Introduction to Categorical Quantum mechanics

Ross Duncan
0. Motivations
Transistors are shrinking

Feature Size (nm) Log-scale

Diameter of a virus

Thickness of spider silk


Transistors are shrinking

Quantum effects become important
Quantum Computing

- **IDEA**: exploit quantum effects for computation
  - Fast algorithms (Shor, Grover)
  - Simulate physical systems (chemistry, materials)
  - Novel cryptographic protocols (QKD, blind computing)
  - .... and more.
Quantum Computing

**target**: 49 qubit machine in the lab by end of 2017
Quantum Computing

What is IBM Q?

IBM Q is an industry-first initiative to build commercially available universal quantum computing systems for business and science. Today, IBM has a sophisticated prototype commercial quantum processor that will form the core of the first IBM Q early-access

5 qubits now; 16 later this year

target: 50 qubit machine for sale by 2021
Quantum Computing

**target:** 400 qubits by 2020

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**Ion Traps**
Ion traps are one of our core technologies, which we are developing for use as quantum information processors.

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**Atom-photon Interfaces**
In order to create a powerful quantum computing device by interconnecting many simple quantum processors we need reliable interfaces at the single-quantum level to establish such a connected network.

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**Photonics**
In NQIT we are developing the optics to wire-up individual atoms, by collecting the particles of light which they emit and building a quantum entanglement connection between the atoms.
Quantum Computing: 17 Years Of Major Startup Financings In One Timeline

Practical Quantum Computers
Advances at Google, Intel, and several research groups indicate that computers with previously unimaginable power are finally within reach.

Availability: 4-5 years
Motivations
Motivations
Motivations
Motivations

```plaintext
fun fact 0 = 1
    | fact x = x * fact (x-1)
```

[Diagram of a circuit]
Motivations

\( \lambda x. \lambda y. \lambda z. xz(yz) \)
Motivations
Motivations
Motivations

Hilbert space, unitary transforms, self-adjoint operators....
Motivations

Hilbert space,
unitary transforms,
self-adjoint operators....
The need for abstraction:

This is an 8-bit adder

\[ D \sim 2^{1764} \]
What's so special about quantum?
What’s so special about quantum?

Seek a mathematical answer.
Superposition?

Will the QC kill the PC?

Last Updated: 12:01am BST 01/07/2008

• Why we should celebrate Charles Darwin

Quantum computers could become a reality very soon, opening up some fascinating possibilities - including teleportation, says Richard Gray

It might be the science of the very small, but quantum computing is on the verge of solving some giant problems. For more than 30 years, physicists have dreamed of harnessing the power of atoms to produce computers that would far outstrip the capabilities of the microchips used in today's PCs.
Superposition?

Traditional computers shuffle information in the form of binary numbers, the digits 1 and 0, which are remembered by the "on" and "off" positions of tiny switches, or "bits", on the circuit boards. Quantum computers use atoms and subatomic particles as the switches that perform the memory and processing tasks.

The difference is that in quantum computing, the switches can be "on" and "off" at the same time. This means the basic component, the "qubit", can be involved in multiple calculations, while its strange properties also allow such computers to skip the step-by-step operations that current PCs use.
Superposition?

Classical waves can exhibit superposition too!
No-Cloning and No-Deleting

Theorem: There are no unitary operations $D$ such that

$$D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$$
$$D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no unitary operations $E$ such that

$$E : |\psi\rangle \mapsto |0\rangle$$
$$E : |\phi\rangle \mapsto |0\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]
No-Cloning and No-Deleting

Linear types have been proposed* to capture this:

\[
!A \quad A \otimes B
\]

Separate classical and quantum data in a hybrid machine

But we need additional axioms to get the QM-like behaviour


** Hensinger et al Nature 2005
Non-locality or Contextuality?

* Aspect et al
PRL 1981
Non-locality or Contextuality?

But!

“generalised probabilistic theories” display even stronger non-locality than quantum theory!

Many aspects of contextuality appear in classical settings like databases and NLP!

