The non-logic of quantum computation

OR:
How I learned to live without propositions-as-types

Ross Duncan
Laboratoire d'Information Quantique
Université Libre de Bruxelles
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Background and Motivations

"I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more."

--John von Neumann, letter to G. Birkhoff, 1935
Motivations
Motivations
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Motivations

\[ \lambda x. \lambda y. \lambda z. xz(yz) \]
Motivations

Hilbert space, unitary transforms, self-adjoint operators....

\[ \lambda x.\lambda y.\lambda z.xz(yz) \]

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Motivations

Hilbert space, unitary transforms, self-adjoint operators....

\( \lambda x. \lambda y. \lambda z. xz(yz) \)

\[
\begin{align*}
\text{fun} & \quad \text{fact} \ 0 = 1 \\
1 & \quad \text{fact} \ x = x \ast \text{fact} \ (x - 1)
\end{align*}
\]

\[
\begin{align*}
\quad & = \quad \quad \quad \quad \quad \quad \quad \quad \\
\quad & = \quad \quad \quad \quad \quad \quad \quad \quad \\
\end{align*}
\]
The need for abstraction:

D \sim 2^{1764}

This is an 8-bit adder
Motivation & Context

New **structural** axiomatisation of QM:
- Hilbert space formalism not a perfect fit for QM
- Can also study alternative quantum-like theories
- Emphasis on composition

We use **Monoidal Categories**:
- Very general framework, including Hilbert spaces
- Well studied in literature
- Beautiful diagrammatic presentation
Quantum Mechanics

Overview of the physical theory
Quantum Mechanics (Abridged)

1. Quantum states are represented by unit vectors in a complex Hilbert space.

\[ |0\rangle, |1\rangle, \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \in \mathbb{C}^2 =: Q \]

2. The state space formed by combining two or more systems is the tensor product of their individual state spaces

\[ |010\rangle := |0\rangle \otimes |1\rangle \otimes |0\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = Q^3 \]

3. For each discrete time step, an undisturbed quantum system evolves according to a unitary operator acting on its state space

\[ X, Z, H : Q \rightarrow Q \quad \wedge X, \wedge Z : Q^2 \rightarrow Q^2 \]

…but the quantum state is not directly accessible... more on this later!

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No-Cloning and No-Deleting

Theorem: There are no quantum operations $D$ such that
\[
D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle \\
D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle
\]
unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no quantum operations $E$ such that
\[
E : |\psi\rangle \mapsto |0\rangle \\
E : |\phi\rangle \mapsto |0\rangle
\]
unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:

Input register $|\Psi\rangle$  

----------------------------------  

$|\Psi\rangle$  

----------------------------------

$|\Psi'\rangle$  

Output register
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:

Input register $|\Psi\rangle$ → Unitary gates → Output register $|\Psi'\rangle$
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:

U : $\mathbb{C}^{2^n}$ $\rightarrow$ $\mathbb{C}^{2^n}$
Universality

We say that a model of quantum computation is universal if it can represent all unitary maps.

The circuit model requires a small set of gates to be universal:

\[ Z_\beta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \]
\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
Universality

We say that a model of quantum computation is universal if it can represent all unitary maps.

The circuit model requires a small set of gates to be universal:

\[
Z_\beta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix}, \quad X_\alpha = \begin{pmatrix} \cos \frac{\alpha}{2} & -i \sin \frac{\alpha}{2} \\ -i \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
Quantum Observables

Each observable quantity $O$ is represented as self-adjoint operator $\hat{O} = \sum_i \lambda_i |e_i\rangle \langle e_i|$

- The possible values of $O$ are the eigenvalues $\lambda_i$ of $\hat{O}$
- When we observe $O$ for a system in state $|\psi\rangle$, there is probability $\langle e_i | \psi \rangle^2$ of observing $\lambda_i$.
- If $\lambda_i$ is the outcome of the measurement, the system is then in state $|e_i\rangle$.

In general, measuring $O_1$ then $O_2$ will give a different answer than measuring $O_2$ first!

*Not every quantum observable is well defined at the same time.*
Pauli Matrices

The following 1-qubit unitaries are the Pauli spin matrices:

\[
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
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Some properties:
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\end{pmatrix} \quad Z = \begin{pmatrix}
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0 & -1 \\
\end{pmatrix}
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3. They form a basis for \( \mathbb{C}^2 \rightarrow \mathbb{C}^2 \)
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\]

Some properties:
1. Paulis are self adjoint - i.e. they define measurements.
2. Their eigenvectors form a set of 3 Mutually Unbiased Bases

3. They form a basis for \( \mathbb{C}^2 \rightarrow \mathbb{C}^2 \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \)
X and Z Spins

We can measure the spin of qubit $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\alpha |0\rangle + \beta |1\rangle$$

- $p = \alpha^2$
- $p = \beta^2$
- $p = (\alpha + \beta)/2^2$
- $p = (\alpha - \beta)/2^2$
X and Z Spins

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X and Z Spins

We can measure the spin of qubit $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$

\[ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ (|0\rangle + |1\rangle)/\sqrt{2} \]

$p = 1/2$

$p = 1/2$

$p = 1$

$p = 0$

$|0\rangle$

$|1\rangle$

$|+\rangle$

$|\rangle$
Complementary Observables

\[ a_0 \]

\[ a_1 \]
Complementary Observables

\[ a_0 \]

\[ a_1 \]
Complementary Observables

$\alpha_0$

$\alpha_1$
Complementary Observables

\[ a_0 \]

\[ a_1 \]
Complementary Observables

\[ a_0 \]
\[ b_1 \]
\[ b_0 \]
\[ a_1 \]
Complementary Observables

\begin{align*}
& b_1 \\
& a_0 \\
& b_0 \\
& a_1
\end{align*}

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Complementary Observables
Complementary Observables

\[ |\langle a_i \mid b_j \rangle| = \frac{1}{\sqrt{D}} \]
Entangled States

A state \( |\psi\rangle \in A \otimes B \) is called \textit{separable} if \( |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \); otherwise it is called \textit{entangled}; i.e. it must be written as:

\[
|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle
\]
Entangled States

A state $|\psi\rangle \in A \otimes B$ is called *separable* if $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$; otherwise it is called *entangled*; i.e. it must be written as:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle$$

Example 1: $|00\rangle = |0\rangle \otimes |0\rangle$ is separable
Entangled States

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$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle$$

Example 1: $|00\rangle = |0\rangle \otimes |0\rangle$ is separable

Example 2: $|00\rangle + |01\rangle + |10\rangle + |11\rangle$ is separable
Entangled States

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$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle$$

Example 1: $|00\rangle = |0\rangle \otimes |0\rangle$ is separable

Example 2: $|00\rangle + |01\rangle + |10\rangle + |11\rangle$ is separable

$$= (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$
Entangled States

A state $|\psi\rangle \in A \otimes B$ is called separable if $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$; otherwise it is called entangled; i.e. it must be written as:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle$$

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Example 2: $|00\rangle + |01\rangle + |10\rangle + |11\rangle$ is separable

$$= (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

$$= |++\rangle$$
Entangled States

A state $|\psi\rangle \in A \otimes B$ is called separable if $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$; otherwise it is called entangled; i.e. it must be written as:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle$$

Example 3: $|\text{Bell}_1\rangle = |00\rangle + |11\rangle$ is entangled
Entangled States

A state $|\psi\rangle \in A \otimes B$ is called **separable** if $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$; otherwise it is called **entangled**; i.e. it must be written as:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle$$

Example 3: $|\text{Bell}_1\rangle = |00\rangle + |11\rangle$ is entangled

Example 4: $|H\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle$ is entangled
Entanglement and Measurement

- Entangled states consist of two or more systems which can be shared between distant parties.
- If one party measures their system the other system can be affected.

\[ |0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \quad \text{Initial shared state} \]
Entanglement and Measurement

- Entangled states consist of two or more systems which can be shared between distant parties.
- If one party measures their system the other system can be affected.

$|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B$  \hspace{1cm} Initial shared state

Alice measures

$p = 1/2$

$|0\rangle_A |0\rangle_B$  \hspace{1cm} $|1\rangle_A |1\rangle_B$
Entanglement and Measurement

- Entangled states consist of two or more systems which can be shared between distant parties.
- If one party measures their system the other system can be affected.

Initial shared state:

\[ |0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \]

Alice measures

-\[ |0\rangle_A |0\rangle_B \]
-\[ |0\rangle_A |0\rangle_B \]

Bob measures

-\[ |1\rangle_A |1\rangle_B \]
-\[ |1\rangle_A |1\rangle_B \]
Map-State Duality

Recall that there is an isomorphism:

\[ A \rightarrow B \cong A \otimes B \]

In particular:

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \iff \quad |00\rangle + |11\rangle =: |Bell_1\rangle \]

\[ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \iff \quad |01\rangle + |10\rangle =: |Bell_2\rangle \]

\[ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \iff \quad |00\rangle - |11\rangle =: |Bell_3\rangle \]

\[ XZ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \iff \quad |01\rangle - |10\rangle =: |Bell_4\rangle \]

Since they form a basis we can measure with them.
Quantum Teleportation

$|\psi\rangle$

$|00\rangle + |11\rangle$
Quantum Teleportation

\[ |\psi\rangle \rightarrow |00\rangle + |11\rangle \]

Alice

Bob
Quantum Teleportation

|ψ⟩

|00⟩ + |11⟩

Alice
Audrey

Bob

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Quantum Teleportation

\[ |\psi\rangle = |00\rangle + |11\rangle \]

Alice

\[ |\psi\rangle \]

Bob

\[ \langle 01| + \langle 10| \]

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Quantum Teleportation

Alice

Audrey

|ψ⟩

Bob

|ψ⟩

⟨01| + ⟨10|

|00⟩ + |11⟩
Quantum Teleportation

\[ |\psi\rangle \]

\[ |00\rangle + |11\rangle \]

\[ \langle 01| - \langle 10| \]

Alice

Audrey

Bob

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Quantum Teleportation

|ψ⟩

Alice

Audrey

|ψ⟩

⟨01| − ⟨10|

Z

X

|00⟩ + |11⟩

Bob

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Channels via entanglement

Bennett at al:

“Note that qubits are a directed channel resource, sent in a particular direction from the sender to the receiver; by contrast [entangled pairs] are an undirected resource shared between the sender and receiver.”

