

# Quadrant marked mesh patterns in alternating permutations

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## Abstract

This paper is continuation of the systematic study of distribution of quadrant marked mesh patterns initiated in [7]. We study quadrant marked mesh patterns on up-down and down-up permutations, also known as alternating and reverse alternating permutations, respectively. In particular, we refine classic enumeration results of André [1, 2] on alternating permutations by showing that the distribution of the quadrant marked mesh pattern of interest is given by  $(\sec(xt))^{1/x}$  on up-down permutations of even length and by  $\int_0^t (\sec(xz))^{1+\frac{1}{x}} dz$  on down-up permutations of odd length.

**Keywords:** permutation statistics, marked mesh pattern, distribution

## 1 Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [4] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns (see [6] for a comprehensive introduction to the theory of permutation patterns). This notion was further studied in [3, 5, 7, 9, 10, 13].

Let  $\sigma = \sigma_1 \dots \sigma_n$  be a permutation in the symmetric group  $S_n$  written in one-line notation. Then we will consider the graph of  $\sigma$ ,  $G(\sigma)$ , to be the set of points  $(i, \sigma_i)$  for  $i = 1, \dots, n$ . For example, the graph of the permutation  $\sigma = 471569283$  is pictured in Figure 1. Then if we draw a coordinate system centered at a point  $(i, \sigma_i)$ , we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as pictured in Figure 1. For any  $a, b, c, d \in \mathbb{N}$  where  $\mathbb{N} = \{0, 1, 2, \dots\}$  is the set of natural numbers and any  $\sigma = \sigma_1 \dots \sigma_n \in S_n$ , we say that  $\sigma_i$  matches the quadrant marked

mesh pattern  $MMP(a, b, c, d)$  in  $\sigma$  if in  $G(\sigma)$  relative to the coordinate system which has the point  $(i, \sigma_i)$  as its origin, there are  $\geq a$  points in quadrant I,  $\geq b$  points in quadrant II,  $\geq c$  points in quadrant III, and  $\geq d$  points in quadrant IV. For example, if  $\sigma = 471569283$ , the point  $\sigma_4 = 5$  matches the quadrant marked mesh pattern  $MMP(2, 1, 2, 1)$  since relative to the coordinate system with origin  $(4, 5)$ , there are 3 points in  $G(\sigma)$  in quadrant I, 1 point in  $G(\sigma)$  in quadrant II, 2 points in  $G(\sigma)$  in quadrant III, and 2 points in  $G(\sigma)$  in quadrant IV. Note that if a coordinate in  $MMP(a, b, c, d)$  is 0, then there is no condition imposed on the points in the corresponding quadrant. In addition, one can consider patterns  $MMP(a, b, c, d)$  where  $a, b, c, d \in \mathbb{N} \cup \{\emptyset\}$ . Here when one of the parameters  $a, b, c,$  or  $d$  in  $MMP(a, b, c, d)$  is the empty set, then for  $\sigma_i$  to match  $MMP(a, b, c, d)$  in  $\sigma = \sigma_1 \dots \sigma_n \in S_n$ , it must be the case that there are no points in  $G(\sigma)$  relative to coordinate system with origin  $(i, \sigma_i)$  in the corresponding quadrant. For example, if  $\sigma = 471569283$ , the point  $\sigma_3 = 1$  matches the marked mesh pattern  $MMP(4, 2, \emptyset, \emptyset)$  since relative to the coordinate system with origin  $(3, 1)$ , there are 6 points in  $G(\sigma)$  in quadrant I, 2 points in  $G(\sigma)$  in quadrant II, no points in  $G(\sigma)$  in quadrant III, and no points in  $G(\sigma)$  in quadrant IV. We let  $\text{mmp}^{(a,b,c,d)}(\sigma)$  denote the number of  $i$  such that  $\sigma_i$  matches the marked mesh pattern  $MMP(a, b, c, d)$  in  $\sigma$ .

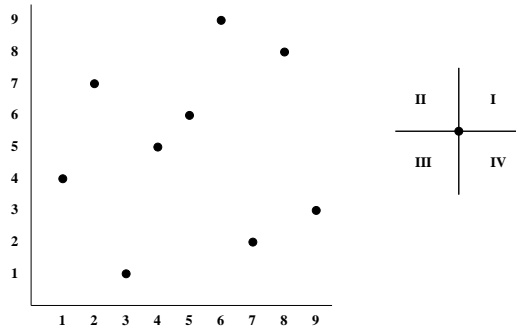


Figure 1: The graph of  $\sigma = 471569283$ .

Note how the (two-dimensional) notation of Úlfarsson [13] for marked mesh patterns corresponds to our (one-line) notation for quadrant marked mesh patterns. For example,

$$\begin{aligned}
 MMP(0, 0, k, 0) &= \begin{array}{c} | \\ \bullet \\ \hline \boxed{k} \end{array}, & MMP(k, 0, 0, 0) &= \begin{array}{c} \boxed{k} \\ \bullet \\ \hline | \end{array}, \\
 MMP(0, a, b, c) &= \begin{array}{c} \boxed{a} \\ \bullet \\ \hline \boxed{b} \quad \boxed{c} \end{array} & \text{and} & MMP(0, 0, \emptyset, k) &= \begin{array}{c} | \\ \bullet \\ \hline \text{shaded} \quad \boxed{k} \end{array}.
 \end{aligned}$$

Kitaev and Remmel [7] studied the distribution of quadrant marked mesh patterns in the symmetric group  $S_n$  and Kitaev, Remmel, and Tiefenbruck [9, 10] studied the distribution of quadrant marked mesh patterns in 132-avoiding permutations in  $S_n$ . The main goal of this paper is to study the distribution of the statistics  $\text{mmp}^{(1,0,0,0)}$ ,  $\text{mmp}^{(0,1,0,0)}$ ,  $\text{mmp}^{(0,0,1,0)}$ , and

$\text{mmp}^{(0,0,0,1)}$  in the set of *up-down* and *down-up permutations*. We say that  $\sigma = \sigma_1 \dots \sigma_n \in S_n$  is an up-down permutation if it is of the form

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \dots ,$$

and  $\sigma$  is a down-up permutation if it is of the form

$$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \sigma_5 > \dots .$$

Let  $UD_n$  denote the set of all up-down permutations in  $S_n$  and  $DU_n$  denote the set of all down-up permutations in  $S_n$ . Given a permutation  $\sigma = \sigma_1 \dots \sigma_n \in S_n$ , we define the reverse of  $\sigma$ ,  $\sigma^r$ , to be  $\sigma_n \sigma_{n-1} \dots \sigma_1$  and the complement of  $\sigma$ ,  $\sigma^c$ , to be  $(n+1-\sigma_1)(n+1-\sigma_2) \dots (n+1-\sigma_n)$ . For  $n \geq 1$ , we let

$$\begin{aligned} A_{2n}(x) &= \sum_{\sigma \in UD_{2n}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}, & B_{2n-1}(x) &= \sum_{\sigma \in UD_{2n-1}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}, \\ C_{2n}(x) &= \sum_{\sigma \in DU_{2n}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}, & \text{and } D_{2n-1}(x) &= \sum_{\sigma \in DU_{2n-1}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}. \end{aligned}$$

We then have the following simple proposition.