* Aspect et al
PRL 1981
Complementarity

Quantum systems have properties which are not accessible at the same time. They are not even simultaneously well-defined!
Quantum Theory

Categorical Quantum Theory

Hilbert spaces
Operators

Monoidal categories
(co)Algebras
Commutation rules
Categorical Quantum Theory

- Monoidal categories
- (co)Algebras
- Commutation rules

CQT as an "algebraic theory"

- PROPs
- Distributive laws
Categorical Quantum Theory

Quantum theory as an *internal* theory in a monoidal category

- No assumption of linear structure
- Algebra: \[ A \otimes A \rightarrow A \]
- Coalgebra: \[ A \rightarrow A \otimes A \]
- And laws for their interaction.
1. Monoidal Categories
1. Symmetric Monoidal Categories

A monoidal category $M$ is a category with a bifunctor, $\otimes$ or $\Box$, 

$$\Box : M \times M \to M$$

written for objects $a, b$ of $M$ variously as a “product” 

$$(a, b) \to a \Box b, a \otimes b, \text{or } ab$$

which is associative up to a natural isomorphism 

$$\alpha : a(bc) \cong (ab)c \tag{1}$$

and is equipped with an element $e$, which is unit up to natural isomorphisms 

$$\lambda : ea \cong a, \quad \rho : ae \cong e. \tag{2}$$

These maps must satisfy certain commutativity requirements; for $\alpha$, a pentagonal diagram 

$$\begin{array}{c}
    a(b(c d)) \xrightarrow{\alpha} (ab)(cd) \xrightarrow{\alpha} ((ab)c)d \\
    1_{\alpha} \downarrow \quad \downarrow 1_{\alpha}
\end{array} \xrightarrow{\alpha} \quad a((bc)d) \xrightarrow{\alpha} (a(bc))d , \tag{3}$$

as in §VII.1.(5), and for $\lambda$ and $\rho$ the two commutativities 

$$\begin{array}{c}
    a(ec) \xrightarrow{\alpha} (ae)c \\
    1_{\lambda} \downarrow \quad \downarrow 1_{\rho}
\end{array} \xrightarrow{\lambda = \rho : ee \to e} \tag{4}$$

A braiding for a monoidal category $M$ consists of a family of isomorphisms 

$$\gamma_{a,b} : a \Box b \cong b \Box a \tag{5}$$

natural in $a$ and $b \in M$, which satisfy for $e$ the commutativity 

$$\begin{array}{ccc}
    a \Box e & \xrightarrow{\gamma} & e \Box a \\
    \rho \downarrow & \Downarrow & \downarrow \lambda \\
    a & = & a
\end{array} \tag{6}$$

and which, with the associativity $\alpha$, make both the following hexagonal diagrams commute (with the symbol $\Box$ omitted): 

$$\begin{array}{cccccc}
    (ab)c & \xrightarrow{\gamma} & c(ab) & a(bc) & \xrightarrow{\gamma} & (bc)a \\
    \downarrow 1_{\gamma} & \quad & \downarrow \alpha & \quad & \downarrow \alpha & \quad & \downarrow 1_{\gamma} \\
    a(bc) & a(cb) & (ab)c & b(ca) & b(ca) \tag{7}
\end{array}$$

Note that the first diagram replaces each $\gamma_{a,b,c}$ which has a product $ab$ as first index by two $\gamma$'s with single indices, while the second hexagonal diagram does the same for $\gamma_{a,b,c}$ with a product as second index. Note also that the first hexagon of (7) for $\gamma$ implies the second diagram for $\gamma^{-1}$, and conversely. Thus, when $\gamma$ is a braiding for $M$, then $\gamma^{-1}$ is also a braiding for $M$.

A symmetric monoidal category, as already defined in §VII. 7, is a category with a braiding $\gamma$ such that every diagram 

$$\begin{array}{c}
    ab \xrightarrow{\gamma_{ab}} ba \\
    \downarrow \gamma_{b,a} \downarrow \\
    ab
\end{array} \tag{8}$$

commutes. For this case, either one of the hexagons (7) implies the other.
Monoidal Categories (Graphically)
Why Diagrams?
Why Diagrams?

\[
\frac{1}{2} \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\gamma} \end{pmatrix} \right) \otimes \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \\
\circ \left( \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right) \circ \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \right) \\
\otimes \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right) \otimes \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{pmatrix} \right)
\]

= ?
Why Diagrams?