Teleporting an unknown quantum state via dual classical and EPR channels, PRL, 1993

This suggests that the type of an entangled pair should be the linear type $Q \otimes Q$ rather than the usual $Q \to Q$. 
More Entanglement

Entanglement can be used for a lot more than just transmitting information:

MBQC is a universal model of computation which is based on the flow of information through large entangled states.
Quantum Logic

The Birkhoff-von Neumann approach and its
A small historical detour

Quantum logic was an attempt to do two things at once:

- Develop a logic that took the limitations of knowledge imposed by quantum mechanics seriously;
- Re-found quantum theory on a more abstract logical basis.

It is a “Tarskian” approach based purely on what the propositions mean, and not at all concerned with the proofs.
Propositions and projectors

A proposition is a question with a yes/no answer:

\[ A = \text{"Is the spin up?"} \]

but the answer will be given by a quantum measurement:

\[ \psi \models A \Leftrightarrow p_A |\psi\rangle = |\psi\rangle \]

hence each proposition corresponds to a pair of orthogonal subspaces.

The “lattice of propositions” is simply the collection of closed subspaces ordered under inclusion.
Distributivity Fails

In general we have $p_A p_B \neq p_B p_A$ which implies the failure of distributivity.

Consider:

we have

$$\bot = (A \land B) \lor (A^\bot \land B) \neq (A \lor A^\bot) \land B = B$$

hence such a lattice is not distributive.

(It does satisfy a weaker law called orthomodularity which I won’t discuss.)
No deduction theorem

**Theorem:** Suppose we can define a connective $\rightarrow$ such that

$$A \land X \leq B \iff X \leq A \rightarrow B$$

then the lattice is distributive.

**Corollary:** Quantum logic does not admit modus ponens.

Note that the sub-lattice defined by any set of commuting projectors is just a boolean lattice.
No “good” tensor product

Given finite dimensional Hilbert spaces $H_1$ and $H_2$, we can construct their subspace lattices $\mathcal{L}(H_1)$ and $\mathcal{L}(H_2)$. In fact this is a functor:

$$\mathcal{L} : \text{FDHilb} \rightarrow \text{OML}$$

But what about the tensor product?

$$\mathcal{L}(H_1) \otimes \mathcal{L}(H_2) = \mathcal{L}(H_1 \otimes H_2) \ ?$$

To date no one has been able to find a tensor product on $\text{OML}$ to make this functor monoidal.
Quantum logic today

The failure of both sequential and parallel modes of composition in quantum logic means that the projection lattice approach cannot support any notion of *process*. Hence for quantum computation, a new approach must be found.
Quantum Logic Today

Some modern developments based on quantum logic:

- The topos approach: essentially aims to get back realism by working in a suitable topos; Isham, Döring, Butterfield; Heunen, Landsman, Spitters.

- Jacobs and Heunen have shown that the lattice of subobjects in a dagger-kernel category is orthomodular; hence we can carry out quantum logic \textit{internally} in a suitable category.

- Coecke, Heunen and Kissinger constructed orthomodular lattices internally using tensor products of certain Frobenius algebras.
Propositions as types for QM

A logic based on processes not properties
Proofs and processes

A long tradition in computer science is to treat the proof as the more important object.

- Propositions are types.
- Many different proofs of the same theorem, processes producing output of that type.
- Different possibilities for equivalence of proofs: denotational/static vs operational/dynamic.
General Scheme

Categorical Structure

Rewriting system

Logic
The Curry-Howard-Lambek correspondence

Cartesian closed categories

Simply typed $\lambda$-calculus

Intuitionistic Logic
What is the quantum version?

- We want a logic of “quantum processes”

Some hints as to what this should be:

- entangled systems can’t be described by a Cartesian product
- map-state duality suggests we should have a “function-type”
- no-cloning and no-deleting imply that the underlying setting should be linear
- ....however we still need some way to represent non-determinism
My (old) approach:

\[ \dagger \text{-compact closed categories with biproducts} \]

\[ \rightarrow \]

\[ \text{Tensor-sum logic} \]

\[ \rightarrow \]

\[ \text{Generalised self-dual proof-nets} \]
The connectives

Classical logic

\[ \neg
\neg A = A \]

\[ \neg (A \land B) = \neg A \lor \neg B \]

\[ \neg (A \lor B) = \neg A \land \neg B \]

<table>
<thead>
<tr>
<th>conjunction</th>
<th>disjunction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \land )</td>
<td>( \lor )</td>
</tr>
</tbody>
</table>
The connectives

Linear logic

(MALL)

\[ A \perp \perp = A \]
\[ (A \otimes B) \perp = A \perp \bowtie B \perp \]
\[ (A \bowtie B) \perp = A \perp \otimes B \perp \]
\[ (A \& B) \perp = A \perp \oplus B \perp \]
\[ (A \oplus B) \perp = A \perp \& B \perp \]

<table>
<thead>
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<td>( \otimes )</td>
<td>( \bowtie )</td>
</tr>
<tr>
<td>additive</td>
<td>( &amp; )</td>
<td>( \oplus )</td>
</tr>
</tbody>
</table>
The connectives

Tensor-sum “logic”

\[ A^{**} = A \]
\[ (A \otimes B)^* = A^* \otimes B^* \]
\[ (A \oplus B)^* = A^* \oplus B^* \]
Tensor-Sum Logic

Tensor-sum logic is a Gentzen system, designed to capture the structure of a certain free category on some generators $A$.

- Essentially it is MALL with self-dual connectives
- Every proof has an interpretation as an arrow of $FA$
- Every arrow of $FA$ has a corresponding proof
- The system is cut-eliminating, and the cut-elimination procedure is sound wrt the interpretation.
Tensor-Sum Logic

Tensor-sum logic is a Gentzen system, designed to capture the structure of a certain free category on some generators $\mathcal{A}$. 

- Essentially it is MALL with self-dual connectives
- Every proof has an interpretation as an arrow of $F\mathcal{A}$
- Every arrow of $F\mathcal{A}$ has a corresponding proof
- The system is cut-eliminating, and the cut-elimination procedure is sound wrt the interpretation.

It has some oddities as a logical system:

- Every entailment $A \vdash B$ is derivable with a zero proof
- Self-duality allows the formation of self-cuts
  - the empty sequent is derivable in many inequivalent ways
Proof-nets for tensor and sum

We update the system of proof-nets with generalised axioms:

The rewrite rules are unchanged.
Proof-nets for tensor and sum

We update the system of proof-nets with generalised axioms:

The rewrite rules are unchanged.
Example: teleportation

The shared Bell state and the input qubit:
Example: teleportation

The Bell basis measurement
Example: teleportation

The classically controlled corrections:
Example: teleportation

The whole protocol:
Example: teleportation

The whole protocol:
Example: teleportation
Example: teleportation
Example: teleportation
Example: teleportation
Example: teleportation

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Example: teleportation
Example: teleportation
Example: teleportation

\[ Z^2 = 1_Q \]

\[ Y^2 = 1_Q \]

\[ X^2 = 1_Q \]
Example: teleportation
The biproduct

We used the biproduct to encode the branching nature of quantum processes.

- The diagonal map shows the possibility of different choices:
  \[ \begin{align*}
  Q & \xrightarrow{\Delta} Q \oplus Q \\
  Q \oplus Q & \xrightarrow{\nabla} Q
  \end{align*} \]

- But what about the codiagonal?

- Semantically this corresponds to superposition rather than probabilistic mixing --- the wrong interpretation

- To properly address the issue of probabilities in QM we use Selinger's CPM construction --- but this kills the biproducts.
The biproduct

The biproduct fixes a particular matrix representation for the arrows, that is *a basis*.

- Bases play represent measurements in quantum mechanics - the spectra of self-adjoint operators
- In order to fully handle quantum measurements we must deal with different, incompatible measurements at the same time.
- Bases cannot be represented by natural transformations - unlikely any “categorical logic” can help here

The biproduct does not offer a good formalisation of this. We will drop the biproduct and find a more useful way to formalise quantum computation.
Normal form theorem

Pure logic, determined by premise

A unique A-labelled circuit

Pure logic determined by conclusion
Types for entanglement?

Can we regain the separation between $\otimes$ and $\mathcal{F}$ to talk about entanglement?

- Entangled states do not form a subspace
- Do double gluing on $\text{fdHilb}$
  - $\otimes$ gives product states
  - $\mathcal{F}$ gives all states
- Hence $\otimes$ is a subtype of $\mathcal{F}$
How many types anyway?

**Defn:** A state $S$ is said to be *SLOCC reachable* from state $S'$ if there is a sequence of stochastic local operations and classical communications producing $S$ from $S'$.

**Defn:** If $S$ and $S'$ are mutually SLOCC reachable then they are *SLOCC equivalent*. 

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How many types anyway?

