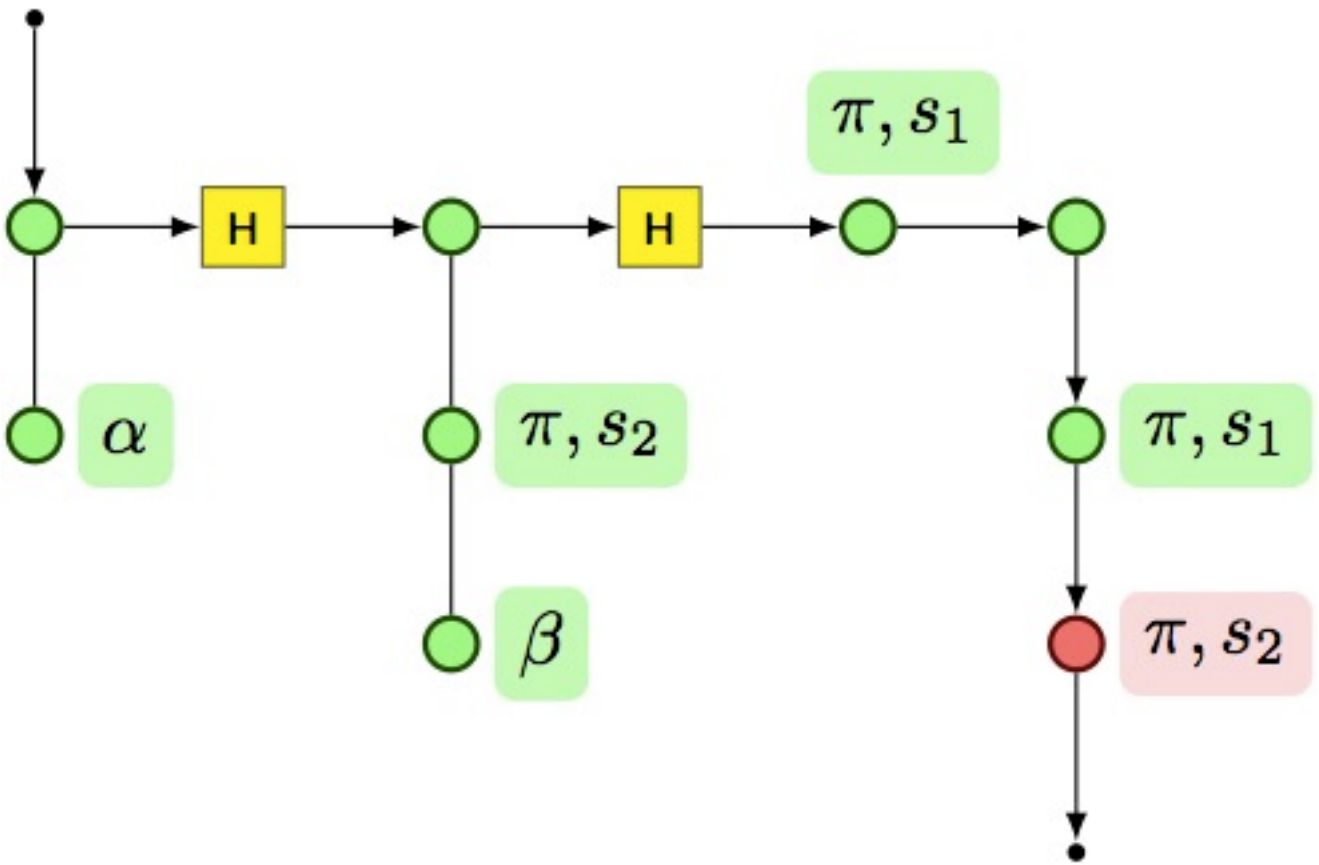
Flow Strategy



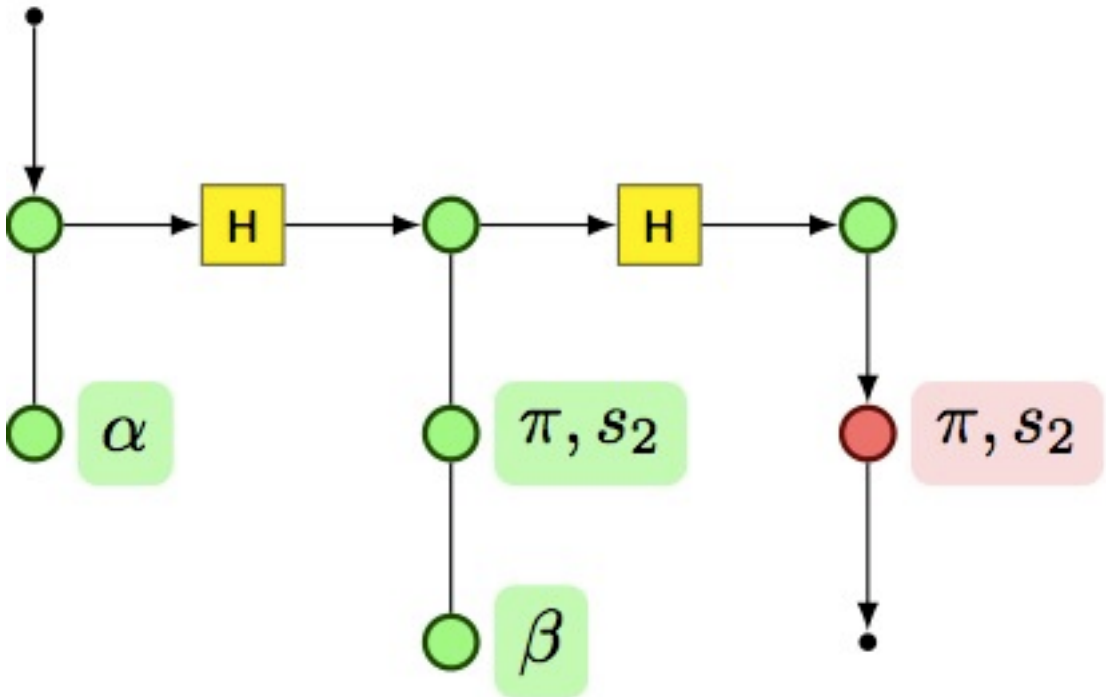
Flow Strategy



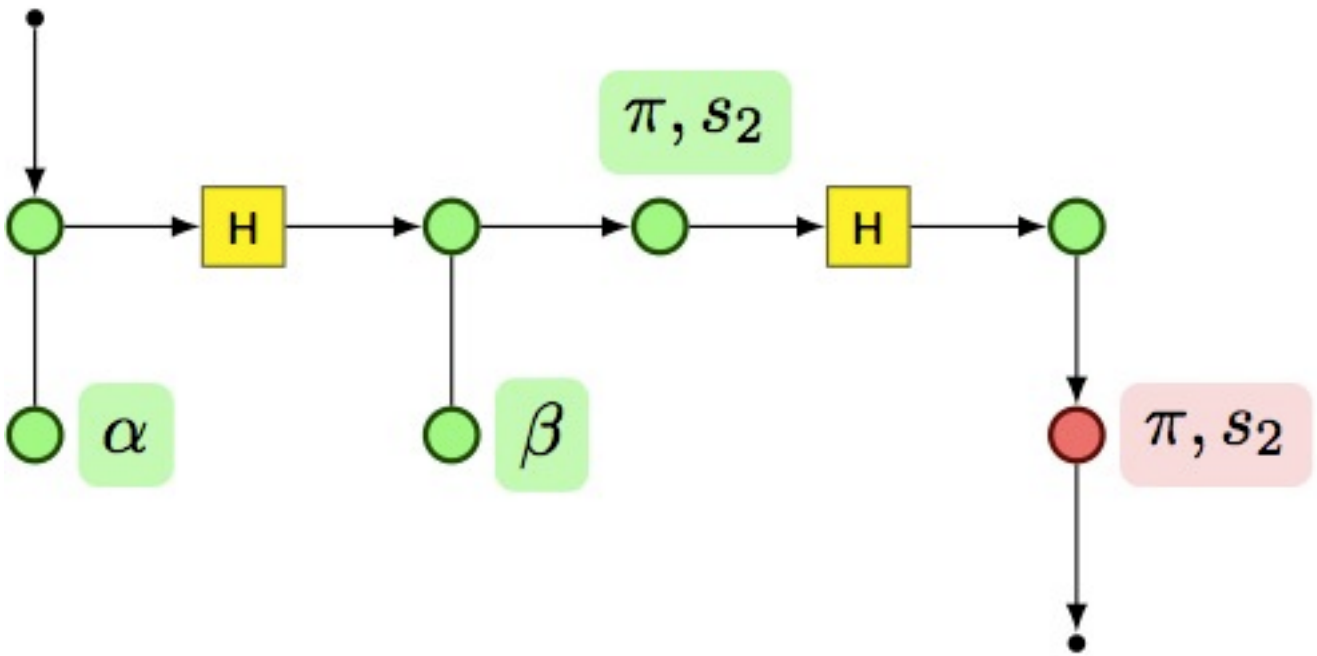
Flow Strategy



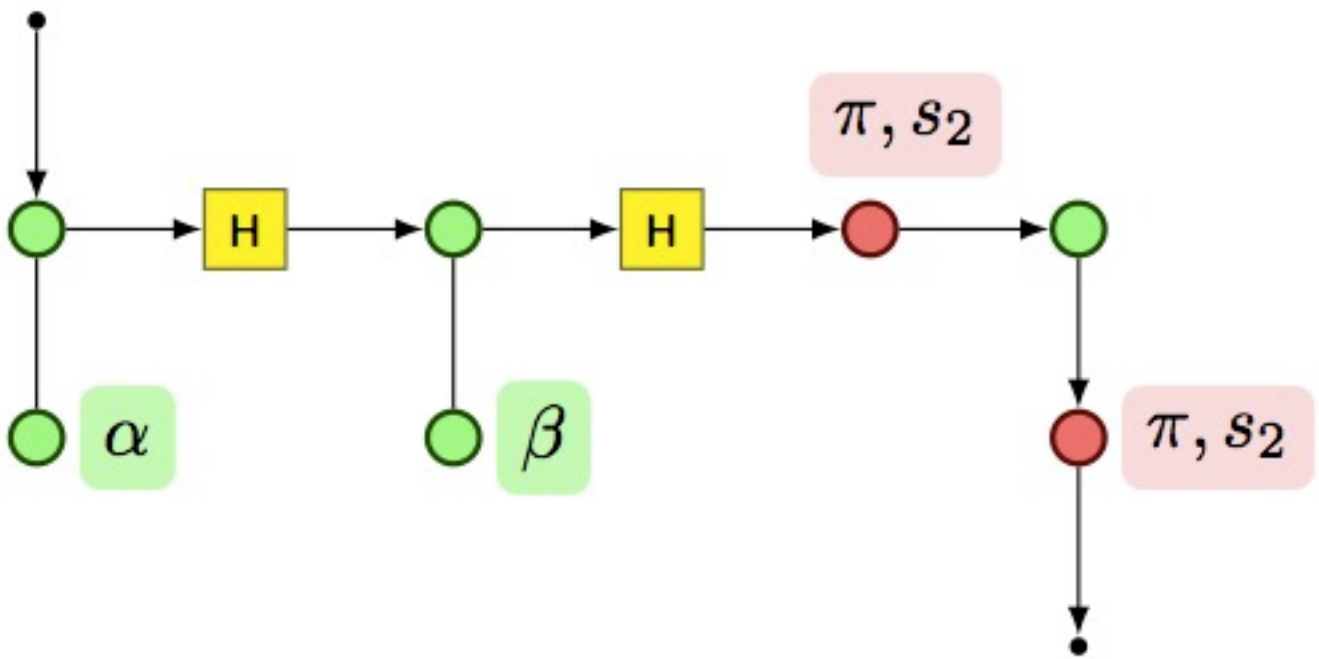