• Great when we have parallel and sequential composition
• Essential for talking about interacting algebraic and coalgebraic things
• Different kinds of diagram give different kinds of monoidal category
Historical Examples

Circuits (classical and quantum) :
Historical Examples

Feynmann Diagrams:
Historical Examples

“Applications of Negative Dimensional Tensors”
(Penrose)
Historical Examples

Proofs-Nets in Linear Logic
(Girard, Danos-Regnier, Tortoro de Falco, many others)

For example, the proof net

A B
\( A \otimes B \) C
\( (A \otimes B) \otimes C \)

\( C \perp \)

\( A \perp \otimes B \perp \)
\( A \otimes B \)
\( B \perp \)
\( A \perp \)

reduces (in three steps) to

A B
\( A \otimes B \) C
\( (A \otimes B) \otimes C \)

\( C \perp \)

\( B \perp \)
\( A \perp \)
Historical Examples

Proofs-Nets in Linear Logic
(Blute-Cocket-Seely-Trimble, 1996)
Historical Examples

Interaction Combinators

(Lafont, 1997)
Historical Examples

“Coherence for Compact Closed Categories”
(Kelly-Laplaza, 1980)
Historical Examples

“Coherence for Compact Closed Categories”
(Kelly-Laplaza, 1980)

Coherence for compact closed categories

By an involution \( \theta \) we mean a category which is a coproduct of copies of the arrow-category \( \mathbb{2} \); these are really the fixed-point-free involutions, but we need no others. Such an involution is a special kind of order on its object-set \( |P| \); and, as with orders in general, we do not distinguish two involutions on \( |P| \) which differ only in the “names” of their maps. This object-set \( |P| \) becomes signed when we attribute \( - \) to the source and \( + \) to the target of each arrow \( \mathbb{2} \); we call \( \theta \) an involution on the signed set \( P \) when this signing agrees with that of \( P \).

By a loop \( L \) we mean the free category on a graph of the form

\[
A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n \rightarrow A_1,
\]

for some \( n \geq 1 \). The composite represented by this string determines a canonical element \( \langle L \rangle \) of the set \( [L] \) of cycles of \( L \), independently of where the loop is started.

We use \( + \) for the coproduct of categories, and henceforth suppose all signed sets and involutions to be finite. If \( \theta \) is an involution on the signed set \( P^* \otimes Q \) and \( \phi \) is one on \( Q^* \otimes R \), consider the pushout \( \theta + \phi \) of \( \phi \) in \( \text{Cat} \), obtained from the coproduct \( \theta + \phi \) by identifying the two copies of the discrete category \( |Q| \), one in \( \theta \) and one in \( \phi \). Write \( \phi \theta \) for the full subcategory of \( \theta + Q \) \( \phi \) determined by the object-set \( |P| + |R| \); it is clearly an involution on \( P^* \otimes R \); moreover we clearly have \( \theta + Q \) \( \phi = \phi \theta + \phi \phi \theta \), where \( \phi \phi \theta \) is a coproduct \( \sum L_i \) of loops with objects in \( |Q| \). If now \( \psi \) is a third involution on \( R^* \otimes S \), it is further immediate that
Historical Examples

“Geometry of Tensor Calculus I”
(Joyal-Street, 1991)
Historical Examples

“Traced Monoidal Categories”
(Joyal-Street-Verity, 1996)
More Recently:

Ghica (2013) - Asynchronous Circuits

Unit: \((U \otimes W); X = (W \otimes U); X = W\).

\[ \begin{array}{c}
\text{X} \\
\end{array} \leftrightarrow \begin{array}{c}
\text{X}
\end{array} \]

Commutativity: \(\gamma_A; X = X\).

\[ \begin{array}{c}
\text{X} \\
\end{array} \leftrightarrow \begin{array}{c}
\text{X}
\end{array} \]

Retract: \(T; X = W\).

\[ \begin{array}{c}
\text{T} \\
\end{array} \leftrightarrow \begin{array}{c}
\text{X}
\end{array} \]

\((A, C, P)\) is a commutative monoid with \(U\) an absorbing element.