**Prop:** For 2-qubit states there are 2 SLOCC classes:
How many types anyway?

**Prop:** For 3-qubit states there are 6 SLOCC classes:
How many types anyway?

**Prop:** For 4-qubit states there are **uncountably many** SLOCC classes
How many types anyway?

**Prop:** For 4-qubit states there are **uncountably many** SLOCC classes

Forget about types to describe entanglement

Just look at the terms
Overview

Hilbert Space QM  Categorical QM
Overview

Hilbert Space QM  Categorical QM

Hilbert space
Overview

Hilbert Space QM  Categorical QM

Hilbert space  †-symmetric monoidal category
Overview

Hilbert Space QM

- Hilbert space
- Observables

Categorical QM

- \dagger\text{-symmetric monoidal category}
Overview

Hilbert Space QM  Categorical QM

Hilbert space  \(\dagger\)-symmetric monoidal category

Observables  special commutative \(\dagger\)-Frobenius algebras
Overview

Hilbert Space QM

Hilbert space

Observables

Unbiasedness & relative phase

Categorical QM

†-symmetric monoidal category

special commutative †-Frobenius algebras
Overview

Hilbert Space QM
- Hilbert space
- Observables
  - Unbiasedness & relative phase

Categorical QM
- $\dagger$-symmetric monoidal category
- Special commutative $\dagger$-Frobenius algebras
- Phase group
Overview

Hilbert Space QM
- Hilbert space
- Observables
- Unbiasedness & relative phase
- Complementarity

Categorical QM
- $\dagger$-symmetric monoidal category
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- Hilbert space
- Observables
- Unbiasedness & relative phase
- Complementarity

Categorical QM
- †-symmetric monoidal category
- special commutative †-Frobenius algebras
- Phase group
- Bialgebras & Hopf algebras
Overview

Hilbert Space QM
- Hilbert space
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- $\dagger$-symmetric monoidal category
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- ZX-calculus
Monoidal Categories & Diagrams

PAST / HEAVEN

FUTURE / HELL

Pessimistic Diagram Convention
Diagrams
a.k.a. Monoidal categories

\[ j : A \otimes B \rightarrow C \otimes D \otimes E \]
Diagrams

a.k.a. Monoidal categories

Input Systems

\[ j : A \otimes B \rightarrow C \otimes D \otimes E \]
Diagrams

a.k.a. Monoidal categories

Input Systems

\[ j : A \otimes B \rightarrow C \otimes D \otimes E \]
Diagrams
a.k.a. Monoidal categories

Input Systems

A → B

C → D → E

Interaction, or process, or state change, or ...

\[ j : A \otimes B \rightarrow C \otimes D \otimes E \]
Diagrams

a.k.a. Monoidal categories

Input Systems

\[ A \to B \]

Output Systems

Interaction, or process, or state change, or ...

\[ j : A \otimes B \to C \otimes D \otimes E \]

"Time"
A hierarchy of categories
A hierarchy of categories

Categories

Monoidal Categories
A hierarchy of categories

Categories

Monoidal Categories

Symmetric Monoidal Categories
A hierarchy of categories

Categories

Monoidal Categories

Symmetric Monoidal Categories

Compact Categories
A hierarchy of categories

Categories

Monoidal Categories

Symmetric Monoidal Categories

Compact Categories

†-Categories
A hierarchy of categories

- Categories
  - Monoidal Categories
    - Symmetric Monoidal Categories
      - Compact Categories
        - †-Category
          - †-Compact Categories
A hierarchy of categories

- Categories
  - Monoidal Categories
    - Symmetric Monoidal Categories
      - Compact Categories
        - †-Compact Categories
          - F.D. Hilbert spaces
          - Sets and Relations
          - Strongly self-dual convex cones
A category consists of objects $A, B, C$, etc, and arrows between them:

$$f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D$$
Categories

A category consists of objects $A$, $B$, $C$, etc, and arrows between them:

$$g \circ f : A \to C$$

$$h : C \to D$$

"Tuesday 11 June 2013"
A category consists of objects $A$, $B$, $C$, etc, and arrows between them:

$$h \circ g \circ f : A \to D$$
A category consists of objects $A$, $B$, $C$, etc, and arrows between them:

$$h \circ g \circ f : A \rightarrow D$$
A category consists of objects $A$, $B$, $C$, etc, and arrows between them:

$h \circ g \circ f : A \to D$
Categories

A *category* consists of *objects* $A, B, C, \text{ etc, and arrows between them:}$

$$h \circ g \circ f : A \rightarrow D$$
Categories

\( \text{id}_A : A \to A \)

\( A \)

\( A \)
Categories

\[ f \circ \text{id}_A : A \to B \]

\[
\begin{array}{c}
A \\
\downarrow \\
f \\
\downarrow \\
B
\end{array}
\]
Categories

\[ \text{id}_B \circ f : A \to B \]

\[ \begin{tikzpicture} 
\node (A) at (0,0) {$A$}; 
\node (B) at (0,-2) {$B$}; 
\node (f) at (0,-1) {$f$}; 
\draw [->] (A) -- (f); 
\draw [->] (f) -- (B); 
\end{tikzpicture} \]
Categories

\[ f : A \to B \]

\[ A \]

\[ f \]

\[ B \]
†-Categories

A category is a †-category if it is equipped with an involutive functor, $(\cdot)^\dagger$ which reverses the arrows while leaving the objects unchanged.

\[ f : A \to B \quad f^\dagger : B \to A \]
An arrow $f : A \rightarrow B$ is called \textit{unitary} when:

$$
(f \otimes g) \dagger = f \dagger \otimes g \dagger \hspace{1cm} \sigma_B \alpha_A = \sigma_A \alpha_B
$$
An arrow \( f : A \to B \) is called \textit{unitary} when:

\[
\sigma^* A, B = \sigma B, A
\]
$\dagger$-Categories

An arrow $f : A \rightarrow B$ is called unitary when:
Monoidal Categories

A strict monoidal category is a category equipped with a tensor product on both objects and arrows:

\[ f \otimes h : A \otimes C \rightarrow B \otimes D \]

\[
\begin{array}{c}
A \\
\downarrow f \\
B \\
\downarrow h \\
C \\
\downarrow D \\
C \\
\downarrow D
\end{array}
\]

The tensor is associative:

\[
(f \otimes g) \otimes h = f \otimes (g \otimes h)
\]

\[
\begin{array}{c}
A \\
\downarrow f \\
B \\
\downarrow g \\
C \\
\downarrow h \\
D
\end{array}
\]
Monoidal Categories

The tensor product is *bifunctorial*, meaning that it preserves composition:

\[(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)\]

\[
\begin{array}{ccc}
A & \bullet & C \\
\downarrow & & \downarrow \\
B & \bullet & D \\
\downarrow & & \downarrow \\
C & \bullet & E
\end{array}
\]

and identities:

\[\text{id}_{A \otimes B} = \text{id}_A \otimes \text{id}_B\]

\[
\begin{array}{ccc}
A & \downarrow & B \\
\downarrow & & \downarrow \\
A & \downarrow & B
\end{array}
\]
Monoidal Categories

Monoidal categories have a special *unit* object called $I$ which is a left and right identity for the tensor:

\[
I \otimes A = A = A \otimes I
\]
\[
\text{id}_I \otimes f = f = f \otimes \text{id}_I
\]

No lines are drawn for $I$ in the graphical notation:

\[
\psi : I \rightarrow A \quad \phi^\dagger : A \rightarrow I \quad \phi^\dagger \circ \psi : I \rightarrow I
\]
Symmetric Monoidal Categories

\[ \sigma_{A,B} : A \otimes B \rightarrow B \otimes A \]
Symmetric Monoidal Categories

\[ \sigma_{A,B} : A \otimes B \rightarrow B \otimes A \]
Symmetric Monoidal Categories

\[ \sigma_{A,B} : A \otimes B \to B \otimes A \]
Symmetric Monoidal Categories

\[ \sigma_{A,B} : A \otimes B \to B \otimes A \]

\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & C \\
\downarrow & & \downarrow \\
C & \xrightarrow{} & B
\end{array} \]
Symmetric Monoidal Categories

\[ \sigma_{A,B} : A \otimes B \rightarrow B \otimes A \]
Compact Closure

A symmetric monoidal category is called compact if, for every object $A$, there exists a dual object $A^*$ and arrows:

$$d : I \rightarrow A^* \otimes A$$
$$e : A \otimes A^* \rightarrow I$$
Compact Closure

A symmetric monoidal category is called compact if, for every object $A$, there exists a dual object $A^*$ and arrows:

\[ d : I \rightarrow A^* \otimes A \quad e : A \otimes A^* \rightarrow I \]
Compact Closure

A symmetric monoidal category is called compact if, for every object $A$, there exists a dual object $A^*$ and arrows:

$$d : I \rightarrow A^* \otimes A \quad e : A \otimes A^* \rightarrow I$$
Compact Closure

Every arrow $f$ in a compact category has a dual $f^*$ defined by

$$f^* = \begin{array}{c}
A \\
\end{array}$$
Compact Closure

Every arrow $f$ in a compact category has a dual $f^*$ defined by

$$f^* = \begin{array}{c}
\begin{array}{c}
A \\
B
\end{array}
\end{array}$$

and

$$f^{**} = \begin{array}{c}
\begin{array}{c}
A \\
B
\end{array}
\end{array}$$
Compact Closure

Every arrow $f$ in a compact category has a dual $f^*$ defined by

$$f^* = \quad f^{**} = \quad B$$

$$A^* \quad \quad \quad \quad B$$
Compact Closure

Every arrow $f$ in a compact category has a dual $f^*$ defined by

$$f^* = \begin{array}{c}
\begin{array}{c}
A \\
\downarrow f \\
B
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
B^* \\
\downarrow f^{**} \\
A
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
A \\
\downarrow f \\
B
\end{array}
\end{array}$$
Compact Closure

Every arrow $f$ in a compact category has a dual $f^*$ defined by

$$f^* = \begin{array}{c} A^* \\ f \end{array} \quad f^{**} = \begin{array}{c} A \\ f \\ B \end{array} = f$$
Compact Closure

Every arrow $f$ in a compact category has a dual $f^*$ defined by

$$f^* = \begin{array}{c}
\begin{array}{c}
A^* \\
\downarrow \\
B \\
\uparrow \\
A
\end{array}
\end{array} =:egin{array}{c}
\begin{array}{c}
B^* \\
\downarrow \\
A^*
\end{array}
\end{array}$$
Compact Closure

\[ f = f \]
Compact Closure

\[ f = \]

\[ f^\dagger = \]
Compact Closure

\[ f = \]

\[ f^\dagger = \]

\[ = f^* \]
Compact Closure

\[
f = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\framebox[0.5cm]{\text{f}}
\end{array}
\end{array}
\hfill
\begin{array}{c}
\begin{array}{c}
\framebox[0.5cm]{\text{f}}
\end{array}
\end{array}
\end{array}
\quad = f^{\ast\dagger} =: f_{\ast}
\end{array}
\]

\[
f^{\dagger} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\framebox[0.5cm]{\text{f}}
\end{array}
\end{array}
\hfill
\begin{array}{c}
\begin{array}{c}
\framebox[0.5cm]{\text{f}}
\end{array}
\end{array}
\end{array}
\quad = f^{\ast}
\end{array}
\]
Graphical Calculus Theorem

**Thm:** one diagram can be deformed to another if and only if their denotations are equal by the structural equations of the category.
**Graphical Calculus Theorem**

Thm: one diagram can be deformed to another if and only if their denotations are equal by the structural equations of the category.
The Category **FRel**

**FRel** is the category of finite relations. It is \(\dagger\)-compact with the following structure.

- **Objects**: finite sets, \(X, Y, Z, \text{ etc}\)
- **Arrows**: Relations \(R \subseteq X \times Y\)
- **Tensor**: Cartesian product; \(X \otimes Y := X \times Y, \quad I = \{\ast\}\)
- \(f^\dagger\) is relational converse: \((x, y) \in f \iff (y, x) \in f^\dagger\)

A relation \(R \subseteq \{\ast\} \times X\) is simply a *subset* of \(X\); its converse is also a subset.