**Proposition 1.** *For all  $n \geq 1$ ,*

$$\begin{aligned} (1) \quad A_{2n}(x) &= \sum_{\sigma \in DU_{2n}} x^{\text{mmp}^{(0,1,0,0)}(\sigma)} = \sum_{\sigma \in DU_{2n}} x^{\text{mmp}^{(0,0,0,1)}(\sigma)} = \sum_{\sigma \in UD_{2n}} x^{\text{mmp}^{(0,0,1,0)}(\sigma)}, \\ (2) \quad C_{2n}(x) &= \sum_{\sigma \in UD_{2n}} x^{\text{mmp}^{(0,1,0,0)}(\sigma)} = \sum_{\sigma \in UD_{2n}} x^{\text{mmp}^{(0,0,0,1)}(\sigma)} = \sum_{\sigma \in DU_{2n}} x^{\text{mmp}^{(0,0,1,0)}(\sigma)}, \\ (3) \quad B_{2n-1}(x) &= \sum_{\sigma \in UD_{2n-1}} x^{\text{mmp}^{(0,1,0,0)}(\sigma)} = \sum_{\sigma \in DU_{2n-1}} x^{\text{mmp}^{(0,0,0,1)}(\sigma)} = \sum_{\sigma \in DU_{2n-1}} x^{\text{mmp}^{(0,0,1,0)}(\sigma)}, \\ &\text{and} \\ (4) \quad D_{2n-1}(x) &= \sum_{\sigma \in DU_{2n-1}} x^{\text{mmp}^{(0,1,0,0)}(\sigma)} = \sum_{\sigma \in UD_{2n-1}} x^{\text{mmp}^{(0,0,0,1)}(\sigma)} = \sum_{\sigma \in UD_{2n-1}} x^{\text{mmp}^{(0,0,1,0)}(\sigma)}. \end{aligned}$$

*Proof.* It is easy to see that for any  $\sigma \in S_n$ ,

$$\text{mmp}^{(1,0,0,0)}(\sigma) = \text{mmp}^{(0,1,0,0)}(\sigma^r) = \text{mmp}^{(0,0,0,1)}(\sigma^c) = \text{mmp}^{(0,0,1,0)}((\sigma^r)^c).$$

Then part 1 easily follows since

$$\sigma \in UD_{2n} \iff \sigma^r \in DU_{2n} \iff \sigma^c \in DU_{2n} \iff (\sigma^r)^c \in UD_{2n}.$$

Parts 2, 3, and 4 are proved in a similar manner.  $\square$

It follows from Propostion 1 that the study of the distribution of the statistics  $\text{mmp}^{(1,0,0,0)}$ ,  $\text{mmp}^{(0,1,0,0)}$ ,  $\text{mmp}^{(0,0,1,0)}$ , and  $\text{mmp}^{(0,0,0,1)}$  in the set of up-down and down-up permutations can be reduced to the study of the following generating functions:

$$\begin{aligned} A(t, x) &= 1 + \sum_{n \geq 1} A_{2n}(x) \frac{t^{2n}}{(2n)!}, \\ B(t, x) &= \sum_{n \geq 1} B_{2n-1}(x) \frac{t^{2n-1}}{(2n-1)!}, \\ C(t, x) &= 1 + \sum_{n \geq 1} C_{2n}(x) \frac{t^{2n}}{(2n)!}, \text{ and} \\ D(t, x) &= \sum_{n \geq 1} D_{2n-1}(x) \frac{t^{2n-1}}{(2n-1)!}. \end{aligned}$$

In the case when  $x = 1$ , these generating functions are well known. That is, the operation of complementation shows that  $A_{2n}(1) = C_{2n}(1)$  and  $B_{2n-1}(1) = D_{2n-1}(1)$  for all  $n \geq 1$  and André [1, 2] proved that

$$\sum_{n \geq 0} A_{2n}(1) \frac{t^{2n}}{(2n)!} = \sec(t)$$

and

$$\sum_{n \geq 1} B_{2n-1}(1) \frac{t^{2n-1}}{(2n-1)!} = \tan(t).$$

Thus, the number of up-down permutations is given by the following exponential generating function

$$\sec(t) + \tan(t) = \tan\left(\frac{t}{2} + \frac{\pi}{4}\right). \quad (1)$$

We shall prove the following theorem.

**Theorem 1.** *We have*

$$\begin{aligned} A(t, x) &= (\sec(xt))^{1/x}, \\ B(t, x) &= (\sec(xt))^{1/x} \int_0^t (\sec(xz))^{-1/x} dz, \\ C(t, x) &= 1 + \int_0^t (\sec(xy))^{1+\frac{1}{x}} \int_0^y (\sec(xz))^{1/x} dz dy, \text{ and} \\ D(t, x) &= \int_0^t (\sec(xz))^{1+\frac{1}{x}} dz. \end{aligned}$$

As an immediate corollary to Theorem 1 we get, for example, that the number of up-down permutations by occurrences of  $MMP(1, 0, 0, 0)$  is given by

$$A(t, x) + B(t, x) = (\sec(xt))^{1/x} \left(1 + \int_0^t (\sec(xz))^{-1/x} dz\right)$$

which refines (1).

One can use these generating functions to find some initial values of the polynomials  $A_{2n}(x)$ ,  $B_{2n-1}(x)$ ,  $C_{2n}(x)$ , and  $D_{2n-1}(x)$ . For example, we have used Mathematica to compute the following tables.

$n$	$A_{2n}(x)$
0	1
1	$x$
2	$x^2(3 + 2x)$
3	$x^3(15 + 30x + 16x^2)$
4	$x^4(105 + 420x + 588x^2 + 272x^3)$
5	$x^5(945 + 6300x + 16380x^2 + 18960x^3 + 7936x^4)$
6	$x^6(10395 + 103950x + 429660x^2 + 893640x^3 + 911328x^4 + 353792x^5)$

$n$	$B_{2n-1}(x)$
1	1
2	$2x$
3	$8x^2(1 + x)$
4	$16x^3(3 + 8x + 6x^2)$
5	$128x^4(3 + 15x + 27x^2 + 17x^3)$
6	$256x^5(15 + 120x + 381x^2 + 556x^3 + 310x^4)$
7	$1024x^6(45 + 525x + 2562x^2 + 6420x^3 + 8146x^4 + 4146x^5)$

$n$	$C_{2n}(x)$
0	1
1	1
2	$x(2 + 3x)$
3	$x^2(8 + 28x + 25x^2)$
4	$x^3(48 + 296x + 614x^2 + 427x^3)$
5	$x^4(384 + 3648x + 13104x^2 + 20920x^3 + 12465x^4)$
6	$x^5(3840 + 51840x + 282336x^2 + 769072x^3 + 1039946x^4 + 555731x^5)$

$n$	$D_{2n-1}(x)$
1	1
2	$x(1 + x)$
3	$x^2(3 + 8x + 5x^2)$
4	$x^3(15 + 75x + 121x^2 + 61x^3)$
5	$x^4(105 + 840x + 2478x^2 + 3128x^3 + 1385x^4)$
6	$x^5(945 + 11025x + 51030x^2 + 115350x^3 + 124921x^4 + 50521x^5)$
7	$x^6(10395 + 166320x + 1105335x^2 + 3859680x^3 + 7365633x^4 + 7158128x^5 + 2702765x^6)$

The outline of this paper is as follows. In Section 2, we shall prove Theorem 1. Then in Section 3, we shall study the entries of the tables above explaining them either explicitly or through recursions.

## 2 Proof of Theorem 1

The proof of all parts of Theorem 1 proceed in the same manner. That is, there are simple recursions satisfied by the polynomials  $A_{2n}(x)$ ,  $B_{2n+1}(x)$ ,  $C_{2n}(x)$ , and  $D_{2n+1}(x)$  based on the position of the largest value in the permutation.