Associativity: \((W \otimes C); C = (C \otimes W); C\).

\[ \begin{array}{c}
\text{C} \\
\end{array} \leftrightarrow \begin{array}{c}
\text{C}
\end{array} \]

Unit: \((P \otimes W); C = (W \otimes P); C = W\).

\[ \begin{array}{c}
P \\
\end{array} \leftrightarrow \begin{array}{c}
\text{C}
\end{array} \]
More Recently:

Mellïès (2014) - Local State

Grov Kissinger Lin (2013) - Proof Planning
More Recently:

Baez and Fong (2014) — Passive Linear Circuits
More Recently:

Bonchi Sobocinski Zanasi (2014) — Signal Flow Graphs
$j : A \otimes B \rightarrow C \otimes D \otimes E$
Input Systems

\[ j : A \otimes B \rightarrow C \otimes D \otimes E \]
Diagrams

\[ j : A \otimes B \rightarrow C \otimes D \otimes E \]
Diagrams

\[ j : A \otimes B \rightarrow C \otimes D \otimes E \]
Monoidal Categories

\[ f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D \]
Monoidal Categories

\[ f : A \to B \quad g : B \to C \quad h : C \to D \]

\[ g \circ f : A \to C \]
Monoidal Categories

\[ f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D \]

\[ g \circ f : A \rightarrow C \]

\[ f \otimes h : A \otimes C \rightarrow B \otimes D \]
Monoidal Categories
Monoidal Categories

Monoidal categories have a special unit object called $I$ which is a left and right identity for the tensor:

$$I \otimes A = A = A \otimes I$$

$$\text{id}_I \otimes f = f = f \otimes \text{id}_I$$

No lines are drawn for $I$ in the graphical notation:

$$\psi : I \rightarrow A \quad \phi^\dagger : A \rightarrow I \quad \phi^\dagger \circ \psi : I \rightarrow I$$
Categories

\[ \text{id}_A : A \rightarrow A \]
Categories

\[ f \circ \text{id}_A : A \to B \]
Categories

$$\text{id}_B \circ f : A \rightarrow B$$
Categories

\[ f : A \to B \]
Symmetric Monoidal Categories

\[ \sigma_{A,B} : A \otimes B \rightarrow B \otimes A \]
Symmetric Monoidal Categories

\[ \sigma_{A,B} : A \otimes B \rightarrow B \otimes A \]
Symmetric Monoidal Categories

$\sigma_{A,B} : A \otimes B \rightarrow B \otimes A$
Compact Closure

A symmetric monoidal category is called \textit{compact} if, for every object $A$, there exists a \textit{dual} object $A^*$ and arrows:

$$d : I \to A^* \otimes A \quad e : A \otimes A^* \to I$$
Compact Closure

A symmetric monoidal category is called *compact* if, for every object $A$, there exists a *dual* object $A^*$ and arrows:

\[
d : I \rightarrow A^* \otimes A \quad \quad e : A \otimes A^* \rightarrow I
\]
Compact Closure

A symmetric monoidal category is called compact if, for every object $A$, there exists a dual object $A^*$ and arrows:

$$d : I \rightarrow A^* \otimes A \quad e : A \otimes A^* \rightarrow I$$
A category is a $\dagger$-category if it is equipped with an involutive functor, $(\cdot)\dagger$ which reverses the arrows while leaving the objects unchanged.

\[ f : A \to B \quad f\dagger : B \to A \]
An arrow \( f : A \rightarrow B \) is called \textit{unitary} when:

\[
\begin{align*}
\overset{\text{†}}{f : A \rightarrow B} & \Rightarrow (f \circ \overset{\text{†}}{g}) \overset{\text{†}}{=} f \overset{\text{†}}{g} \\
A \overset{f}{\rightarrow} A & \Rightarrow B \overset{f}{\rightarrow} B
\end{align*}
\]
An arrow $f : A \rightarrow B$ is called \textit{unitary} when:
An arrow $f : A \to B$ is called unitary when:

\[
\begin{align*}
\tau \circ (f \circ g) &= (\tau \circ f) \circ \tau \\
A, B &= B, A
\end{align*}
\]
Thm: one diagram can be deformed to another iff their denotations are equal by the structural equations of the category.
**Theorem**: one diagram can be deformed to another iff their denotations are equal by the structural equations of the category.
The Category $\text{FDHilb}$

$\text{FDHilb}$ is the category of finite dimensional complex Hilbert spaces. It is $\dagger$-monoidal with the following structure.