Hence \(S^\dagger \circ R : I \to I\) is non-empty iff \(R \cap S \neq \emptyset\).
The Category **FDHilb**

**FDHilb** is the category of finite dimensional complex Hilbert spaces. It is $\dagger$-monoidal with the following structure.

- **Objects:** finite dimensional Hilbert spaces, $A, B, C$, etc
- **Arrows:** all linear maps
- **Tensor:** usual (Kronecker) tensor product; $I = \mathbb{C}$
- $f^\dagger$ is the usual adjoint (conjugate transpose)

A linear map $\psi : I \to A$ picks out exactly one vector. It is a ket and $\psi^\dagger : A \to I$ is the corresponding bra.

Hence $\psi^\dagger \circ \phi : I \to I$ is the inner product $\langle \psi | \phi \rangle$
Compact Structure of \( \text{FDHilb} \)

In \( \text{FDHilb} \) the compact structure is given by the maps:

whenever \( \{ a_i \}_i \) is a basis for \( A \) and \( \{ a_i^* \}_i \) is the corresponding basis for the dual space \( A^* \)
Compact Structure of $\text{FDHilb}$

In $\text{FDHilb}$ the compact structure is given by the maps:

$$d : 1 \mapsto \sum_i a_i \otimes a_i \quad \quad \quad e : a_i \otimes a_i \mapsto 1$$

whenever $\{a_i\}_i$ is a basis for $A$ and $\{a_i\}_i$ is the corresponding basis for the dual space $A^*$.
Compact Structure of \textbf{FDHilb}

In \textbf{FDHilb} the compact structure is given by the maps:

\[ d : 1 \mapsto \sum_i a_i \otimes \overline{a_i} \quad e : \overline{a_i} \otimes a_i \mapsto 1 \]

whenever \( \{a_i\}_i \) is a basis for \( A \) and \( \{\overline{a_i}\}_i \) is the corresponding basis for the dual space \( A^* \).

In the case of \( \mathbb{C}^2 \) the map \( d \) picks out the Bell state

\[
\frac{|00\rangle + |11\rangle}{\sqrt{2}}
\]

which is the simplest example of \textit{quantum entanglement}.
Example: Quantum Teleportation

Aleks → Bob

Bennett et al 1993; Abramsky & Coecke 2004
Example: Quantum Teleportation

\[ \psi : I \rightarrow A \]
\[ \phi^* : A \rightarrow I \]
\[ \psi : I \rightarrow I \]

Bob

Aleks

Bennett et al 1993; Abramsky & Coecke 2004
Example: Quantum Teleportation

\[ \psi: I \rightarrow A \]
\[ \phi^\dagger: A \rightarrow I \]
\[ \phi^\dagger \circ \psi: I \rightarrow I \]

Bob

Aleks

\[ \langle 00 \rangle + \langle 11 \rangle \]
\[ \frac{\sqrt{2}}{\sqrt{2}} \]

Bennett et al 1993; Abramsky & Coecke 2004
Example: Quantum Teleportation

ψ: I → A
φ†: A → I
ψ: I → I

Bennett et al 1993; Abramsky & Coecke 2004
Example: Quantum Teleportation

$\psi$: $I \rightarrow A$

$\phi^\dagger$: $A \rightarrow I$

$\phi^\dagger \circ \psi$: $I \rightarrow I$

Bennett et al 1993; Abramsky & Coecke 2004

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Example: Quantum Teleportation

\[ \psi : I \rightarrow A \]
\[ \phi^\dagger : A \rightarrow I \]
\[ \phi^\dagger \circ \psi : I \rightarrow I \]

Bennett et al 1993; Abramsky & Coecke 2004
Example: Quantum Teleportation

\[ \psi : I \rightarrow A \]
\[ \phi^\dagger : A \rightarrow I \]

\[ \phi^\dagger \circ \psi : I \rightarrow I \]

Bennett et al 1993; Abramsky & Coecke 2004
Example: Quantum Teleportation

\[
\psi : I \rightarrow A
\]

\[
\phi^\dagger : A \rightarrow I
\]

\[
\phi^\dagger \circ \psi : I \rightarrow I
\]

Aleks

Bob

Bennett et al 1993; Abramsky & Coecke 2004
Example: Quantum Teleportation

Aleks

Bob

ψ:

I→A

φ:

A→I

φ†:

I→I

Bennett et al 1993; Abramsky & Coecke 2004

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A compact caveat

If compact structure exists, it need not be unique.

For example, in $\text{FDHilb}$, we have compact structures:

\[ d_Z = 1 \mapsto |00\rangle + |11\rangle \]

and

\[ d_Y = 1 \mapsto |00\rangle - |11\rangle \]

Therefore the $(\cdot)^*$ and $(\cdot)_*$ operations are not defined uniquely either.
Example: Preparing a Bell state
Example: Preparing a Bell state

\[ |0\rangle : I \rightarrow Q \]
Example: Preparing a Bell state

$|0\rangle : I \rightarrow Q$

$H$

$\land X$

$|0\rangle : I \rightarrow Q$
Example: Preparing a Bell state

$|0\rangle : I \rightarrow Q$

$H : Q \rightarrow Q$

$\wedge X$

$|0\rangle : I \rightarrow Q$
Example: Preparing a Bell state

|0⟩ : I → Q

H : Q → Q

∧X : Q ⊗ Q → Q ⊗ Q

∧X

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Classical Structure

Copying, deleting, and all that
No-Cloning and No-Deleting

Theorem: There are no unitary operations $D$ such that

\[ D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle \]
\[ D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle \]

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no unitary operations $E$ such that

\[ E : |\psi\rangle \mapsto |0\rangle \]
\[ E : |\phi\rangle \mapsto |0\rangle \]

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]
No-Cloning and No-Deleting, abstractly

**Theorem:** if a †-compact category has natural transformations

\[ \delta : - \Rightarrow - \otimes - \]

\[ \epsilon : - \Rightarrow I \]

then the category collapses [Abramsky 2007]

(Translation: in our abstract setting there are no universal cloning or deleting operations, just like in quantum mechanics)
“Classical” Quantum States

When can a quantum state be treated as if classical?

- no-go theorems allow copying and deleting of orthogonal states;

In other words:

- A quantum state may be copied and deleted if it is an eigenstate of some known observable.

We’ll use this property to formalise observables in terms of copying and deleting operations.
Classical Structures

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Comonoid Laws

\[ \quad = \quad = \quad = \quad = \]

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Classical Structures

\[\delta = \begin{array}{c}
\phantom{\delta}\\
\end{array} \quad \epsilon = \begin{array}{c}
\phantom{\epsilon}\\
\end{array} \quad \delta^\dagger = \begin{array}{c}
\phantom{\delta^\dagger}\\
\end{array} \quad \epsilon^\dagger = \begin{array}{c}
\phantom{\epsilon^\dagger}\\
\end{array}\]

Monoid Laws

\[\begin{array}{c}
\phantom{\delta}\\
\end{array} = \begin{array}{c}
\phantom{\delta}\\
\end{array} \quad \begin{array}{c}
\phantom{\epsilon}\\
\end{array} = \begin{array}{c}
\phantom{\epsilon}\\
\end{array} \quad \begin{array}{c}
\phantom{\delta^\dagger}\\
\end{array} = \begin{array}{c}
\phantom{\delta^\dagger}\\
\end{array} \quad \begin{array}{c}
\phantom{\epsilon^\dagger}\\
\end{array} = \begin{array}{c}
\phantom{\epsilon^\dagger}\\
\end{array}\]

\[\begin{array}{c}
\phantom{\delta}\\
\end{array} = \begin{array}{c}
\phantom{\delta}\\
\end{array} \quad \begin{array}{c}
\phantom{\epsilon}\\
\end{array} = \begin{array}{c}
\phantom{\epsilon}\\
\end{array} \quad \begin{array}{c}
\phantom{\delta^\dagger}\\
\end{array} = \begin{array}{c}
\phantom{\delta^\dagger}\\
\end{array} \quad \begin{array}{c}
\phantom{\epsilon^\dagger}\\
\end{array} = \begin{array}{c}
\phantom{\epsilon^\dagger}\\
\end{array}\]

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Classical Structures

\[ \delta = \begin{array}{c}
\end{array} \quad \epsilon = \begin{array}{c}
\end{array} \quad \delta^\dagger = \begin{array}{c}
\end{array} \quad \epsilon^\dagger = \begin{array}{c}
\end{array} \]

**Isometry Law**

**Frobenius Law**
In other words: a classical structure is a special commutative $\dagger$-Frobenius algebra.
Classical Structures