Flow Strategy



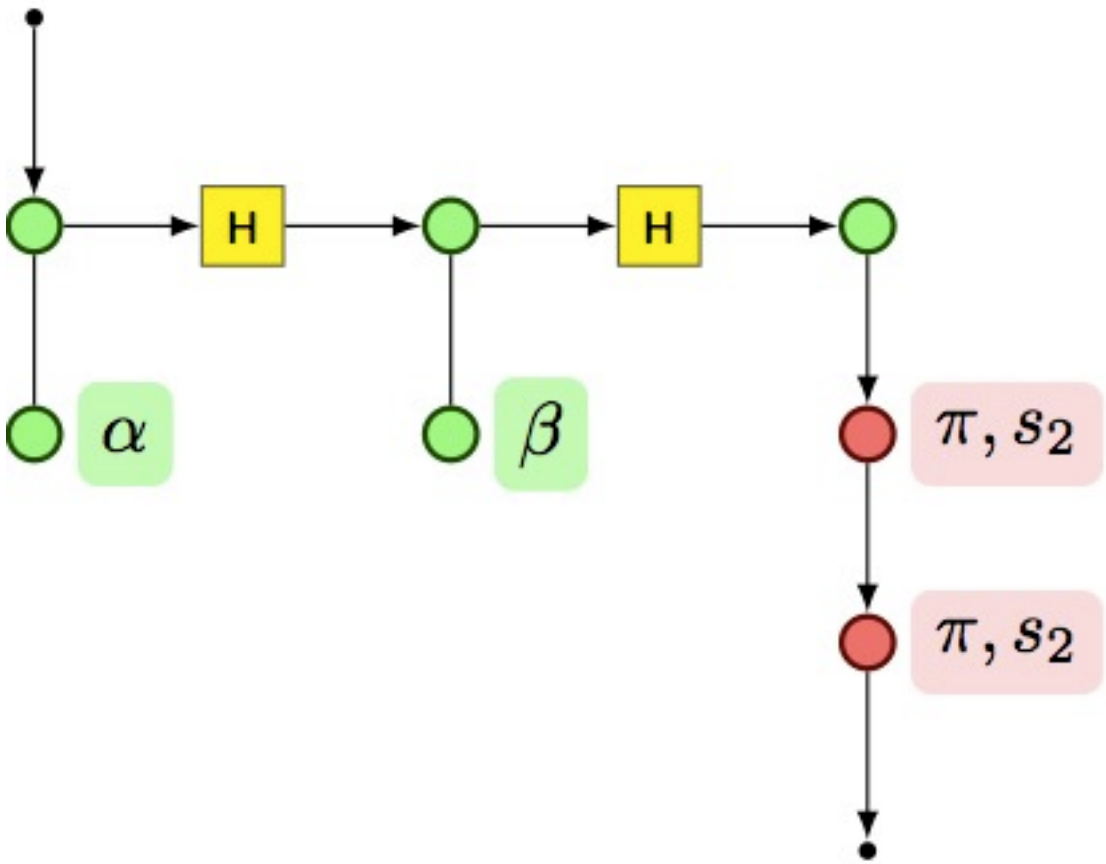
Flow Strategy



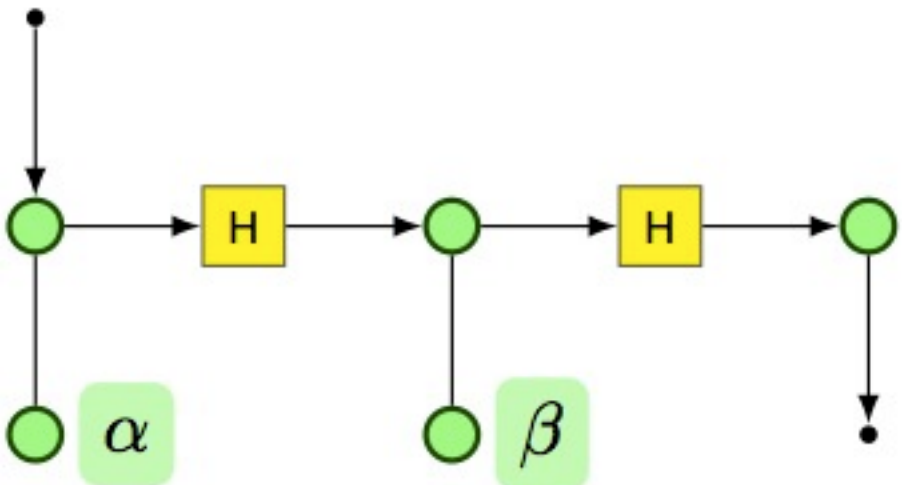
Flow Strategy



Flow Strategy



Flow Strategy

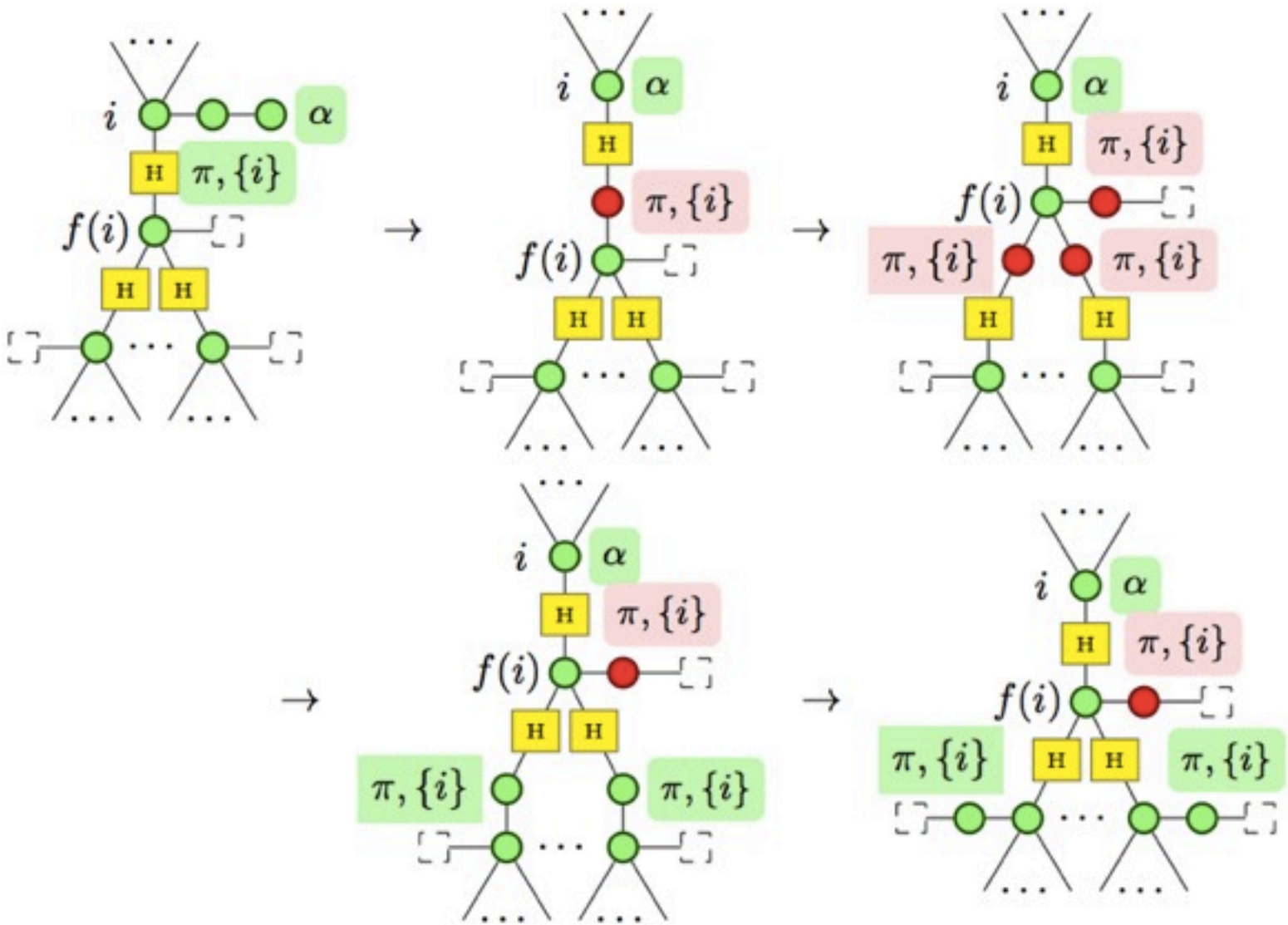


