

Polygon-circle and word-representable graphs

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Abstract

We describe work on the relationship between the independently-studied polygon-circle graphs and word-representable graphs.

A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that letters x and y form a subword of the form $xyxy \cdots$ or $yxyx \cdots$ iff xy is an edge in E . Word-representable graphs generalise several well-known and well-studied classes of graphs [2,3]. It is known that any word-representable graph is k -word-representable, that is, can be represented by a word having exactly k copies of each letter for some k dependent on the graph. Recognising whether a graph is word-representable is NP-complete ([3, Theorem 4.2.15]). A *polygon-circle graph* (also known as a *spider graph*) is the intersection graph of a set of polygons inscribed in a circle [4]. That is, two vertices of a graph are adjacent if their respective polygons have a non-empty intersection, and the set of polygons that correspond to vertices in this way are said to *represent* the graph. Recognising whether an input graph is a polygon-circle graph is NP-complete [5]. We show that neither of these two classes is included in the other one by showing that the word-representable Petersen graph and crown graphs are not polygon-circle, while the non-word-representable wheel graph W_5 is polygon-circle. We also provide a more

refined result showing that for any $k \geq 3$, there are k -word-representable graphs which are neither $(k - 1)$ -word-representable nor polygon-circle.

Keywords: polygon-circle graph, word-representable graph, Petersen graph

1 Introduction

Polygon-circle graphs are the intersection graphs of polygons inscribed in a circle, and are one of many well-studied classes of intersection graphs. While recognising whether a graph is a polygon-circle graph is NP-complete, many problems that are NP-Complete in general can be resolved in polynomial time given a polygon-circle representation of a graph [5,6]. This algorithmic advantage, as well as the position of the polygon-circle graphs in the broad hierarchy of intersection classes, has been the focus of much of the study on this class. Polygon-circle graphs can also be defined in terms of a string that has a direct correspondence to a polygon-circle representation.

Word-representable graphs are defined primarily in terms of a string representation: we say that two letters x, y *alternate* in a string S if, when we delete all letters other than x and y , the string is either of the form $xyxy\dots$ or $yxyx\dots$, of either even or odd length. Word-representable graphs can be characterised by a *semi-transitive orientation* of their edges (as described in [2], not to be defined here), and generalise the *comparability graphs* (which are graphs admitting transitive orientations), and the *3-colourable graphs*.

It is often of interest to consider polygon-circle graphs of limited polygons, or word-representable graphs of limited numbers of copies of each letter: if a graph is representable by polygons of at most k corners inscribed in a circle, then it is a *k-polygon-circle graph*, and if a graph is word-representable by a string in which each letter appears at most k times, then it is *k-word-representable*. We note that originally k -word-representable graphs are defined as graphs admitting representation by a string in which each letter appears *exactly* k times, but “exactly” here is equivalent to “at most”, and in fact for any word-representable graph G there exists k such that G is k -word-representable [2,3].

2-polygon-circle graphs are the intersection graphs of chords in a circle, and are also known as *circle graphs*. These coincide exactly with the

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2-word-representable graphs, and so 2-polygon-circle graphs are exactly 2-word-representable graphs [2]. Beyond this result, little further was known on interrelations between these two classes of graphs. In this note, we outline progress in understanding the relationship between polygon-circle and word-representable graphs, showing that neither is a subset of the other.