Given any finite dimensional Hilbert space we can define a classical structure by

$$\delta : A \rightarrow A \otimes A :: a_i \mapsto a_i \otimes a_i$$

$$\epsilon : A \rightarrow I :: \sum_i a_i \mapsto 1$$

Example:

$$\delta : \begin{align*}
|0\rangle & \mapsto |00\rangle \\
|1\rangle & \mapsto |11\rangle
\end{align*}$$

$$\epsilon : |0\rangle + |1\rangle \mapsto 1$$

define a classical structure over qubits; the standard basis is copied and erased. Note however that:

$$\delta(|+\rangle) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

showing that not every state can be cloned.
Given any finite dimensional Hilbert space we can define a classical structure by

\[ \delta : A \rightarrow A \otimes A :: a_i \mapsto a_i \otimes a_i \]
\[ \epsilon : A \rightarrow I :: \sum_i a_i \mapsto 1 \]

**Example:**

\[ \delta : |0\rangle \mapsto |00\rangle \]
\[ |1\rangle \mapsto |11\rangle \]
\[ \epsilon : |0\rangle + |1\rangle \mapsto 1 \]

define a classical structure over qubits; the standard basis is copied and erased. Note however that:

\[
\begin{array}{c}
\text{green node} \\
\otimes \\
\text{blue node}
\end{array} = \bigcup
\]

showing that not every state can be cloned.
Classical Structures

Given any finite dimensional Hilbert space we can define a classical structure by

\[ \delta : A \to A \otimes A :: a_i \mapsto a_i \otimes a_i \]

\[ \epsilon : A \to I :: \sum_i a_i \mapsto 1 \]

**Theorem:** in \( \text{FDHilb} \), classical structures are in bijective correspondence to bases. [Coecke, Pavlovic, Vicary]
Each (well behaved) observable defines a basis, hence each observable defines a classical structure!

Theorem: in $\text{FDHilb}$, classical structures are in bijective correspondence to bases. [Coecke, Pavlovic, Vicary]
Example: FRel

For any set $X$ we can define a classical structure by:

$$D : x \sim (x, x) \quad \forall x \in X$$
$$E : x \sim *$$

Perhaps surprisingly, there are others. Let $X = \{0, 1\}$ then we have:

$$D_X : 0 \sim \{(0, 0), (1, 1)\}$$
$$1 \sim \{(0, 1), (1, 0)\}$$

$$E_X : 0 \sim *$$
Spider Theorem

**Theorem**: any maps constructed from $\delta$ and $\varepsilon$, and their adjoints, whose graph is connected, is determined uniquely by the number of inputs and outputs.

Coecke & Paquette 2006
Theorem: any maps constructed from $\delta$ and $\varepsilon$, and their adjoints, whose graph is connected, is determined uniquely by the number of inputs and outputs.

Coecke & Paquette 2006
Theorem: any maps constructed from $\delta$ and $\epsilon$, and their adjoints, whose graph is connected, is determined uniquely by the number of inputs and outputs.
Classical Structure begets Compact Structure
Classical Structure begets Compact Structure

Frobenius
Classical Structure begets Compact Structure
Classical Structure begets Compact Structure
Classical Structure begets Compact Structure
Classical Structure begets Compact Structure

\[
\begin{array}{ccc}
\text{Each classical structure induces a self-dual compact structure.}
\end{array}
\]

Self duality is addressed in Coecke, Paquette, & Perdrix 2008
Phase Maps

Spinning around an observable
Using the monoid operation

Let $\psi, \phi : I \to A$ be points of $A$; we can combine them using the monoid operation $\delta^\dagger : A \otimes A \to A$

$$\psi \circ \phi := \delta^\dagger \circ (\psi \otimes \phi)$$
Using the monoid operation

Let $\psi, \phi : I \to A$ be points of $A$; we can combine them using the monoid operation $\delta^\dagger : A \otimes A \to A$

$$\psi \odot \phi := \delta^\dagger \circ (\psi \otimes \phi)$$
Example: qubits

\[ \delta^+_{Z} : Q \otimes Q \rightarrow Q \]

\[ |\phi\rangle : I \rightarrow Q \]
\[ |\psi\rangle : I \rightarrow Q \]

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Example: qubits

\[ \delta^\dagger_Z : Q \otimes Q \rightarrow Q \]

\[ |\phi\rangle : I \rightarrow Q \]

\[ |\psi\rangle : I \rightarrow Q \]

\[ \delta^\dagger_Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]
Example: qubits

\[\delta^\dagger_Z : Q \otimes Q \rightarrow Q\]

\[|\phi\rangle : I \rightarrow Q\]

\[|\psi\rangle : I \rightarrow Q\]

\[\delta^\dagger_Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\]

\[|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\]

\[|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\]
Example: qubits

\[ \delta^\dagger_Z : Q \otimes Q \rightarrow Q \]

\[ \psi \otimes \phi = \begin{pmatrix} \psi_1 \phi_1 \\ \psi_1 \phi_2 \\ \psi_2 \phi_1 \\ \psi_2 \phi_2 \end{pmatrix} \]
Example: qubits

\[ \delta^\dagger_Z : Q \otimes Q \rightarrow Q \]

\[ |\phi\rangle : I \rightarrow Q \]
\[ |\psi\rangle : I \rightarrow Q \]

\[ |\psi\rangle \otimes |\phi\rangle := \delta^\dagger_Z \circ (|\psi\rangle \otimes |\phi\rangle) = \begin{pmatrix} \psi_1 \phi_1 \\ \psi_2 \phi_2 \end{pmatrix} \]
Example: qubits

\[ \delta_Z^\dagger : Q \otimes Q \to Q \]

\[ |\phi\rangle : I \to Q \]
\[ |\psi\rangle : I \to Q \]

\[ |\psi\rangle \otimes |\phi\rangle := \delta_Z^\dagger \circ (|\psi\rangle \otimes |\phi\rangle) = \begin{pmatrix} \psi_1 \phi_1 \\ \psi_2 \phi_2 \end{pmatrix} \]

a.k.a. the Schur product or convolution product
Using the monoid operation

Moreover, each point $\psi : I \to A$ can be lifted to an endomorphism $\Lambda(\psi) : A \to A$

$$\Lambda(\psi) := \delta^\dagger \circ (\psi \otimes \text{id}_A)$$

This yields a homomorphism of monoids so we have:

\[ \begin{array}{c}
\phi \phi = \psi \circ \phi = \psi \\
\end{array} \]
Example: qubits

$$\delta^+_Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$
Example: qubits

\[ \delta_Z^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \]

\[ \text{id} \otimes |\psi\rangle = \begin{pmatrix} \psi_1 & 0 \\ \psi_1 & 0 \\ 0 & \psi_2 \\ 0 & \psi_2 \end{pmatrix} \]
Example: qubits

\[
\delta_{Z}^{\dagger} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
|\psi\rangle = \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]

\[
id \otimes |\psi\rangle = \begin{pmatrix}
\psi_1 & 0 \\
\psi_1 & 0 \\
0 & \psi_2 \\
0 & \psi_2
\end{pmatrix}
\]

\[
\Lambda_{Z}^{Z}(\psi) := \delta_{Z}^{\dagger} \circ (id \otimes |\psi\rangle) = \begin{pmatrix}
\psi_1 & 0 \\
0 & \psi_2
\end{pmatrix}
\]
Generalised Spider Theorem

Theorem: any maps constructed from \( \delta, \varepsilon \), some points \( \psi_i : I \to A \) and their adjoints, whose graph is connected, is determined by the number of inputs and outputs and the product \( \bigotimes_i \psi_i \).
Generalised Spider Theorem

Theorem: any maps constructed from $\delta$, $\varepsilon$, some points $\psi_i : I \to A$ and their adjoints, whose graph is connected, is determined by the number of inputs and outputs and the product $\bigotimes_i \psi_i$.
Generalised Spider Theorem

Theorem: any maps constructed from $\delta$, $\varepsilon$, some points $\psi_i : I \rightarrow A$ and their adjoints, whose graph is connected, is determined by the number of inputs and outputs and the product $\bigodot_i \psi_i$. 
Unbiased Points

Q: When is $\psi$ unitary?

A: In Hilbert spaces, $\Lambda(\psi)$ is unitary iff $|\psi\rangle$ is unbiased w.r.t. the basis copied by $\delta$. 
Unbiased Points

Q: When is $\psi$ unitary?

A: In Hilbert spaces, $\Lambda(\psi)$ is unitary iff $|\psi\rangle$ is unbiased w.r.t. the basis copied by $\delta$. 

\[ \psi = \Lambda(\psi) = \delta \]
Unbiased Points

Q: When is $\Psi$ unitary?

A: In Hilbert spaces, $\Lambda(\psi)$ is unitary iff $|\psi\rangle$ is unbiased w.r.t. the basis copied by $\delta$.
Q: When is $\psi$ unitary?

A: In Hilbert spaces, $\Lambda(\psi)$ is unitary iff $|\psi\rangle$ is unbiased w.r.t. the basis copied by $\delta$. 

\[
\begin{array}{c}
\alpha \\
= \\
\alpha
\end{array}
\]
Unbiased Points

Q: When is \( \psi \) unitary?

A: In Hilbert spaces, \( \Lambda(\psi) \) is unitary iff \( |\psi\rangle \) is unbiased w.r.t. the basis copied by \( \delta \).