### 2.1 The generating function $A(t, x)$

If  $\sigma = \sigma_1 \dots \sigma_{2n} \in UD_{2n}$ , then  $2n$  must occur in one of the positions  $2, 4, \dots, 2n$ . Let  $UD_{2n}^{(2k)}$  denote the set of permutations  $\sigma \in UD_{2n}$  such that  $\sigma_{2k} = 2n$ . A schematic diagram of an element in  $UD_{2n}^{(2k)}$  is pictured in Figure 2.

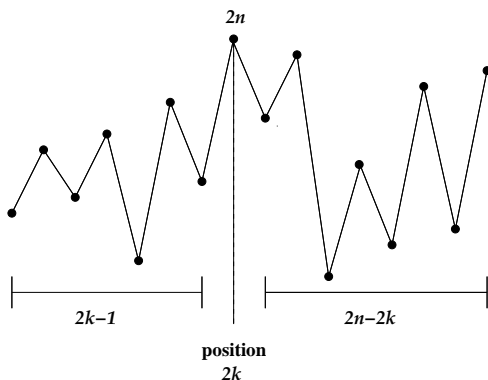


Figure 2: The graph of a  $\sigma \in UD_{2n}^{(2k)}$ .

Note that there are  $\binom{2n-1}{2k-1}$  ways to pick the elements which occur to the left of position  $2k$  in such  $\sigma$  and there are  $B_{2k-1}(1)$  ways to order them since the elements to the left of position  $2k$  form an up-down permutation of length  $2k-1$ . Each of the elements to the left of position  $2k$  contributes to  $\text{mmp}^{(1,0,0,0)}(\sigma)$ . Thus the contribution of the elements to the left of position  $2k$  in  $\sum_{\sigma \in UD_{2n}^{(2k)}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}$  is  $B_{2k-1}(1)x^{2k-1}$ . There are  $A_{2n-2k}(1)$  ways to order the elements to the right of position  $2k$  since they must form an up-down permutation of length  $2n-2k$ . Since the elements to the left of position  $2k$  have no effect on whether an element to the right of position  $2k$  contributes to  $\text{mmp}^{(1,0,0,0)}(\sigma)$ , it follows that the contribution of the elements to the right of position  $2k$  in  $\sum_{\sigma \in UD_{2n}^{(2k)}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}$  is  $A_{2n-2k}(x)$ . It thus follows that

$$A_{2n}(x) = \sum_{k=1}^n \binom{2n-1}{2k-1} B_{2k-1}(1) x^{2k-1} A_{2n-2k}(x)$$

or, equivalently,

$$\frac{A_{2n}(x)}{(2n-1)!} = \sum_{k=1}^n \frac{B_{2k-1}(1)x^{2k-1}}{(2k-1)!} \frac{A_{2n-2k}(x)}{(2n-2k)!}. \quad (2)$$

Multiplying both sides of (2) by  $t^{2n-1}$  and summing for  $n \geq 1$ , we see that

$$\sum_{n \geq 1} \frac{A_{2n}(x)t^{2n-1}}{(2n-1)!} = \left( \sum_{n \geq 1} \frac{B_{2n-1}(1)x^{2n-1}t^{2n-1}}{(2n-1)!} \right) \left( \sum_{n \geq 0} \frac{A_{2n}(x)t^{2n}}{(2n)!} \right).$$

By André's result,

$$\sum_{n \geq 1} \frac{B_{2n-1}(1)x^{2n-1}t^{2n-1}}{(2n-1)!} = \tan(xt)$$

so that

$$\frac{\partial}{\partial t} A(t, x) = \tan(xt)A(t, x).$$

Our initial condition is that  $A(0, x) = 1$ . It is easy to check that the solution to this differential equation is

$$A(t, x) = (\sec(xt))^{1/x}.$$

## 2.2 The generating function $B(t, x)$

If  $\sigma = \sigma_1 \dots \sigma_{2n+1} \in UD_{2n+1}$ , then  $2n+1$  must occur in one of the positions  $2, 4, \dots, 2n$ . Let  $UD_{2n+1}^{(2k)}$  denote the set of permutations  $\sigma \in UD_{2n+1}$  such that  $\sigma_{2k} = 2n+1$ . A schematic diagram of an element in  $UD_{2n+1}^{(2k)}$  is pictured in Figure 3.

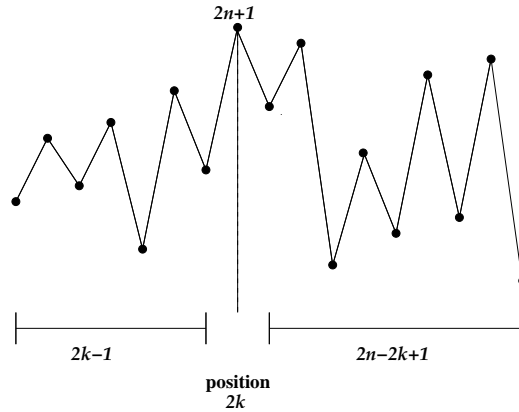


Figure 3: The graph of a  $\sigma \in UD_{2n+1}^{(2k)}$ .

Again there are  $\binom{2n}{2k-1}$  ways to pick the elements which occur to the left of position  $2k$  in such  $\sigma$  and the contribution of the elements to the left of position  $2k$  in  $\sum_{\sigma \in UD_{2n+1}^{(2k)}} x^{\text{mmp}^{(1,0,0)}(\sigma)}$  is  $B_{2k-1}(1)x^{2k-1}$ . There are  $B_{2n-2k+1}(1)$  ways to order the elements to the right of position  $2k$  since they must form an up-down permutation of length

$2n - 2k + 1$ . Since the elements to the left of position  $2k$  have no effect on whether an element to the right of position  $2k$  contributes to  $\text{mmp}^{(1,0,0,0)}(\sigma)$ , it follows that the contribution of the elements to the right of position  $2k$  in  $\sum_{\sigma \in UD_{2n+1}^{(2k)}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}$  is  $B_{2n-2k+1}(x)$ . It thus follows that if  $n \geq 1$ , then

$$B_{2n+1}(x) = \sum_{k=1}^n \binom{2n}{2k-1} B_{2k-1}(1) x^{2k-1} B_{2n-2k+1}(x).$$

Hence for  $n \geq 1$ ,

$$\frac{B_{2n+1}(x)}{(2n)!} = \sum_{k=1}^n \frac{B_{2k-1}(1) x^{2k-1}}{(2k-1)!} \frac{B_{2n-2k+1}(x)}{(2n-2k+1)!}. \quad (3)$$

Multiplying both sides of (3) by  $t^{2n}$ , summing for  $n \geq 1$ , and taking into account that  $B_1(x) = 1$ , we see that

$$\sum_{n \geq 0} \frac{B_{2n+1}(x) t^{2n}}{(2n)!} = 1 + \left( \sum_{n \geq 0} \frac{B_{2n+1}(1) x^{2n+1} t^{2n+1}}{(2n+1)!} \right) \left( \sum_{n \geq 0} \frac{B_{2n+1}(x) t^{2n+1}}{(2n+1)!} \right).$$

Since

$$\sum_{n \geq 1} \frac{B_{2n-1}(1) x^{2n-1} t^{2n-1}}{(2n-1)!} = \tan(xt),$$

we see that

$$\frac{\partial}{\partial t} B(t, x) = 1 + \tan(xt) B(t, x).$$

Our initial condition is that  $B(0, x) = 0$ . It is easy to check that the solution to this differential equation is

$$B(t, x) = (\sec(xt))^{1/x} \int_0^t (\sec(xz))^{-1/x} dz.$$

### 2.3 The generating function $C(t, x)$

If  $\sigma = \sigma_1 \dots \sigma_{2n} \in DU_{2n}$ , then  $2n$  must occur in one of the positions  $1, 3, \dots, 2n-1$ . Let  $DU_{2n}^{(2k+1)}$  denote the set of permutations  $\sigma \in DU_{2n}$  such that  $\sigma_{2k+1} = 2n$ . A schematic diagram of an element in  $DU_{2n}^{(2k+1)}$  is pictured in Figure 4.