Flow Strategy

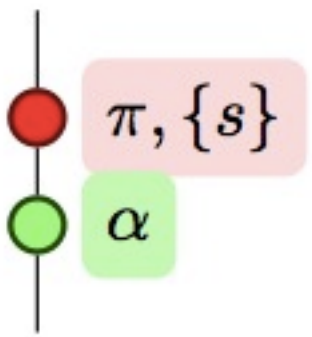
Thm: This strategy terminates in one of two states:

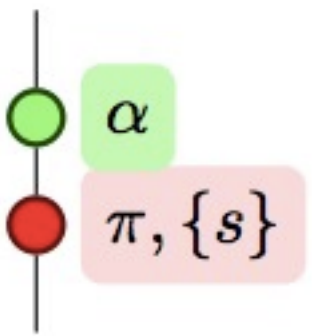
- All conditional operations removed => deterministic
- Not all removed, but their location reveals where extra corrections are needed.

Flow Strategy



Flow Strategy

$D_1 =$ 

$D_2 =$ 

$$[\![D_1]\!] \neq [\![D_2]\!] \text{ unless } \alpha = 0 \text{ or } \alpha = \pi$$

From here we can reconstruct the original pattern given by DKP as the uniformly deterministic pattern based on G.

Generalised Flow

Generalised Flow is the generalisation of flow