Prop:
1. the unbiased points for \((\delta, \epsilon)\) form an abelian group w.r.t. \(\circ\)
2. the arrows generated by the unbiased points form an abelian group w.r.t. composition.
Example: qubits
Example: qubits
Example: qubits
Example: qubits

\[ |0\rangle + e^{i\alpha} |1\rangle \]

Unbiased points
Example: qubits

Unbiased points
Example: qubits

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]

Unbiased points
Example: FRel

\[ D^\dagger : (x, x) \sim x, \forall x \in X \]

\[ D^\dagger \circ (\text{id} \otimes \psi) \text{ is unitary iff } \psi = X \]

Hence the phase group is trivial.
Example: FRel

Suppose that we are working with a two element set \( \{0, 1\} \). Then:

\[
D : \quad 0 \sim \{(0, 0), (1, 1)\} \\
1 \sim \{(0, 1), (1, 0)\}
\]

\[
E : 0 \sim * 
\]

Defines a classical structure whose phase group is \( \mathbb{Z}_2 \)
Example: $\text{FRel}$

**Theorem** (Pavlovic) : The classical structures of $\text{FRel}$ are exactly biproducts of Abelian groups.

It’s easy to check that in this gives a phase group which is just the product group
Complementary Observables

A very general theory of interference
Two kinds of points

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]
Two kinds of points

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Classical Points

Those points which can be copied by \( \delta \)
Two kinds of points

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Classical Points

\[ \quad = \quad \]

Those points which can be copied by \( \delta \)
Two kinds of points

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Classical Points

Those points which can be copied by \( \delta \)

* to keep the pictures tidy, a scalar factor has been omitted (and everywhere from here on)
Two kinds of points

\[ \delta = \text{Classical Points} \]

\[ \epsilon = \text{Unbiased Points} \]

Those points which can be copied by \( \delta \)
Two kinds of points

\[ \delta = \quad \epsilon = \]

\[ \delta^\dagger = \quad \epsilon^\dagger = \]

Classical Points

Those points which can be copied by \( \delta \)

Unbiased Points
Two kinds of points

\[ \delta = \quad \epsilon = \]

\[ \delta^\dagger = \quad \epsilon^\dagger = \]

Classical Points

Unbiased Points

Those points which can be copied by \( \delta \)
Complementary Classical Structures

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\[ |i\rangle \mapsto |ii\rangle \quad |+\rangle \mapsto 1 \quad |\pm\rangle \mapsto |\pm\pm\rangle \quad |0\rangle \mapsto 1 \]
Complementary Classical Structures

\[ \delta_Z = \text{green structure} \quad \epsilon_Z = \text{green structure} \quad \delta_X = \text{red structure} \quad \epsilon_X = \text{red structure} \]

\[ \text{green structure} = \text{red structures} \quad \Rightarrow \quad \text{red structures} = \]
Complementary Classical Structures

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\[ \Rightarrow \]

\[ i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i \quad i 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Complementary Classical Structures

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\[ = \quad = \quad = \quad = \]
Example: qubits

Unbiased points

Classical points
Example: qubits

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]

Unbiased points

Classical points

|0\rangle = \bullet

|1\rangle = \pi

\[ |0\rangle = \pi \]

\[ |1\rangle = \pi \]

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Example: qubits

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]

\[ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

Unbiased points

Classical points

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Example: qubits

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]

\[ |0\rangle = \pi \]

\[ |1\rangle = \pi \]

\[ |\rangle = \pi \]

\[ |\rangle = \pi \]

Unbiased points

Classical points
Classical points are eigenvectors
Classical points are eigenvectors

\[ \alpha \quad i \]
Classical points are eigenvectors
A pair of complementary observables are called *strongly complementary* when they satisfy this equation:

\[ \delta_z = \epsilon_z = \delta_x = \epsilon_x = \]
**Strong Complementarity**

**Corollary:** strongly complementary observables form a *bialgebra*
**Strong Complementarity**

*Corollary:* strongly complementary observables form a *bialgebra*
The antipode

\[ \delta_Z = \text{[Diagram]} \quad \epsilon_Z = \text{[Diagram]} \quad \delta_X = \text{[Diagram]} \quad \epsilon_X = \text{[Diagram]} \]

Define the map \( S \) by:

\[ S = \text{[Diagram]} \quad := \text{[Diagram]} \]
The antipode
Strongly complementary observables form Hopf algebras

Theorem:

Remark: under the assumption of *enough classical points* the "strong" assumption is not needed; simple complementarity suffices
Strong Complementarity

The following are equivalent characterisations of strong complementarity:
Comonoid Homomorphism

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

The classical maps are comonoid homomorphisms
Comonoid Homomorphism

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

The classical maps are comonoid homomorphisms
Classical Phases Commute

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

The classical maps satisfy canonical commutation relations
Closedness Property

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

Defn: complemntary classical structures are called closed when...
Closedness Property

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\[ i \quad = \quad i \quad = \quad i \qquad i \circ j \quad = \quad i \circ j \quad = \quad i \circ j \]

\[ j \quad = \quad j \quad = \quad j \]
Closedness Property

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

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Closedness Property

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

In \textbf{fdHilb} it is always possible to construct a pair of strong complementary observables...

... but in big enough dimension, it is possible to construct MUBs which are not closed.
Classical Points form a Subgroup

By the definition of complementarity, we have that:

\[ C_X \subseteq U_Z \]
\[ C_Z \subseteq U_X \]

**Prop**: If the observable structure is closed, and there are finitely many classical points, then they form a *subgroup* of the unbiased points, i.e.

\[ (C_Z, \odot_X) \leq (U_X, \odot_X) \]
\[ (C_X, \odot_Z) \leq (U_Z, \odot_Z) \]

In particular, each \( \Lambda^X(z_i) \) is a permutation on \( C_Z \).
Example: qubits

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]

\[ |0\rangle = \begin{array}{c} \bullet \\ \pi \end{array} \]

\[ |1\rangle = \begin{array}{c} \bullet \\ \pi \end{array} \]

\[ |\pm\rangle = \begin{array}{c} \bullet \\ \pi \end{array} \]

| Unbiased points |

| Classical points |

\[ \alpha = \begin{pmatrix} \cos \frac{\alpha}{2} & i \sin \frac{\alpha}{2} \\ i \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \]

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Example: qubits

\[
\pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[
\begin{align*}
|0\rangle &= \pi \\
|1\rangle &= \pi
\end{align*}
\]

Classical points

Unbiased points

\[
\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
\begin{align*}
|+\rangle &= \pi \\
|-\rangle &= \pi
\end{align*}
\]
Automorphism Action

**Theorem:** The classical maps are group automorphisms of the unbiased points:

\[ \alpha \]

\[ \mathbf{i} \]
Automorphism Action

**Theorem:** The classical maps are group automorphisms of the unbiased points:

\[
\begin{align*}
\alpha & \rightarrow \alpha \\
1 & \rightarrow -i \\
-i & \rightarrow 1 \\
-1 & \rightarrow -1
\end{align*}
\]
Automorphism Action

**Theorem**: The classical maps are group automorphisms of the unbiased points:

\[
\alpha \quad i \quad -i
\]

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Automorphism Action

**Theorem:** The classical maps are group automorphisms of the unbiased points:
Automorphism Action

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Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:
Automorphism Action

**Theorem:** The classical maps are group automorphisms of the unbiased points:
Automorphism Action

**Theorem:** The classical maps are group automorphisms of the unbiased points:

\[
\begin{align*}
\alpha \beta &= \alpha \beta \\
\kappa &= \kappa \\
\alpha \beta &= \alpha \beta \\
\kappa &= \kappa
\end{align*}
\]
Automorphism Action

**Theorem:** The classical maps are group automorphisms of the unbiased points:

Classical points are symmetries of the phase group.
Why care about phase groups?

The phase group seems like a rather random structure. Why do we care? The phase group:

- ... determines the underlying arithmetic
- ... fixes the possibility for interference
- ... determines the possible results of CHSH-type experiments, and hence the possibility for non-locality.
The ZX-calculus

Quantum processes, diagrammatically

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Z and X Observables

The Pauli $\mathbf{Z}$ and $\mathbf{X}$ observables are strongly complementary, with some additional features:

- The phase group is $[0, 2\pi)$
- Dimension 2 implies two classical points
- The action of the classical group is $\alpha \mapsto -\alpha$
Z and X Observables

- Both observables generate the same compact structure

- can just treat the diagram as an undirected graph

- the Z and X are related by a definable unitary

- gives rise to colour change rule
**ZX-calculus syntax**

**Defn:** A *diagram* is an undirected open graph generated by the above vertices.

**Defn:** Let $\mathcal{D}$ be the dagger compact category of diagrams s.t.

$$ (\cdot)\dagger : \alpha \mapsto -\alpha $$
Equations

(spider) \( \alpha \) \( + \beta \) = \( \alpha + \beta \)

(anti-loop) \( \alpha \) = \( \alpha \)

(identity) \( 0 = \)

(\( \pi \)-commute) \( \alpha \) \( = \) \( -\alpha \)

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Equations

\[
\begin{align*}
\pi_2 & + \alpha + \frac{n \pi}{2} = \\
-\pi_2 & - \alpha - \frac{\pi}{2} = \\
\end{align*}
\]

(colour change)
**Defn:** Let $[\cdot] : \mathcal{D} \to \mathbf{FDHilb}$ be the traced monoidal functor such that $[A] = \mathbb{C}^{2^n}$ and define its action on generators by:

$$
\begin{align*}
\begin{bmatrix} \alpha \end{bmatrix} = \begin{cases}
|0\rangle^\otimes m &\mapsto |0\rangle^\otimes n \\
|1\rangle^\otimes m &\mapsto e^{i\alpha} |1\rangle^\otimes n
\end{cases}
\end{align*}
$$

$$
\begin{align*}
\begin{bmatrix} \alpha \end{bmatrix} = \begin{cases}
|+\rangle^\otimes m &\mapsto |+\rangle^\otimes n \\
|\rangle^\otimes m &\mapsto e^{i\alpha} |\rangle^\otimes n
\end{cases}
\end{align*}
$$
Representing Qubits

\[
\begin{align*}
\left[ \begin{array}{c} \textcolor{red}{\bullet} \\ \bullet \end{array} \right] &= \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = |0\rangle \\
\left[ \begin{array}{c} \textcolor{red}{\pi} \\ \bullet \end{array} \right] &= \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = |1\rangle \\
\left[ \begin{array}{c} \textcolor{green}{\bullet} \\ \bullet \end{array} \right] &= \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = |+\rangle \\
\left[ \begin{array}{c} \textcolor{green}{\pi} \\ \bullet \end{array} \right] &= \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) = |-\rangle
\end{align*}
\]
Representing Phase shifts

\[
\begin{pmatrix}
\alpha
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & e^{i\alpha}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\beta
\end{pmatrix}
= \begin{pmatrix}
\cos \frac{\beta}{2} & -i \sin \frac{\beta}{2} \\
-i \sin \frac{\beta}{2} & \cos \frac{\beta}{2}
\end{pmatrix}
\]
Representing Paulis

\[
\begin{bmatrix}
\pi
\end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
\begin{bmatrix}
\pi
\end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
Representing CNot