Note that there are  $\binom{2n-1}{2k}$  ways to pick the elements which occur to the left of position  $2k+1$  in such  $\sigma$  and there are  $C_{2k}(1) = A_{2k}(1)$  ways to order them since the elements to the left of position  $2k+1$  form a down-up permutation of length  $2k$ . Each of the elements to the left of position  $2k+1$  contributes to  $\text{mmp}^{(1,0,0,0)}(\sigma)$ . Thus the contribution of the elements to the left of position  $2k+1$  in  $\sum \sigma \in UD_{2n}^{(2k)} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}$  is  $A_{2k}(1) x^{2k}$ . There are  $B_{2n-2k-1}(1)$  ways to order the elements to the right of position  $2k+1$  since they must form an up-down permutation of length  $2n-2k+1$ . Since the elements to the left of position  $2k+1$  have no effect on whether an element to the right of position  $2k+1$  contributes to



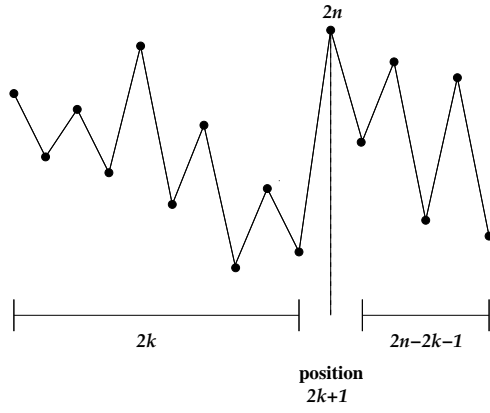


Figure 4: The graph of a  $\sigma \in DU_{2n}^{(2k+1)}$ .

$\text{mmp}^{(1,0,0,0)}(\sigma)$ , it follows that the contribution of the elements to the right of position  $2k$  in  $\sum_{\sigma \in UD_{2n}^{(2k)}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}$  is  $B_{2n-2k-1}(x)$ . It thus follows that

$$C_{2n}(x) = \sum_{k=0}^{n-1} \binom{2n-1}{2k} A_{2k}(1) x^{2k} B_{2n-2k-1}(x),$$

or, equivalently,

$$\frac{C_{2n}(x)}{(2n-1)!} = \sum_{k=0}^{n-1} \frac{A_{2k}(1) x^{2k}}{(2k)!} \frac{B_{2n-2k-1}(x)}{(2n-2k-1)!}. \quad (4)$$

Multiplying both sides of (4) by  $t^{2n-1}$  and summing for  $n \geq 1$ , we see that

$$\sum_{n \geq 1} \frac{C_{2n}(x) t^{2n-1}}{(2n-1)!} = \left( \sum_{n \geq 1} \frac{B_{2n-1}(1) x^{2n-1} t^{2n-1}}{(2n-1)!} \right) \left( \sum_{n \geq 0} \frac{A_{2n}(x) t^{2n}}{(2n)!} \right).$$

By André's result,

$$\sum_{n \geq 0} \frac{A_{2n}(1) x^{2n} t^{2n}}{(2n)!} = \sec(xt)$$

so that

$$\frac{\partial}{\partial t} C(t, x) = \sec(xt) B(t, x) = (\sec(xt))^{1+\frac{1}{x}} \int_0^t (\sec(xz))^{\frac{-1}{x}} dz. \quad (5)$$

Our initial condition is that  $C(0, x) = 1$ . Both Maple and Mathematica will solve this differential equation but the final expressions are complicated and not particularly useful for enumeration purposes. Thus we actually used the RHS of (5) to find the entries of the table for the initial values of  $C_{2n}(x)$  given in the introduction. Nevertheless, we can record the solution of (5) as

$$C(t, x) = 1 + \int_0^t (\sec(xy))^{1+\frac{1}{x}} \int_0^y (\sec(xz))^{\frac{-1}{x}} dz dy.$$

## 2.4 The generating function $D(t, x)$

If  $\sigma = \sigma_1 \dots \sigma_{2n+1} \in DU_{2n+1}$ , then  $2n+1$  must occur in one of the positions  $1, 3, \dots, 2n+1$ . Let  $DU_{2n+1}^{(2k+1)}$  denote the set of permutations  $\sigma \in DU_{2n+1}$  such that  $\sigma_{2k+1} = 2n+1$ . A schematic diagram of an element in  $DU_{2n+1}^{(2k+1)}$  is pictured in Figure 5.

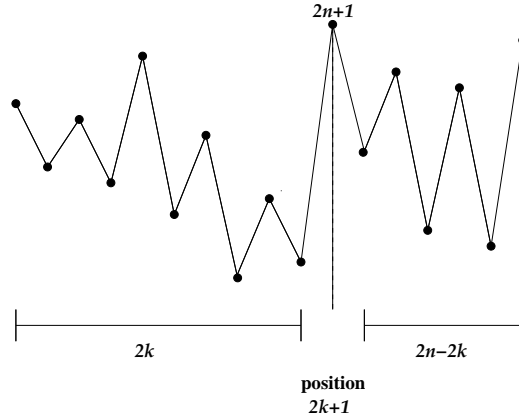


Figure 5: The graph of a  $\sigma \in DU_{2n+1}^{(2k+1)}$ .

Note that there are  $\binom{2n}{2k}$  ways to pick the elements which occur to the left of position  $2k+1$  in such  $\sigma$  and there are  $C_{2k}(1) = A_{2k}(1)$  ways to order them since the elements to the right of position  $2k+1$  form a down-up permutation of length  $2k$ . Each of the elements to the left of position  $2k+1$  contributes to  $\text{mmp}^{(1,0,0,0)}(\sigma)$ . Thus the contribution of the elements to the left of position  $2k+1$  in  $\sum_{\sigma \in DU_{2n+1}^{(2k+1)}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}$  is  $A_{2k}(1)x^{2k}$ . There are  $A_{2n-2k}(1)$  ways to order the elements to the right of position  $2k+1$  since they must form an up-down permutation of length  $2n-2k$ . Since the elements to the left of position  $2k+1$  have no effect on whether an element to the right of position  $2k+1$  contributes to  $\text{mmp}^{(1,0,0,0)}(\sigma)$ , it follows that the contribution of the elements to the right of position  $2k+1$  in  $\sum_{\sigma \in DU_{2n+1}^{(2k+1)}} x^{\text{mmp}^{(1,0,0,0)}(\sigma)}$  is  $A_{2n-2k}(x)$ . It thus follows that if  $n \geq 1$ , then

$$D_{2n+1}(x) = \sum_{k=0}^n \binom{2n}{2k} A_{2k}(1)x^{2k} A_{2n-2k}(x).$$

Hence for  $n \geq 1$ ,

$$\frac{D_{2n+1}(x)}{(2n)!} = \sum_{k=0}^n \frac{A_{2k}(1)x^{2k}}{(2k)!} \frac{A_{2n-2k}(x)}{(2n-2k)!}. \quad (6)$$