2 Definitions and preliminaries

A *polygon-circle representation* of a graph $G = (V, E)$ is a mapping \mathcal{P} of vertices in V to polygons in a circle such that $(u, v) \in E$ if and only if $\mathcal{P}(u)$ intersects $\mathcal{P}(v)$. Sometimes for convenience we will use subscripting, as in P_i , to denote the polygon $\mathcal{P}(v_i)$. A *corner-string* of a polygon-circle representation is a string produced by starting at an arbitrary point on the circle that hosts the representation, and proceeding around the circle, adding a letter denoting the vertex represented by a polygon each time a corner of a polygon encountered (example in Figure 1). Note that a single polygon-circle representation has many possible corner-strings, depending on the starting point. It is known [6] that two polygons representing vertices u and v intersect if and only if, when we delete all corners other than u and v from the corner-string, the result has as a substring either $uvuv$ or $vuvu$, in which case we say that u and v *loosely alternate*, and that the segments *overlap*. Note that this loose alternation is less strict than the notion of alternation used to define word-representable graphs: in the string $xyxxyx$ the letters x and y loosely alternate, but they do not alternate in the word-representation sense because there are two copies of x between two consecutive copies of y .

If a vertex v_i is represented by a polygon P_i in a polygon-circle representation that gives a corner-string \mathcal{S} , then the *segment* of u (denoted $s(u)$) in \mathcal{S} is the minimal contiguous substring of \mathcal{S} that contains all occurrences of u .

We say that a segment $s(u)$ is *contained in* $s(v)$ (denoted $s(u) \subset s(v)$) if $s(u)$ is entirely between two consecutive occurrences of v . Contrast this to the idea of a subsegment: we say $s(u)$ is a *subsegment* of $s(v)$ if $s(u)$ is a substring of $s(v)$. Segment containment is transitive and antisymmetric. Given a corner-string \mathcal{S} , we say that a segment s is *outermost* if it is not contained in any other segment. A *path* from vertex a to b is a sequence of vertices starting with a and ending with b such that each vertex is adjacent to the one before it in the sequence.

We use several relatively straightforward lemmas (proofs omitted for space):

Lemma 2.1 *If two segments intersect but their corresponding polygons do*

not, then one is contained in the other.

Lemma 2.2 *If $s(u) \subset s(v)$ and $s(v) \subset s(w)$, then all paths from u to w must pass through the neighbourhood of v .*

Lemma 2.3 *In a corner-string from a polygon-circle representation of a chordless cycle on ≥ 4 vertices at least one vertex's segment contains another.*

2.1 Word-representable graphs

The *wheel graph* W_n is the graph on $n + 1$ vertices obtained from the cycle graph C_n by adding an all-adjacent vertex.

Lemma 2.4 ([3]) *Wheel graphs W_{2n+1} are non-word-representable for $n \geq 2$.*

Lemma 2.5 ([2,3]) *The Petersen graph (shown in Figure 2) is 3-word-representable but not 2-word-representable.*

A *crown graph* $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. See Figure 2 for $H_{4,4}$.

Lemma 2.6 ([1]) *For $n \geq 5$, the crown graph $H_{n,n}$ is $\lceil n/2 \rceil$ -word-representable but not $(\lceil n/2 \rceil - 1)$ -word-representable.*

3 Polygon-circle graphs vs word-representable graphs

In this section we will show that the word-representable Petersen graph and crown graphs are not polygon-circle, while the non-word-representable wheel graph W_5 is polygon-circle. We also provide a more refined result showing that for any $k \geq 3$, there are k -word-representable graphs which are neither $(k - 1)$ -word-representable nor polygon-circle.

Lemma 3.1 *The wheel graph W_n is a polygon-circle graph for $n \geq 3$.*

Proof. Essentially, for a wheel W_n we represent the vertices on the large outer cycle $v_1 \dots v_n$ with chords on the circle. We can place their corners starting at an arbitrary point on the circle in the sequence $v_0, v_n, v_1, v_0, v_2, v_1, v_3, \dots, v_{n-1}, v_n$. Then the central high-degree vertex is represented by a polygon with corners between the first corner of v_2 and the second corner of v_1 , the first corner of v_4 and the second corner of v_3 , etc - essentially between every second pair of chord corners, giving this polygon representing the central vertex of the wheel on n vertices $\lceil n/2 \rceil$ corners, producing a $\lceil n/2 \rceil$ -polygon-circle representation. An example of this representation for W_5 in particular is given in Figure 1. \square