- Objects: finite dimensional Hilbert spaces, $A$, $B$, $C$ etc
- Arrows: all linear maps
- Tensor: usual (Kronecker) tensor product; $I = \mathbb{C}$
- $f^\dagger$ is the usual adjoint (conjugate transpose)

A linear map $\psi : I \to A$ picks out exactly one vector. It is a ket and $\psi^\dagger : A \to I$ is the corresponding bra.

Hence $\psi^\dagger \circ \phi : I \to I$ is the inner product $\langle \psi \mid \phi \rangle$. 
2.
Quantum Theory in 5 minutes
2. Quantum Theory in 5* minutes

*Might be more than 5
1. Quantum states are represented by **unit vectors** in a complex Hilbert space.

\[ |0\rangle, |1\rangle, \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \in \mathbb{C}^2 =: Q \]
2. The state space formed by combining two or more systems is the **tensor** product of their individual state spaces.

\[ |010\rangle := |0\rangle \otimes |1\rangle \otimes |0\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = Q^3 \]
3. For each discrete time step, an undisturbed quantum system evolves according to a **unitary** operator acting on its state space

\[
X, Z, H : Q \rightarrow Q \\
\wedge X, \wedge Z : Q^2 \rightarrow Q^2
\]

**but** the quantum state is not directly accessible... more on this later!
Example: Z-Rotation

\[
Z_\beta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix}
\]
Example: X-Rotation

\[ X_\alpha = \begin{pmatrix} \cos \frac{\alpha}{2} & -i \sin \frac{\alpha}{2} \\ -i \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \]
Quantum Circuits

Unitary gates

“Time”

$|\Psi\rangle$ Input register

$U : \mathbb{C}^{2^n} \rightarrow \mathbb{C}^{2^n}$

$|\Psi'\rangle$ Output register
Universality

We say that a model of quantum computation is \textit{universal} if it can represent all unitary maps.

The circuit model requires a small set of gates to be universal:

\[
\begin{align*}
\begin{bmatrix} Z_\beta \end{bmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \\
\begin{bmatrix} H \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
\end{align*}
\]
We say that a model of quantum computation is universal if it can represent all unitary maps.

The circuit model requires a small set of gates to be universal:

\[
\begin{align*}
Z_\beta &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \\
X_\alpha &= \begin{pmatrix} \cos \frac{\alpha}{2} & -i \sin \frac{\alpha}{2} \\ -i \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \\
\text{hadamard} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]
While the preceding has demonstrated the ease with which quantum circuits can be translated into diagrams, there are many diagrams which do not correspond to circuit elements.

3.2 Circuit-like diagrams

The corresponding circuit is shown below:

Quantum Circuits

Preparing a Bell state

Controlled-Z gate
Quantum Observables

4. Each observable quantity $O$ is represented as self-adjoint operator $\hat{O} = \sum_i \lambda_i |e_i\rangle \langle e_i|:

- The possible values of $O$ are the eigenvalues $\lambda_i$ of $\hat{O}$.
- When we observe $O$ for a system in state $|\psi\rangle$ there is probability $\langle e_i | \psi \rangle^2$ of observing $\lambda_i$.
- If $\lambda_i$ is the outcome of the measurement, the system is then in state $|e_i\rangle$. 
X and Z Spins

We can measure the spin of qubit $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\alpha |0\rangle + \beta |1\rangle$

$p = \alpha^2 \quad \quad p = \beta^2 \quad \quad p = (\alpha + \beta)/2^2 \quad \quad p = (\alpha - \beta)/2^2$

$|0\rangle \quad |1\rangle \quad |+\rangle \quad |-\rangle$
We can measure the spin of qubit $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
Quantum Observables

Measuring $A$ then $B$ may give a different answer than measuring $B$ first then $A$!