Multiplying both sides of (6) by  $t^{2n}$  and summing for  $n \geq 0$ , we see that

$$\sum_{n \geq 0} \frac{D_{2n+1}(x)t^{2n}}{(2n)!} = \left( \sum_{n \geq 0} \frac{A_{2n}(1)x^{2n}t^{2n}}{(2n)!} \right) \left( \sum_{n \geq 0} \frac{A_{2n}(x)t^{2n}}{(2n)!} \right)$$

so that

$$\frac{\partial}{\partial t} D(t, x) = \sec(x, t) A(t, x) = (\sec(xt))^{1+\frac{1}{x}}.$$

Our initial condition is that  $D(0, x) = 0$  so that the solution to this differential equation is

$$D(t, x) = \int_0^t (\sec(xz))^{1+\frac{1}{x}} dz.$$

## 2.5 A remark on $MMP(k, 0, 0, 0)$ for $k \geq 2$

We note that we cannot apply the same techniques to find the distribution of marked mesh patterns  $MMP(k, 0, 0, 0)$  in up-down and down-up permutations when  $k \geq 2$ . For example, suppose that we try to develop a recursion for  $A_{2n}^{(2,0,0,0)}(x) = \sum_{\sigma \in UD_{2n}} x^{\text{mmp}^{(2,0,0,0)}(\sigma)}$ . Then if we consider the permutations  $\sigma = \sigma_1 \dots \sigma_{2n} \in UD_{2n}$  such that  $\sigma_{2k} = 2n$ , we still have  $\binom{2n-1}{2k-1}$  ways to pick the elements for  $\sigma_1 \dots \sigma_{2k-1}$ . However, in this case the question of whether some  $\sigma_i$  with  $i < 2k$  matches the marked mesh pattern  $MMP(2, 0, 0, 0)$  in  $\sigma$  is dependent on what values occur in  $\sigma_{2k+1} \dots \sigma_{2n}$ . For example, if  $2n-1 \in \{\sigma_{2k+1}, \dots, \sigma_{2n}\}$ , then every  $\sigma_i$  with  $i \leq k$  will match the marked mesh pattern  $MMP(2, 0, 0, 0)$  in  $\sigma$ . However, if  $2n-1 \in \{\sigma_1, \dots, \sigma_{2k-1}\}$ , this will not be the case. Thus we cannot develop a simple recursion for  $A_{2n}^{(2,0,0,0)}(x)$ .

## 3 The coefficients of the polynomials $A_{2n}(x)$ , $B_{2n+1}(x)$ , $C_{2n}(x)$ , and $D_{2n+1}(x)$ .

The main goal of this section is to explain several of the coefficients of the polynomials  $A_{2n}(x)$ ,  $B_{2n+1}(x)$ ,  $C_{2n}(x)$ , and  $D_{2n+1}(x)$ . First it is easy to understand the coefficients of the lowest power of  $x$  in each of these polynomials. That is, we have the following theorem, where  $0!! = 1$  and, for  $n \geq 1$ ,  $(2n)!! = \prod_{i=1}^n (2i)$  and  $(2n-1)!! = \prod_{i=1}^n (2i-1)$ .

**Theorem 2.**

(1) For all  $n \geq 1$ ,

$$A_{2n}(x)|_{x^k} = \begin{cases} 0 & \text{if } 0 \leq k < n \\ (2n-1)!! & \text{if } k = n. \end{cases}$$

(2) For all  $n \geq 1$ ,

$$B_{2n+1}(x)|_{x^k} = \begin{cases} 0 & \text{if } 0 \leq k < n \\ (2n)!! & \text{if } k = n. \end{cases}$$

(3) For all  $n \geq 1$ ,

$$C_{2n}(x)|_{x^k} = \begin{cases} 0 & \text{if } 0 \leq k < n-1 \\ (2(n-1))!! & \text{if } k = n-1. \end{cases}$$

(4) For all  $n \geq 1$ ,

$$D_{2n+1}(x)|_{x^k} = \begin{cases} 0 & \text{if } 0 \leq k < n \\ (2n-1)!! & \text{if } k = n. \end{cases}$$

*Proof.* For (1), note that if  $\sigma = \sigma_1 \dots \sigma_{2n} \in UD_{2n}$ , then  $\sigma_{2i+1}$  matches the pattern  $MMP(1, 0, 0, 0)$  for  $i = 0, \dots, n-1$ . Thus  $mpp^{(1,0,0,0)}(\sigma) \geq n$ . We now proceed by induction to prove that  $A_{2n}(x)|_{x^n} = (2n-1)!!$  for all  $n \geq 1$ . This is clear for  $n = 1$  since  $A_2(x) = x$ . Now suppose that  $\sigma = \sigma_1 \dots \sigma_{2n} \in UD_{2n}$  and  $mpp^{(1,0,0,0)}(\sigma) = n$ . It is then easy to see that it must be the case that  $\sigma_2 = 2n$  (otherwise  $\sigma_2$  is an unwanted occurrence of the pattern  $MMP(1, 0, 0, 0)$ ). Moreover, if  $\tau = \text{red}(\sigma_3 \dots \sigma_{2n})$ , then  $\tau \in UD_{2n-2}$  and  $mmp^{(1,0,0,0)}(\tau) = n-1$ . Thus since we are assuming by induction that  $A_{2n-2}(x)|_{x^{n-1}} = (2n-3)!!$ , we have  $2n-1$  choices of  $\sigma_1$  and  $(2n-3)!!$  choices for  $\tau$ . Hence  $A_{2n}(x)|_{x^n} = (2n-1)!!$ .

For (2), note that if  $\sigma = \sigma_1 \dots \sigma_{2n+1} \in UD_{2n+1}$ , then  $\sigma_{2i+1}$  matches the pattern  $MMP(1, 0, 0, 0)$  for  $i = 0, \dots, n-1$ . Thus  $mpp^{(1,0,0,0)}(\sigma) \geq n$ . We now proceed by induction to prove that  $B_{2n+1}(x)|_{x^n} = (2n)!!$  for all  $n \geq 1$ . This is clear for  $n = 1$  since  $B_3(x) = 2x$ . Now suppose that  $\sigma = \sigma_1 \dots \sigma_{2n+1} \in UD_{2n+1}$  and  $mpp^{(1,0,0,0)}(\sigma) = n$ . It is then easy to see that it must be the case that  $\sigma_2 = 2n+1$ . Moreover if,  $\tau = \text{red}(\sigma_3 \dots \sigma_{2n+1})$ , then  $\tau \in UD_{2n-1}$  and  $mmp^{(1,0,0,0)}(\tau) = n-1$ . Thus since we are assuming by induction that  $B_{2n-1}(x)|_{x^{n-1}} = (2n-2)!!$ , we have  $2n$  choices of  $\sigma_1$  and  $(2n-2)!!$  choices for  $\tau$ . Hence  $B_{2n+1}(x)|_{x^n} = (2n)!!$  for  $n \geq 1$ .

For (3), note that if  $\sigma = \sigma_1 \dots \sigma_{2n} \in DU_{2n}$ , then  $\sigma_{2i}$  matches the pattern  $MMP(1, 0, 0, 0)$  for  $i = 1, \dots, n-1$ . Thus  $mpp^{(1,0,0,0)}(\sigma) \geq n-1$ . Suppose that  $mpp^{(1,0,0,0)}(\sigma) = n-1$ . It is then easy to see that it must be the case that  $\sigma_1 = 2n$ . Moreover, if  $\tau = \sigma_2 \dots \sigma_{2n}$ , then  $\tau \in UD_{2n-1}$  and  $mmp^{(1,0,0,0)}(\tau) = n-1$ . Thus we have  $(2(n-1))!!$  choices for  $\tau$  by part (2). Hence  $C_{2n}(x)|_{x^{n-1}} = (2(n-1))!!$ .