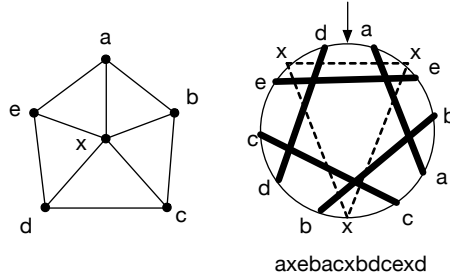


Fig. 1. The wheel W_5 with a polygon-circle representation. All vertices except for x are represented with lines, and the vertex x is represented by the dashed triangle. The corner-string below the representation is produced by starting at the top of the circle (indicated by the arrow) and reading clockwise around the circle.

Theorem 3.2 *There are non-word-representable polygon-circle graphs.*

Proof. This is an immediate corollary of Lemmas 2.4 and 3.1. □

Lemma 3.3 *The Petersen graph is not a polygon-circle graph.*

Proof. We proceed by contradiction. Let \mathcal{S} be a corner-string of a representation of the Petersen graph $P = (V_P, E_P)$, as in Figure 2: we denote the segment corresponding to vertex v as $s(v)$. By Lemma 2 there is at least one segment on each of the inner and outer cycles of five vertices that contains all segments corresponding to non-neighbours. Without loss of generality, let vertex a be a vertex on the outer cycle represented by a segment that is outermost and contains the segments corresponding to d and c .

Let $N[a]$ be the closed neighbourhood of a . We now argue that no segment corresponding to a vertex in $V \setminus N[a]$ can contain any other: let x, y be non-adjacent vertices in $V \setminus N[a]$. Both $s(x) \subset s(a)$ and $s(y) \subset s(a)$. If $s(x) \subset s(y)$, then by Lemma 2.2 every path from x to a must pass through the neighbourhood of y , a contradiction to the fact that every member of $V \setminus N[a]$ has a path to a avoiding the neighbourhood of each other member of $V \setminus N[a]$.

Then no segment representing a vertex in $V \setminus N[a]$ contains any other. However, the graph $P[V \setminus N[a]]$ is a chordless cycle on six vertices, which by Lemma 2 requires containment, a contradiction. □

Theorem 3.4 *The are word-representable non-polygon-circle graphs.*

Proof. This is an immediate corollary of Lemmas 3.3 and 2.5. □

Lemma 3.5 *Crown graphs $H_{n,n}$ for $n \geq 4$ are not polygon-circle graphs.*

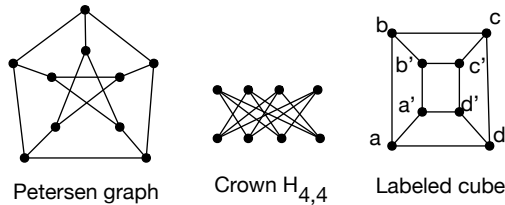


Fig. 2. Three graphs that we show are not polygon-circle graphs, some labeled for convenience. Note that the crown and cube on the right are isomorphic.

Proof. Because $H_{n+1,n+1}$ contains $H_{n,n}$ as an induced subgraph, it suffices to show that $H_{4,4}$ (isomorphic to the cube) is not a polygon-circle graph. It is already known that the cube is not a member of a superclass of the polygon-circle graphs [6]. A direct argument in the polygon-circle setting is also possible, and hinges on the repeated use of Lemma 2 along with the antisymmetry of segment containment to show that representing the many 4-cycles in the cube leads to contradictory constraints. \square

Theorem 3.6 *For every $k \geq 3$, there are k -word-representable graphs that are neither $(k - 1)$ -word-representable nor polygon-circle.*

Proof. The desired result follows from Lemmas 2.6 and 3.5, since $H_{2k,2k}$ is k -word-representable but not $(k - 1)$ -word-representable for $k \geq 3$. \square

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