Not all observables are well defined at the same time:

- Two observables are compatible if their operators commute
- Two operators commute if they have the same eigenvectors
- Identify a non-degenerate observable with its eigenbasis
Incompatible Observables
Complementary Observables

\[ |\langle a_i | b_j \rangle| = \frac{1}{\sqrt{D}} \]
Complementary Observables

Mutually Unbiased Bases

\[ |\langle a_i | b_j \rangle| = \frac{1}{\sqrt{D}} \]
3. Categorical Quantum Mechanics
No-Cloning and No-Deleting

Theorem: There are no unitary operations $D$ such that

$D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$

$D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no unitary operations $E$ such that

$E : |\psi\rangle \mapsto |0\rangle$

$E : |\phi\rangle \mapsto |0\rangle$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]
“Classical” Quantum States

When can a quantum state be treated as if classical?

- no-go theorems allow copying and deleting of orthogonal states;

In other words:

- A quantum state may be copied and deleted if it is an eigenstate of some known observable.

We’ll use this property to formalise observables in terms of copying and deleting operations.
Observables

\[ \delta = \quad \epsilon = \]

Comonoid Laws

\[ \quad = \quad = \quad = \]
Observables

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Comonoid Laws

\[ = \quad = \quad = \quad = \]

\[ = \quad = \quad = \]
Observables

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Monoid Laws
Observables

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Isometry Law

Frobenius Law
Observables

In other words: an observable is a special commutative $\dagger$-Frobenius algebra
Observables

Given any finite dimensional Hilbert space we can define a Frobenius algebra by

\[
\delta : A \to A \otimes A :: a_i \mapsto a_i \otimes a_i \\
\epsilon : A \to I :: \sum_i a_i \mapsto 1
\]
Observables

Given any finite dimensional Hilbert space we can define a Frobenius algebra by

\[ \delta : A \rightarrow A \otimes A :: a_i \mapsto a_i \otimes a_i \]
\[ \epsilon : A \rightarrow I :: \sum_i a_i \mapsto 1 \]

Example:
\[ \delta : \begin{align*}
|0\rangle & \mapsto |00\rangle \\
|1\rangle & \mapsto |11\rangle
\end{align*} \]
\[ \epsilon : |0\rangle + |1\rangle \mapsto 1 \]

define a Frobenius algebra over qubits; the standard basis is copied and erased.
Observables

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\end{align*} \quad \epsilon : |0\rangle + |1\rangle \mapsto 1
\]

define a Frobenius algebra over qubits; the standard basis is copied and erased.

Theorem: in \textbf{FDHilb}, \dagger-SCFAs are in bijective correspondence to bases. [Coecke, Pavlovic, Vicary]
Spider Theorem

Theorem: any maps constructed from $\delta$ and $\varepsilon$, and their adjoints, whose graph is connected, is determined uniquely by the number of inputs and outputs.

Coecke & Paquette 2006
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Coecke & Paquette 2006
Theorem: any maps constructed from $\delta$ and $\varepsilon$, and their adjoints, whose graph is connected, is determined uniquely by the number of inputs and outputs.
Phases

**Defn.** a map $\alpha:A \to A$ is called a *pre-phase* if:

\[
\begin{array}{c}
\alpha \\
\downarrow \\
\alpha
\end{array}
= 
\begin{array}{c}
\alpha \\
\downarrow \\
\alpha
\end{array}
\]

A pre-phase is a *phase* if it is unitary.

**Defn:** a point $\psi:I \to A$ is called *unbiased* if the map

\[
\begin{array}{c}
\psi \\
\downarrow \\
\psi\quad \leftrightarrow \quad \psi
\end{array}
\]

is a phase.

\[\Lambda(\psi) : \psi \mapsto \mu \circ (\psi \otimes \text{id})\]
Phases

Lemma. Let $\alpha: A \rightarrow A$ be a phase; then there exists an unbiased point $\psi: I \rightarrow A$ such that:

1. $\alpha = \Lambda(\psi)$;
2. $\alpha^\bullet = \alpha$;
3. $\alpha^\dagger = \Lambda(\psi^\bullet)$;
4. $\mu(\psi \otimes \psi^\bullet) = \eta$.