For (4), note that if  $\sigma = \sigma_1 \dots \sigma_{2n+1} \in DU_{2n+1}$ , then  $\sigma_{2i}$  matches  $MMP(1, 0, 0, 0)$  for  $i = 1, \dots, n$ . Thus  $mpp^{(1,0,0,0)}(\sigma) \geq n$ . Suppose that  $mpp^{(1,0,0,0)}(\sigma) = n$ . It is then easy to see that it must be the case that  $\sigma_1 = 2n+1$ . Moreover, if  $\tau = \sigma_2 \dots \sigma_{2n+1}$ , then  $\tau \in UD_{2n}$  and  $mmp^{(1,0,0,0)}(\tau) = n$ . Thus we have  $(2n-1)!!$  choices for  $\tau$  by part (1). Hence  $D_{2n+1}(x)|_{x^n} = (2n-1)!!$  for  $n \geq 1$ .  $\square$

We can easily explain the coefficients of the highest power of  $x$  in each of the polynomials  $A_{2n}(x)$ ,  $B_{2n+1}(x)$ ,  $C_{2n}(x)$ , and  $D_{2n+1}(x)$ . That is, we have the following proposition.

**Proposition 2.**

- (1) For all  $n \geq 1$ , the highest power of  $x$  that appears in  $A_{2n}(x)$  is  $x^{2n-1}$  which appears with coefficient  $B_{2n-1}(1)$ .
- (2) For all  $n \geq 1$ , the highest power of  $x$  that appears in  $B_{2n+1}(x)$  is  $x^{2n-1}$  which appears with coefficient  $(2n)B_{2n-1}(1)$ .
- (3) For all  $n \geq 1$ , the highest power of  $x$  that appears in  $C_{2n}(x)$  is  $x^{2n-2}$  which appears with coefficient  $(2n-1)A_{2n-2}(1)$ .
- (4) For all  $n \geq 1$ , the highest power of  $x$  that appears in  $D_{2n+1}(x)$  is  $x^{2n}$  which appears with coefficient  $A_{2n}(1)$ .

*Proof.* For (1), it is easy to see that  $mmp^{(1,0,0,0)}(\sigma)$  is maximized for a  $\sigma = \sigma_1 \dots \sigma_{2n} \in UD_{2n}$  when  $\sigma_{2n} = 2n$ . In such a case  $mmp^{(1,0,0,0)}(\sigma) = 2n - 1$  and  $\sigma_1 \dots \sigma_{2n-1}$  can be any element of  $UD_{2n-1}$ .

For (2), it is easy to see that  $mmp^{(1,0,0,0)}(\sigma)$  is maximized for a  $\sigma = \sigma_1 \dots \sigma_{2n+1} \in UD_{2n+1}$  when  $\sigma_{2n} = 2n + 1$ . In such a case  $mmp^{(1,0,0,0)}(\sigma) = 2n - 1$ . We then have  $2n$  choices for  $\sigma_{2n+1}$  and  $\text{red}(\sigma_1 \dots \sigma_{2n-1})$  can be any element of  $UD_{2n-1}$ . Thus  $B_{2n+1}(x)|_{x^{2n-1}} = (2n)B_{2n-1}(1)$ .

For (3), it is easy to see that  $mmp^{(1,0,0,0)}(\sigma)$  is maximized for a  $\sigma = \sigma_1 \dots \sigma_{2n} \in DU_{2n}$  when  $\sigma_{2n-1} = 2n$ . In such a case  $mmp^{(1,0,0,0)}(\sigma) = 2n - 2$ . We then have  $2n - 1$  choices for  $\sigma_{2n}$  and  $\text{red}(\sigma_1 \dots \sigma_{2n-2})$  can be any element of  $DU_{2n-2}$ . Thus  $C_{2n}(x)|_{x^{2n-2}} = (2n - 1)C_{2n-2}(1) = (2n - 1)A_{2n-2}(1)$ .

For (4), it is easy to see that  $mmp^{(1,0,0,0)}(\sigma)$  is maximized for a  $\sigma = \sigma_1 \dots \sigma_{2n+1} \in DU_{2n+1}$  when  $\sigma_{2n+1} = 2n + 1$ . In such a case  $mmp^{(1,0,0,0)}(\sigma) = 2n$ . Then  $\sigma_1 \dots \sigma_{2n}$  can be any element of  $DU_{2n}$ . Thus  $D_{2n+1}(x)|_{x^{2n}} = C_{2n}(1) = A_{2n}(1)$ . □

### 3.1 Recursions on up-down permutations of even length

By Theorem 2, the lowest power of  $x$  that appears with a non-zero coefficient in  $A_{2n}(x)$  is  $x^n$ . Next we consider  $A_{2n}(x)|_{x^{n+k}}$  for fixed  $k$ . That is, we let

$$A_{2n}^{\overline{n+k}} = \{|\sigma \in UD_{2n} : mmp^{(1,0,0,0)}(\sigma) = n + k|\}$$

for fixed  $k \geq 1$ . Our goal is to show that  $A_{2n}^{\overline{n+k}} = p_k(n)(2n - 1)!!$  for some fixed polynomial  $p_k(n)$  in  $n$ . That is, we shall prove the following theorem, where we let

$$(x) \downarrow_n = x(x - 1) \cdots (x - n + 1) \text{ if } n \geq 1 \text{ and } (x) \downarrow_0 = 1.$$

**Theorem 3.** *There is a sequence of polynomials  $p_0(x), p_1(x), \dots$  such that for all  $k \geq 0$ ,*

$$A_{2n}^{\overline{n+k}} = p_k(n)(2n - 1)!! \text{ for all } n \geq k + 1.$$

Moreover for  $k \geq 1$ , the values  $p_k(n)$  are defined by the recursion

$$p_k(n) = \frac{B_{2k+1}(1)}{(2k + 1)!!} + \sum_{j=1}^k \sum_{t=k+2}^n \frac{B_{2j+1}(1)2^j(t - 1) \downarrow_j}{(2j + 1)!} p_{k-j}(t - j - 1) \quad (7)$$

where  $p_0(x) = 1$ .

*Proof.* We proceed by induction on  $k$ . For  $k = 0$ , we know by Theorem 2 that  $A_{2n}^{\overline{n}} = (2n - 1)!!$  for all  $n \geq 1$  so that we can let  $p_0(x) = 1$ .

Now assume that  $k \geq 1$  and the theorem is true for  $s < k$ . That is, assume that for  $0 \leq s < k$ , there is a polynomial  $p_s(x)$  such that for  $n \geq s + 1$ ,  $A_{2n}^{\overline{n+s}} = p_s(n)(2n - 1)!!$ .