$\wedge X = [ \begin{array} \bullet & \mathbf{N} & \bullet \\
\bullet & \mathbf{N} & \bullet \\
\bullet & \mathbf{N} & \bullet \\
\end{array} ] = [ \begin{array} \bullet & \mathbf{N} & \bullet \\
\bullet & \mathbf{N} & \bullet \\
\bullet & \mathbf{N} & \bullet \\
\end{array} ] = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}$
Representing Logic Gates

\[ \land X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
Representing Logic Gates

\[ \wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
Representing Logic Gates

\[ \wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
Representing Logic Gates

\[ \wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
Representing Logic Gates

\[ \land X = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \]
Representing Logic Gates

\[ \wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
The ZX-calculus is universal

**Theorem:** Let \( U \) be a unitary map on \( n \) qubits; then there exists a ZX-calculus term \( D \) such that:

\[
[ D ] = U
\]
Representing Hadamard

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{bmatrix} \pi/2 \\ \pi/2 \end{bmatrix} \]
More on the Hadamard

\[ H := \begin{align*}
\pi/2 \\
\pi/2 \\
\pi/2
\end{align*} \]
More on the Hadamard

\[ H := \begin{bmatrix} \frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & -\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} \end{bmatrix} \]

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More on the Hadamard

\[ H := \pi/2 = \pi/2 = \pi/2 = \pi/2 \]
More on the Hadamard

\[ H := -\frac{\pi}{2} = \frac{\pi}{2} = -\frac{\pi}{2} = -\frac{\pi}{2} \]
More on the Hadamard

\[ H = -\frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} \]
More on the Hadamard

\[
H = \frac{\pi}{2} = - \frac{\pi}{2} = \frac{\pi}{2} = - \frac{\pi}{2}
\]

colour change
More on the Hadamard

\[
\begin{align*}
H &= \pi/2 \quad = \quad -\pi/2 \\
&= \pi/2 \quad = \quad -\pi/2 \\
&= \pi/2 \quad = \quad -\pi/2 \\
&= \pi/2 \quad = \quad -\pi/2 \\
&= \pi/2 \quad = \quad -\pi/2 \\
\end{align*}
\]

colour change

colour change
More on the Hadamard

Corollary 1:

$$H = \left( \begin{array}{c} H \\ H \end{array} \right) \dagger = \left( \begin{array}{c} H \\ H \end{array} \right)$$
More on the Hadamard

Corollary 2:
More on the Hadamard

Corollary 2:

\[ \alpha + \frac{n\pi}{2} \]
More on the Hadamard

Corollary 2:

\[ \alpha \]

\[ \alpha + \frac{n\pi}{2} \]

\[ \alpha \]

\[ \alpha - \frac{n\pi}{2} \]
More on the Hadamard

Corollary 2:

\[ \alpha + \frac{n \pi}{2} = \alpha - \frac{\pi}{2} \]

Corollary 2.1: total symmetry between red and green

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More on the Hadamard

Corollary 3:

\[ \frac{\pi}{2} \]
Corollary 3:

\[
\pi - \frac{\pi}{2} = \frac{\pi}{2} - \frac{\pi}{2}
\]
More on the Hadamard

Corollary 3:

\[ \frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2} = -\frac{\pi}{2} \]
More on the Hadamard

Corollary 3:

\[
\frac{\pi}{2} = -\frac{\pi}{2} = \frac{\pi}{2} = -\frac{\pi}{2}
\]
More on the Hadamard

Corollary 3:

\[
\frac{\pi}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \frac{-\pi}{2} = \frac{-\pi}{2} = \frac{-\pi}{2} = \frac{-\pi}{2}
\]
More on the Hadamard

Corollary 3:

\[ \frac{\pi}{2} = -\frac{\pi}{2} \]
More on the Hadamard

Corollary 3:

$$\frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2} = -\frac{\pi}{2}$$
More on the Hadamard

Corollary 3:

\[
\frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2} = -\frac{\pi}{2}
\]
More on the Hadamard

Corollary 4: Van den Nest’s theorem

$$R_x(\pi/2)^{(u)} R_z(-\pi/2)^{(N_G(u))} |G\rangle = |G \ast u\rangle$$


Representing Logic Gates

\[ \wedge Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]
Example: 2 CZs
Example: 2 CZs
Example: 2 CZs
Example: 2 CZs
Example: 2 CZs
Example: 2 CZs
Example: 2 CZs
Example: 2 CZs
Example: Preparing a Bell state

\[ H \wedge X \]
Example: Preparing a Bell state
Example: Preparing a Bell state

\[ H \]
Example: Preparing a Bell state

\[ H \]
Example: Preparing a Bell state
Example: Preparing a Bell state
Example: 2-Qubit Quantum Fourier Transform

\[ j_1 = |1\rangle \]
\[ j_0 = |0\rangle \]

Input qubits

\[ \pi \]

\[ \frac{-\pi}{4} \quad \frac{\pi}{4} \]

Controlled gate \( Z_{\pi/2} \)
Example: 2-Qubit Quantum Fourier Transform

\[ \pi \]

\[ -\frac{\pi}{4} \quad \frac{\pi}{4} \]
Example: 2-Qubit Quantum Fourier Transform
Example: 2-Qubit Quantum Fourier Transform

\[ H \pi \frac{\pi}{4} \]

\[ H \pi \pi \pi \pi \frac{-\pi}{4} \frac{\pi}{4} \]
Example: 2-Qubit Quantum Fourier Transform
Example: 2-Qubit Quantum Fourier Transform
Example: 2-Qubit Quantum Fourier Transform

\begin{align*}
\pi - \frac{\pi}{4} & \quad \frac{\pi}{4}
\end{align*}
Example: 2-Qubit Quantum Fourier Transform
Example: 2-Qubit Quantum Fourier Transform
Example: 2-Qubit Quantum Fourier Transform
Example: 2-Qubit Quantum Fourier Transform

\[
\begin{align*}
j_0 &= |0\rangle + e^{i\pi/2} |1\rangle \\
j_1 &= |0\rangle + e^{i\pi} |1\rangle
\end{align*}
\]
Which diagrams are circuits?

**Defn:** a diagram is called *circuit-like* if:

- All of its \( \alpha \), \( \beta \), and boundary vertices can be covered by set of disjoint paths \( P \), each of which ends in an output.

- Every cycle in the diagram which overlaps with 2 paths in \( P \) traverses an edge in the opposite direction to \( P \).

- It is 3-coloured.
Which diagrams are circuits?
Which diagrams are circuits?

YES!
Which diagrams are circuits?
Which diagrams are circuits?
Which diagrams are circuits?

NO :(

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Which diagrams are circuits?

**Thm:** if a diagram \( D \) is circuit-like then \([D]\) is a unitary embedding.

Being circuit-like is a little stronger than being a unitary embedding: we require a particular gate set, and a particular encoding of the gate set as diagrams, and the minimality w.r.t. to spider. Up to these conditions, the theorem is *if and only if*. 
Information Flow in the 1-Way Quantum Computer

Using measurements to compute
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:

Input register $|\Psi\rangle$
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:

Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:
Quantum Circuits

Model of quantum computation analogous to classical Boolean circuits:

\[ U : \mathbb{C}^{2^n} \rightarrow \mathbb{C}^{2^n} \]
Quantum Circuits

Circuits can be generalised:

$|\Psi\rangle$ $|\Psi'\rangle$
Quantum Circuits

Circuits can be generalised:

$|\Psi\rangle$ $|\Psi'\rangle$

Ancillas
Quantum Circuits

Circuits can be generalised:

\[ |\Psi\rangle - |\Psi'\rangle \]

Ancillas
Quantum Circuits

Circuits can be generalised:

|ψ⟩  |ψ'⟩  
|---|---|
|0⟩ |0⟩

Ancilllas

Measurements
Quantum Circuits

Circuits can be generalised:

\[ \mathcal{C} : \mathbb{C}^{2^n \times 2^n} \rightarrow \mathbb{C}^{2^m \times 2^m} \]
Quantum Circuits

Extreme case:

\[ |\Psi\rangle \]
\[ |0\rangle \]
\[ |0\rangle \]
\[ |0\rangle \]
\[ |0\rangle \]
\[ |0\rangle \]

Ancillas

Measurements

\[ C : \mathbb{C}^{2^n \times 2^n} \rightarrow \mathbb{C}^{2^m \times 2^m} \]
Quantum Circuits

Extreme case:

\[ C : \mathbb{C}^{2^n \times 2^n} \rightarrow \mathbb{C}^{2^m \times 2^m} \]

Ancillas

Measurements
The One-Way Model

Some qubits, each initialised in the state $|+\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
The One-Way Model

Some qubits, each initialised in the state $|+\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Entangling operation on pairs of qubits to make a **graph state**.

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
The One-Way Model

A graph state, coupled to some input qubits:

The only operation is to measure single qubits in the basis:

\[ |+\alpha\rangle := \frac{1}{\sqrt{2}} (|0\rangle + e^{i\alpha}|1\rangle) \]

\[ |-\alpha\rangle := \frac{1}{\sqrt{2}} (|0\rangle - e^{i\alpha}|1\rangle) \]
The One-Way Model

A graph state, coupled to some input qubits:

* Measured qubits are removed from the cluster;
* The outcome of measurement alters the remaining state.
The One-Way Model

A graph state, coupled to some input qubits:

* Measured qubits are removed from the cluster;
* The outcome of measurement alters the remaining state.
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The One-Way Model

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Raussendorf and Briegel. PRL (86) 2001

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The One-Way Model

A graph state, coupled to some input qubits:

* Measured qubits are removed from the cluster;
* The outcome of measurement alters the remaining state.