Corollary: $\alpha$ is a phase iff $\alpha^\dagger$ is a phase.
Phases

**Lemma.** Let \( \alpha : A \rightarrow A \) be a phase; then there exists an unbiased point \( \psi : I \rightarrow A \) such that:

**Proposition 6:**
1. The phases are an abelian group
2. The unbiased points are an abelian group
3. They’re isomorphic

\[
\begin{align*}
3. \quad & \alpha^\dagger = \Lambda(\psi) \\
4. \quad & \mu(\psi \otimes \psi) = \eta.
\end{align*}
\]

**Corollary:** \( \alpha \) is a phase iff \( \alpha^\dagger \) is a phase.
Generalised Spider

**Thm:** Any connected monochrome diagram with phases is determined completely by its arity and the sum of its phases.
Example: qubits
Example: qubits
Example: qubits

Unbiased points

$|0 \rangle + e^{i\alpha} |1 \rangle$
Example: qubits

Unbiased points
Example: qubits

\[
\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}
\]

Unbiased points
Two kinds of points

$\delta = \text{Diagram of point 1}$

$\epsilon = \text{Diagram of point 2}$

$\delta^\dagger = \text{Diagram of point 3}$

$\epsilon^\dagger = \text{Diagram of point 4}$
Two kinds of points

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Classical Points

Those points which can be copied by \( \delta \)
Two kinds of points

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Classical Points

Those points which can be copied by \( \delta \)
Two kinds of points

$\delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \quad$

Classical Points

Those points which can be copied by $\delta$

* to keep the pictures tidy, a scalar factor has been omitted (and everywhere from here on)
Two kinds of points

\[ \delta = \quad \epsilon = \quad \delta^\dagger = \quad \epsilon^\dagger = \]

Classical Points

Those points which can be copied by \( \delta \)

Unbiased Points
Two kinds of points

\[ \delta = \quad \epsilon = \]

\[ \delta^\dagger = \quad \epsilon^\dagger = \]

**Classical Points**

Those points which can be copied by \( \delta \)

**Unbiased Points**
Two kinds of points

$\delta = \begin{array}{c}
\text{Classical Points}
\\text{Those points which can be copied by } \delta
\end{array}$

$\epsilon = \begin{array}{c}
\text{Unbiased Points}
\end{array}$
Classical points are eigenvectors
Classical points are eigenvectors
Classical points are eigenvectors
Complementary Observables

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\[ |i\rangle \leftrightarrow |ii\rangle \quad |+\rangle \leftrightarrow 1 \quad |\pm\rangle \leftrightarrow |\pm\pm\rangle \quad |0\rangle \leftrightarrow 1 \]
Complementary Observables

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\[ \begin{array}{c}
\begin{array}{c}
\delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \\
\end{array}
\end{array} \]
Complementary Observables

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\[
\begin{align*}
\delta_Z = & \quad \epsilon_Z = \\
\delta_X = & \quad \epsilon_X =
\end{align*}
\]
Complementary Observables

$\delta_z = \begin{array}{c} \Delta \end{array}$ $\epsilon_z = \begin{array}{c} \Delta \end{array}$ $\delta_x = \begin{array}{c} \Delta \end{array}$ $\epsilon_x = \begin{array}{c} \Delta \end{array}$
Example: qubits

Classical points

Unbiased points

π
Example: qubits

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]

Unbiased points

Classical points

\[ |0\rangle = \pi \]
\[ |1\rangle = \pi \]
Example: qubits

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]

Unbiased points

Classical points

\[ |0\rangle = \begin{array}{l} \text{Red dot} \\ \pi \end{array} \]

\[ |1\rangle = \begin{array}{l} \text{Red dot} \\ \pi \end{array} \]
Example: qubits

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]

\[ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

Classical points

Unbiased points
Example: qubits

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \]

\[ \pi \]

Classical points

\[ |0\rangle = \pi \]

\[ |1\rangle = \pi \]

Unbiased points

\[ \alpha = \begin{pmatrix} \cos \frac{\alpha}{2} & i \sin \frac{\alpha}{2} \\ i \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \]

\[ \pi \]

\[ |+\rangle = \pi \]

\[ |-\rangle = \pi \]
Strong Complementarity

A pair of complementary observables are called *strongly complementary* when they satisfy this equation:

\[
\delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X =
\]

A pair of complementary observables are called *strongly complementary* when they satisfy this equation:
Strong Complementarity