It is easy to see that for  $\sigma = \sigma_1 \dots \sigma_{2n} \in UD_{2n}$ ,  $mmp^{(1,0,0,0)}(\sigma) > n + k$  if  $\sigma_{2j} = 2n$  where  $j \geq k + 2$  because then  $\sigma_2, \sigma_4, \dots, \sigma_{2k+2}$  as well as  $\sigma_{2i+1}$  such that  $i = 0, \dots, n - 1$

will match the pattern  $MMP(1, 0, 0, 0)$  in  $\sigma$ . Thus if  $mmp^{(1,0,0,0)}(\sigma) = n + k$ , then  $2n \in \{\sigma_2, \sigma_4, \dots, \sigma_{2k+2}\}$ . Now suppose that  $j \leq k + 1$  and  $\sigma_{2j} = 2n$ . Then we have  $\binom{2n-1}{2j-1}$  ways to choose the elements  $\sigma_1, \dots, \sigma_{2j-1}$  and we have  $B_{2j-1}(1)$  ways to order them. Then we know that  $\sigma_i$  matches the marked mesh pattern  $MMP(1, 0, 0, 0)$  in  $\sigma$  for  $i$  odd and for  $i \in \{2, 4, \dots, 2j - 2\}$ . Hence, it must be the case that  $mmp^{(1,0,0,0)}(\text{red}(\sigma_{2j+1} \dots \sigma_{2n})) = n - j + k - j + 1$ . Thus it follows that for  $n \geq k + 1$ ,

$$A_{2n}^{=n+k} = \sum_{j=1}^{k+1} \binom{2n-1}{2j-1} B_{2j-1}(1) A_{2(n-j)}^{=(n-j)+k-j+1}. \quad (8)$$

Now define  $p_k(n) = \frac{A_{2n}^{=n+k}}{(2n-1)!!}$  for  $n \geq k + 1$ . Note that  $A_{2k+2}^{(k+1)+k} = B_{2k+1}(1)$  since for a  $\tau = \tau_1 \dots \tau_{2k+2} \in UD_{2k+2}$  to have  $mmp^{(1,0,0,0)}(\tau) = 2k + 1$ , it must be the case that  $\tau_{2k+2} = 2k + 2$  and, hence, we have  $B_{2k+1}(1)$  choices for  $\tau_1 \dots \tau_{2k+1}$ . Hence,  $p_k(k + 1) = \frac{B_{2k+1}(1)}{(2k+1)!!}$ .

We can rewrite (8) as

$$\begin{aligned} p_k(n)(2n-1)!! &= (2n-1)p_k(n-1)(2n-3)!! + \\ &\sum_{j=2}^{k+1} \frac{\prod_{i=0}^{2j-2} (2n-1-j)}{(2j-1)!} B_{2j-1}(1) p_{k-j+1}(n-j)(2n-2j-1)!!. \end{aligned} \quad (9)$$

Dividing (9) by  $(2n-1)!!$ , we obtain that

$$\begin{aligned} p_k(n) - p_k(n-1) &= \sum_{j=2}^{k+1} \frac{B_{2j-1}(1) \prod_{s=1}^{j-1} (2n-2s)}{(2j-1)!} p_{k-j+1}(n-j) \\ &= \sum_{j=1}^k \frac{B_{2j+1}(1) 2^j (n-1) \downarrow_j}{(2j+1)!} p_{k-j}(n-j-1). \end{aligned}$$

Hence for  $n \geq k + 1$ ,

$$\begin{aligned} p_k(n) - p_k(k+1) &= \sum_{t=k+2}^n p_k(t) - p_k(t-1) \\ &= \sum_{t=k+2}^n \sum_{j=1}^k \frac{B_{2j+1}(1) 2^j (t-1) \downarrow_j}{(2j+1)!} p_{k-j}(t-j-1). \end{aligned}$$

It follows that for  $n \geq k + 1$ ,

$$p_k(n) = \frac{B_{2k+1}(1)}{(2k+1)!!} + \sum_{j=1}^k \sum_{t=k+2}^n \frac{B_{2j+1}(1) 2^j (t-1) \downarrow_j}{(2j+1)!} p_{k-j}(t-j-1).$$

This proves (7).

Since  $p_s(x)$  is a polynomial for  $s < k$ , it is easy to see that

$$\sum_{t=k+2}^n \frac{B_{2j+1}(1)2^j(t-1) \downarrow_j}{(2j+1)!} p_{k-j}(t-j-1)$$

is a polynomial in  $n$  for  $j = 1, \dots, k$ . Thus  $p_k(n)$  is a polynomial in  $n$ .  $\square$

One can use Mathematica and (7) to compute the first few expressions for  $p_k(n)$ . For example, we have computed that

$$\begin{aligned} p_0(n) &= 1, \\ p_1(n) &= \frac{2}{3} \binom{n}{2}, \\ p_2(n) &= \frac{n(2+7n-14n^2+5n^3)}{90}, \text{ and} \\ p_3(n) &= \frac{n(192-478n+213n^2+227n^3-198n^4+35n^5)}{5670}. \end{aligned}$$

### 3.2 Recursions on up-down permutations of odd length

**Theorem 4.** *There is a sequence of polynomials  $q_0(x), q_1(x), \dots$  such that for all  $k \geq 0$ ,*

$$B_{2n+1}^{=n+k} = q_k(n)(2n)!! \text{ for all } n \geq k+1.$$

Moreover for  $k \geq 1$ , the values  $q_k(n)$  are defined by the recursion

$$q_k(n) = \frac{B_{2k+1}(1)}{(2k)!!} + \sum_{j=1}^k \sum_{t=k+2}^n \frac{B_{2j+1}(1) \prod_{s=0}^{j-1} (2t-1-2s)}{(2j+1)!} q_{k-j}(t-j-1) \quad (10)$$

where  $q_0(x) = 1$ .

*Proof.* We proceed by induction on  $k$ . For  $k = 0$ , we know by Theorem 2 that  $B_{2n+1}^{=n} = (2n)!!$  for all  $n \geq 1$  so that we can let  $q_0(x) = 1$ .

Now assume that  $k \geq 1$  and the theorem is true for  $s < k$ . That is, assume that for  $0 \leq s < k$ , there is a polynomial  $q_s(x)$  such that for  $n \geq s+1$ ,  $B_{2n+1}^{=n+s} = q_s(n)(2n)!!$ .

We can argue as in Theorem 3 that if  $mmp^{(1,0,0,0)}(\sigma) = n+k$ , then  $2n \in \{\sigma_2, \sigma_4, \dots, \sigma_{2k+2}\}$ . Now suppose that  $j \leq k+1$  and  $\sigma_{2j} = 2n$ . Then we have  $\binom{2n}{2j-1}$  ways to choose the elements  $\sigma_1, \dots, \sigma_{2j-1}$  and we have  $B_{2j-1}(1)$  ways to order them. Then we know that  $\sigma_i$  matches the marked mesh pattern  $MMP(1,0,0,0)$  in  $\sigma$  for  $i \in \{2, 4, \dots, 2j-2\} \cup \{1, 3, \dots, 2n-1\}$ . Hence, it must be the case that  $mmp^{(1,0,0,0)}(\text{red}(\sigma_{2j+1} \dots \sigma_{2n+1})) = n-j+k-j+1$ . Thus it follows that for  $n \geq k+2$ ,

$$B_{2n+1}^{=n+k} = \sum_{j=1}^{k+1} \binom{2n}{2j-1} B_{2j-1}(1) B_{2(n-j)+1}^{=(n-j)+k-j+1}. \quad (11)$$

Now define  $q_k(n) = \frac{B_{2n+1}^{=n+k}}{(2n)!!}$  for  $n \geq k+1$ . Note that  $B_{2k+3}^{(k+1)+k} = (2k+2)B_{2k+1}(1)$  since for a  $\tau = \tau_1 \dots \tau_{2k+3} \in UD_{2k+3}$  to have  $mmp^{(1,0,0,0)}(\tau) = 2k+1$ , it must be the case that  $\tau_{2k+2} = 2k+3$  and, hence, we have  $2k+2$  choices for  $\tau_{2k+3}$  and  $B_{2k+1}(1)$  choices for  $\tau_1 \dots \tau_{2k+1}$ . Thus,  $q_k(k+1) = \frac{(2k+2)B_{2k+1}(1)}{(2k+2)!!} = \frac{B_{2k+1}(1)}{(2k)!!}$ .