where each qubit has a set of qubits.

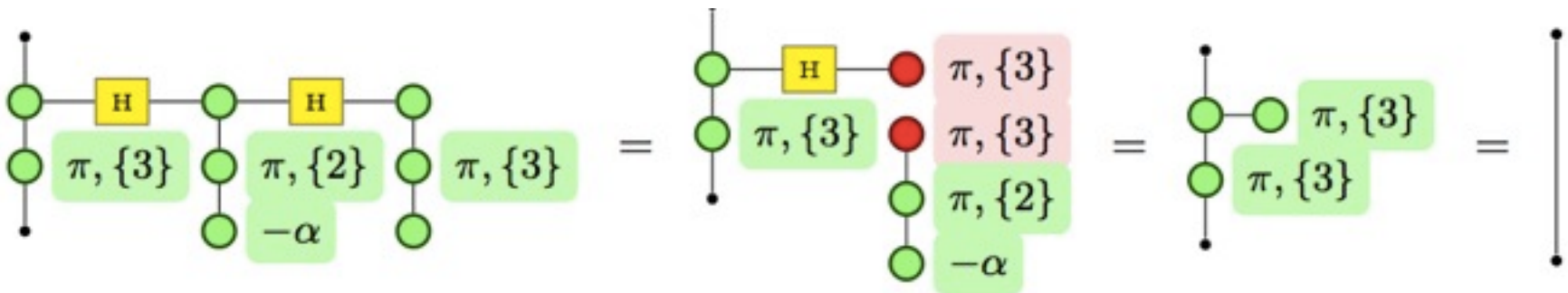
Theorem [BKMP 2007]: A geometry has generalised flow iff it supports a step-wise strongly uniformly deterministic pattern.