**Corollary:** strongly complementary observables form a bialgebra
Strong Complementarity

Corollary: strongly complementary observables form a bialgebra
The antipode

Define the map $S$ by:

\[
\delta_Z = \begin{array}{c}
\text{Diagram}
\end{array} \quad \epsilon_Z = \begin{array}{c}
\text{Diagram}
\end{array} \quad \delta_X = \begin{array}{c}
\text{Diagram}
\end{array} \quad \epsilon_X = \begin{array}{c}
\text{Diagram}
\end{array}
\]

\[
S = \begin{array}{c}
\text{Diagram}
\end{array} := \begin{array}{c}
\text{Diagram}
\end{array}
\]
The antipode

* If we have “enough classical points” this can be proved without strong complementarity
Strongly complementary observables form Hopf algebras

Theorem:

\[ \begin{align*}
\text{Diagram 1} & \quad = \quad \text{Diagram 2} \\
\text{Diagram 3} & \quad = \quad \text{Diagram 4}
\end{align*} \]
Strong Complementarity

The following are equivalent characterisations of strong complementarity:
Comonoid Homomorphism

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

The classical maps are comonoid homomorphisms
Comonoid Homomorphism

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

The classical maps are comonoid homomorphisms.
The classical maps satisfy canonical commutation relations
Closedness Property

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\[ \text{[Diagram with nodes and edges]} \]
Closedness Property

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\[ i = j = \quad i \circ j = \quad i \circ j = \quad i \circ j = \]
Closedness Property

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]

\( i \) \quad \( j \) \quad \( i \odot j \) \quad \( i \odot j \) \quad \( i \odot j \)

\( i \) \quad \( i \) \quad \( i \) \quad \( i \) \quad \( i \odot j \) \quad \( i \odot j \) \quad \( i \odot j \)
Closedness Property

\[ \delta_Z = \quad \epsilon_Z = \quad \delta_X = \quad \epsilon_X = \]
Theorem:
In $\text{fdHilb}$, strongly complementary observables exist for every dimension.

... but in big enough dimension, it is possible to construct MUBs which are not closed.
“Interacting Frobenius Algebras are Hopf”

RD + Kevin Dunne
4. The ZX-calculus
**ZX-calculus**

**Good Points:**
- Graphical language for reasoning about QC
- Universal
- Derived from the basic algebra of complementarity
- Powerful algebraic theory
  + **Complete!**

**Bad point:**
- Need to impose operational meaning post-hoc
ZX-calculus syntax

\[ \alpha \in [0, 2\pi) \]

**Defn:** A diagram is an undirected open graph generated by the above vertices.
Example: CNOTS
Example: CNOTS
Equational Reasoning
Equational Reasoning
Equational Reasoning

\[
\begin{align*}
\alpha \quad \equiv \quad \alpha \\
\beta \quad \equiv \quad \beta \\
\text{(hopf)}
\end{align*}
\]
Equational Reasoning
Equational Reasoning
Equational Reasoning
Equational Reasoning
Equational Reasoning
Advertising

Graphical tool for doing graphical calculations:

http://quantomatic.github.io
Application: QECC

ZX-calculus can demonstrate the correctness Quantum Error Correcting Codes:
5. Compiling.

Oh look category theory can do something useful.
Circuit Perspective
Circuit Perspective

Inputs

Outputs
Circuit Perspective
Circuit Perspective

\[ \delta \alpha \beta \gamma \]
???: Perspective

\[\delta \quad \gamma \quad \alpha \quad \beta\]
??? Perspective

\[ \alpha, \beta, \gamma, \delta \]
Hopf algebra expression
??? Perspective

Hopf algebra
normal form
MBQC Perspective
MBQC Perspective

Physical qubits
Prepared qubits
MBQC Perspective

Prepared qubits

Measured qubits
Any ZX-calculus term can be interpreted as an MBQC in this way.
NQIT Perspective

Diagram with nodes labeled $\delta$, $\gamma$, $\alpha$, and $\beta$. The diagram shows connections between these nodes.
NQIT Perspective
Few qubit ion traps
NQIT Perspective

Optical interconnect

Few qubit ion traps
NQIT Perspective

Optical interconnect

Few qubit ion traps
SUMMARY

• Quantum computing needs a different logico-algebraic basis to classical computing.
• Categorical analysis of concepts of QM reveals the key structures behind algorithms etc.
• Diagrammatic syntax and DPO rewriting give tools to work efficiently with this theory.
• New structural insights give new approaches and techniques to working with quantum computers.