Finally, any output qubits, they can be corrected by a Pauli
The One-Way Model

A graph state, coupled to some input qubits:

* Measured qubits are removed from the cluster;
* The outcome of measurement alters the remaining state.

Finally, any output qubits, they can be corrected by a Pauli
Non-determinism

Non-determinism of measurements leads to probabilistic

Attempt to control branching by using *adaptive measurements*:

(choice of measurements depend on the outcome of earlier ones)
Non-determinism

Non-determinism of measurements leads to probabilistic

Can we carry out measurement-based computation deterministically?

Attempt to control branching by using *adaptive measurements*:

(choice of measurements depend on the outcome of earlier ones)
Measurement Calculus

The measurement calculus is a formal syntax for programming the one-way model.

- $N_i$: initialise qubit $i$ in the $|+\rangle$ state
- $E_{ij}$: entangle qubits $i$ and $j$ using a CZ operation
- $s[M_i^{\alpha}]^t$: measure qubit $i$ in the basis $|0\rangle \pm e^{(-1)^s \alpha + t\pi} |1\rangle$
- $X_i^s, Z_i^s$: apply a Pauli X or Z operator to qubit $i$

The boolean variables $s, t$ are \textit{signals}; their values are determined by the results of measuring the corresponding qubit.

- Standard form: operations occur in the order above
  - Every pattern can be standardised
How to Measure

Suppose we have a measuring device for the standard 1-qubit basis:

|0⟩

Yay!

|1⟩

Boo!
How to Measure

Suppose we have a measuring device for the standard 1-qubit basis:
Should be this:

Yay!  $|+\rangle$

Yay!  $|--\rangle$

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Example: Hadamard

Prepared $|+\rangle$ qubit

$\wedge Z$-gate

Projective measurement
Example: Hadamard
Example: Hadamard

Yay!
Example: Hadamard

Yay!
Example: Hadamard
Example: Hadamard
Example: Hadamard

Boo!
Example: Hadamard

Boo!
Example: Hadamard
Example: Hadamard

Add correction here:
Example: Hadamard

Add correction here:

Boo!
Example: Hadamard

Add correction here:
Conditional Operations

Defn: Let $S$ be a set of variables ("signals"); define $\mathcal{D}(S)$ to be the category of diagrams generated by:

where $S' \subseteq S$
Semantics

**Defn:** a *valuation* of $S$ is a function $v : S \rightarrow \{0, 1\}$. For each valuation we can define a map $\hat{v} : \mathcal{D}(S) \rightarrow \mathcal{D}$ by replacing the angle alpha with 0 if

$$\prod_{s \in S'} v(s) = 0$$

and leaving if unchanged otherwise.

**Defn:** The *denotation* of a diagram $D$ in $\mathcal{D}(S)$ is a superoperator:

$$\rho \mapsto \sum_{v \in 2^S} [\hat{v}(D)] \rho [\hat{v}(D)]^\dagger$$
Example

A measurement in the $|+\rangle, |-\rangle$ basis:

$$\rho \mapsto \langle + | \rho | + \rangle + \langle + | Z \rho Z | + \rangle = \langle + | \rho | + \rangle + \langle - | \rho | - \rangle$$
Equations for conditional diagrams

(spider)

(anti-loop)

(identity)

(α-commute)

(π-commute)
Equations for conditional diagrams
Equations for conditional diagrams

.... plus the same again with the colours exchanged
Translation from MC to ZX-calculus

We can translate any measurement pattern to a diagram using the table below:

<table>
<thead>
<tr>
<th>$N_i$</th>
<th>$E_{ij}$</th>
<th>$M_i^\alpha$</th>
<th>$X_i^s$</th>
<th>$Z_i^s$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Diagrams" /></td>
<td><img src="image" alt="Diagrams" /></td>
<td><img src="image" alt="Diagrams" /></td>
<td><img src="image" alt="Diagrams" /></td>
<td><img src="image" alt="Diagrams" /></td>
</tr>
</tbody>
</table>
Example

The CNOT is computed by the pattern:

\[ \psi = X_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 E_{13} E_{23} E_{34} N_3 N_4 \]

Which yields the diagram:
We removed all the conditional operations, therefore this pattern is deterministic.

Can we do this in general?
Geometry of a pattern

The following pattern computes the CNOT gate:

$$\Psi = \bar{X}_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 E_{13} E_{23} E_{34} N_3 N_4$$

Each pattern implicitly defines a geometry \((G, I, O)\)
Geometry of a pattern

The following pattern computes the CNOT gate:

$$\Psi = \overline{X_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0}^{E_{13} E_{23} E_{34}} N_3 N_4$$

Each pattern implicitly defines a *geometry* \((G, I, O)\)
Geometry of a pattern

The following pattern computes the CNOT gate:

\[ \Psi = \overline{X_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 E_{13} E_{23} E_{34} N_3 N_4} \]

Each pattern implicitly defines a geometry \((G, I, O)\)
Geometry of a pattern

The following pattern computes the CNOT gate:

\[ \Psi = X_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 E_{13} E_{23} E_{34} N_3 N_4 \]

Each pattern implicitly defines a \textit{geometry} \((G, I, O)\)

Output vertices

Input vertices

Entanglement graph

Input vertices

Output vertices
Flow and determinism

**Defn:** let \((G, I, O)\) be a geometry; a flow on \(G\) is a pair \((f, <)\) where \(f: O^C \rightarrow I^C\) is a function and \(<\) is a partial order s.t.

- \(f(u) \sim u\)
- \(u < f(u)\)
- If \(f(u) \sim v, v \neq u\) then \(u < v\)

Intuitively \(f(u)\) is the qubit which must correct the error produced by measuring \(u\). The partial order guarantees that this is consistent with causality.

**Thm:** if a geometry has a flow there exists a uniformly deterministic pattern on it.
Flow

Input vertices

Output vertices
Flow

Input vertices

Output vertices

Boo!
Flow

Input vertices

Output vertices

$\text{Boo!}$

$\text{f(v)}$
Flow

Input vertices

Output vertices

Boo!

$v$

$f(v)$
Flow

Input vertices

Output vertices

$v \rightarrow f(v)$
Flow

The sequence \( u, f(u), f^2(u), \ldots \) determines a path from inputs to outputs.

Effectively the flow path” is a “logical qubit” in a circuit equivalent to the pattern.
Geometry of a pattern

We can define a diagram directly from the geometry \((G, I, O)\)
Geometry of a pattern

We can define a diagram directly from the geometry \((G, I, O)\)

\[
\Gamma_{\mathcal{P}_{\wedge} x} = \begin{array}{c}
\text{inputs} \\
1 \\
\text{outputs} \\
3 \\
2 \\
4
\end{array}
\]

\[
D(\Gamma_{\mathcal{P}_{\wedge} x}) = \begin{array}{c}
1 \\
H \\
H \\
H \\
H \\
3 \\
4
\end{array}
\]

**Theorem**: If geometry \(G\) has a flow then \(D(G)\) is (equivalent to) a circuit-like diagram.
Getting back to the pattern
Getting back to the pattern

We have $D(G(P))$ and $D(P)$. What is the relation?
Getting back to the pattern

We have $D(G(P))$ and $D(P)$. What is the relation?

Define $D^*(P)$ by adding pieces to $D(G(P))$:
Getting back to the pattern

We have $D(G(P))$ and $D(P)$. What is the relation?

Define $D^*(P)$ by adding pieces to $D(G(P))$:

Prop: $D(P)$ rewrites to $D^*(P)$
Flow Strategy

If the geometry has a flow we can define a rewrite strategy to propagate the “errors” forward:

'π-green is attracted by the flow':

'π-red is pushed by the flow':
Flow Strategy

\[ \begin{align*}
\alpha & \quad \pi, s_1 \\
\pi, s_2 & \quad \beta \\
\pi, s_1 & \quad \pi, s_2 \\
\pi, s_1 & \quad \pi, s_1 \\
\end{align*} \]
Flow Strategy
Flow Strategy
Flow Strategy
Flow Strategy
Flow Strategy
Flow Strategy
Flow Strategy
Flow Strategy

\[ \begin{array}{c}
\alpha \\
H \\
H \\
\beta \\
\pi, S_2
\end{array} \]
Ross Duncan • Réunion LOGO! • Paris 2013

Flow Strategy

α → H → β → H

π, s₂

π, s₂
Flow Strategy
Flow Strategy

**Thm:** This strategy terminates in one of two states:

- All conditional operations removed $\Rightarrow$ deterministic
- Not all removed, but their location reveals where extra corrections are needed.
Flow Strategy

\[ D_1 = \pi, \{s\} \]
\[ D_2 = \alpha \]

\[ [D_1] \neq [D_2] \text{ unless } \alpha = 0 \text{ or } \alpha = \pi \]

From here we can reconstruct the original pattern given by DKP as the uniformly deterministic pattern based on \( G \).
Generalised Flow is the generalisation of flow where each qubit has a set of qubits.

Theorem [BKMP 2007]: A geometry has generalised flow iff it supports a step-wise strongly uniformly deterministic pattern.

Theorem [DP 2010] If geometry has flow then the corresponding diagram can rewrite to a circuit-like diagram.

(No time to discuss => relies very strongly upon the bialgebra rule)
Non-uniform determinism

Note the ZX-calculus works with the pattern, not the geometry, so it can also treat non-uniform cases. E.g.:
Papers

Monoidal Categories and Diagrams


†-Compact Categories and QM

Papers

Observable Structures


Complementary Observables


Non-locality and the phase group

Papers

MBQC - classic


MBQC - categorical