Theorem [DP 2010] If geometry has flow then the corresponding diagram can rewrite to a circuit-like diagram.

(No time to discuss => relies very strongly upon the bialgebra rule)

Non-uniform determinism

Note the ZX-calculus works with the pattern, not the geometry, so it can also treat non-uniform cases. E.g.:



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- A. Joyal and R. Street. Geometry of Tensor Categories. *Advances in Mathematics* 88, pp55-113, 1991.
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- P. Selinger. A survey of graphical languages for monoidal categories. In *New Structures for Physics*, Springer Lecture Notes in Physics 813, pp 289-355, 2011.

†-Compact Categories and QM

- S. Abramsky and B. Coecke. A categorical semantics of quantum protocols. *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science: LICS 2004*, IEEE Computer Society, 2004.
- S. Abramsky. No-Cloning In Categorical Quantum Mechanics. In *Semantic Techniques in Quantum Computation, ed. S. Gay and I. Mackie*, pages 1--28, Cambridge University Press 2010. arXiv:0910.2401v1.
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- H. Barnum, R. Duncan, and A. Wilce. Convexity, Categorical Semantics and the Foundations of Physics. *Proceedings of 7th Workshop on Quantum Physics and Logic (QPL 2010)* 2010.

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- A. Carboni. Matrices, relations, and group representations. *Journal of Algebra*, 136(2), 1991.
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- B. Coecke, E. O. Paquette and S. Perdrix. Bases in diagrammatic quantum protocols. *Proceedings of Mathematical Foundations of Programming Semantics XXIV*, Electronic Notes in Theoretical Computer Science, Elsevier, 2008.

Complementary Observables

- B. Coecke and R. Duncan. Interacting quantum observables. Interacting Quantum Observables: Categorical Algebra and Diagrammatics. *New J. Physics* 13 (043016) 2011. arXiv:0906.4725

Non-locality and the phase group

- B. Coecke, W. Edwards, and R. W. Spekkens Phase groups and the origin of non-locality for qubits Electronic Notes in Theoretical Computer Science 270, 15 (2011) arXiv:1003.5005
- W. Edwards. Non-locality in Categorical Quantum Mechanics. DPhil Thesis, Oxford University, 2009.
- B. Coecke and W. Edwards. Spekkens's toy theory as a category of processes. 2011. arxiv:1108.1978.
- B. Coecke, R. Duncan, A. Kissinger and Q. Wang. Strong Complementarity and non-locality in categorical quantum mechanics. Submitted. 2012.

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MBQC - classic

- R. Raussendorf and H. J. Briegel. A One-Way Quantum Computer *PRL* 86 2001.
- R. Raussendorf, D. E. Browne and H. J. Briegel. Measurement-based quantum computation with cluster states, *Phys. Rev. A*. 68, 2003.
- V. Danos, E. Kashefi and P. Panangaden. The Measurement Calculus. *Journal of ACM*, 54 (2) , 2007.
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- R. Duncan and S. Perdrix. Graph states and the necessity of Euler decomposition. In K. Ambos-Spies, B. Löwe, and W. Merkle, editors, *Computability in Europe: Mathematical Theory and Computational Practice (CiE'09)*, volume 5635 of Lecture Notes in Computer Science, pages 167–177. Springer, 2009.
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