We can rewrite (11) as

$$\begin{aligned} q_k(n)(2n)!! &= (2n)q_k(n-1)(2n-2)!! + \\ &\sum_{j=2}^{k+1} \frac{\prod_{i=0}^{2j-2} (2n-j)}{(2j-1)!} B_{2j-1}(1) q_{k-j+1}(n-j)(2n-2j)!!. \end{aligned} \quad (12)$$

Dividing (12) by  $(2n)!!$ , we obtain that

$$q_k(n) - q_k(n-1) = \sum_{j=2}^{k+1} \frac{B_{2j-1}(1) \prod_{s=1}^{j-1} (2n-2s-1)}{(2j-1)!} q_{k-j+1}(n-j). \quad (13)$$

Hence for  $n \geq k+2$ ,

$$\begin{aligned} q_k(n) - q_k(k+1) &= \sum_{t=k+2}^n q_k(t) - q_k(t-1) \\ &= \sum_{t=k+2}^n \sum_{j=1}^k \frac{B_{2j+1}(1) 2^j \prod_{s=1}^{j-1} (2n-2s-1)}{(2j+1)!} q_{k-j}(t-j-1). \end{aligned}$$

It follows that for  $n \geq k+1$ ,

$$q_k(n) = \frac{B_{2k+1}(1)}{(2k)!!} + \sum_{j=1}^k \sum_{t=k+2}^n \frac{B_{2j+1}(1) 2^j \prod_{s=1}^{j-1} (2n-2s-1)}{(2j+1)!} q_{k-j}(t-j-1). \quad (14)$$

This proves (10).

Since  $q_s(x)$  is a polynomial for  $s < k$ , it is easy to see that  $\sum_{t=k+2}^n \frac{B_{2j+1}(1) \prod_{s=1}^{j-1} (2n-2s-1)}{(2j+1)!} q_{k-j}(t-j-1)$  is a polynomial in  $n$  for  $j = 1, \dots, k$ . Thus  $q_k(n)$  is a polynomial in  $n$ .  $\square$

One can use Mathematica and (10) to compute the first few examples of  $q_k(n)$ . For example, we have computed that

$$\begin{aligned} q_0(n) &= 1, \\ q_1(n) &= \frac{n^2 - 1}{3}, \\ q_2(n) &= \frac{(n-2)(n-1)(5n^2 + n - 3)}{90}, \text{ and} \\ q_3(n) &= \frac{35n^6 - 84n^5 - 193n^4 + 345n^3 + 140n^2 - 81n + 198}{5670}. \end{aligned}$$



### 3.3 Recursions on down-up permutations

Similar results hold for down-up permutations.

**Theorem 5.** *There are sequences of polynomials  $r_0(x), r_1(x), \dots$  and  $s_0(x), s_1(x), \dots$  such that for all  $k \geq 0$ ,*

$$C_{2n}^{\overline{=n-1+k}} = r_k(n)(2n-2)!! \text{ for all } n \geq k+1. \quad (15)$$

and

$$D_{2n+1}^{\overline{=n-1+k}} = s_k(n)(2n-1)!! \text{ for all } n \geq k+1. \quad (16)$$

*Proof.* By Theorem 2,  $C_{2n}^{\overline{=n-1}} = (2n-2)!!$  and  $D_{2n+1}^{\overline{=n}} = (2n-1)!!$  for all  $n \geq 1$ . Thus we can let  $r_0(x) = s_0(x) = 1$ .

For a permutation  $\sigma = \sigma_1 \dots \sigma_{2n} \in DU_{2n}$  to have  $mmp^{(1,0,0,0)}(\sigma) = n-1+k$ , it must be the case that  $2n \in \{\sigma_1, \sigma_3, \dots, \sigma_{2k+1}\}$ . If  $\sigma_{2j+1} = 2n$  where  $j \in \{0, 1, \dots, k\}$ , then there are  $\binom{2n-1}{2j}$  ways to pick the elements of  $\sigma_1 \dots \sigma_{2j}$  and  $C_{2j}(1)$  ways to order them. Then  $\text{red}(\sigma_{2j+2} \dots \sigma_{2n}) \in UD_{2(n-j-1)+1}$  and must have  $n-1+k-(2j)$  matches of  $MMP(1, 0, 0, 0)$ . Thus we have  $B_{2(n-j-1)+1}^{\overline{=(n-j-1)+k-j}}$  ways to order  $\sigma_{2j+2} \dots \sigma_{2n}$ . It follows that for  $n \geq k+1$ ,

$$C_{2n}^{\overline{=n-1+k}} = \sum_{j=0}^k \binom{2n-1}{2j} C_{2j}(1) B_{2(n-j-1)+1}^{\overline{=n-j-1+k-j}}. \quad (17)$$

But  $C_{2j}(1) = A_{2j}(1)$  and  $B_{2(n-j-1)+1}^{\overline{=n-j-1+k-j}} = (2(n-j-1))!! q_{k-j}(n-j-1)$ . Thus for  $n \geq k+1$ ,

$$\begin{aligned} C_{2n}^{\overline{=n-1+k}} &= \sum_{j=0}^k \binom{2n-1}{2j} A_{2j}(1) (2(n-j-1))!! q_{k-j}(n-j-1) \\ &= (2n-2)!! \sum_{j=0}^k \frac{A_{2j}(1) \prod_{s=1}^j (2n+1-2s)}{(2j)!} (2(n-j-1))!! q_{k-j}(n-j-1). \end{aligned}$$

Thus  $C_{2n}^{\overline{=n-1+k}} = (2n-2)!! r_k(n)$  where

$$r_k(n) = \sum_{j=0}^k \frac{A_{2j}(1) \prod_{s=1}^j (2n+1-2s)}{(2j)!} q_{k-j}(n-j-1). \quad (18)$$

A similar argument will show that for  $n \geq k+1$ ,

$$D_{2n+1}^{\overline{=n+k}} = \sum_{j=0}^k \binom{2n}{2j} C_{2j}(1) A_{2(n-j)}^{\overline{=n-j+k-j}}.$$

Since  $A_{2(n-j)}^{\overline{=n-j+k-j}} = (2(n-j)-1)!! p_{k-j}(n-j)$ , we obtain that

$$\begin{aligned} D_{2n+1}^{\overline{=n+k}} &= \sum_{j=0}^k \binom{2n}{2j} A_{2j}(1) (2(n-j)-1)!! p_{k-j}(n-j) \\ &= (2n-1)!! \sum_{j=0}^k \frac{A_{2j}(1) \prod_{s=1}^j (2n+2-2s)}{(2j)!} p_{k-j}(n-j). \end{aligned}$$

Thus  $D_{2n+1}^{\overline{n+k}} = (2n-2)!!s_k(n)$  where

$$s_k(n) = \sum_{j=0}^k \frac{A_{2j}(1) \prod_{s=1}^j (2n+2-2s)}{(2j)!} p_{k-j}(n-j). \quad (19)$$

□

One can use (18) and (19) to compute  $r_k(n)$  and  $s_k(n)$  for the first few values of  $k$ . For example, we have that

$$\begin{aligned} r_0(n) &= 1, \\ r_1(n) &= \frac{2n^2 + 2n - 3}{6}, \\ r_2(n) &= \frac{20n^4 + 24n^3 - 128n^2 - 12n + 45}{360}, \text{ and} \\ r_3(n) &= \frac{280n^6 + 168n^5 - 4820n^4 + 3168n^3 + 8734n^2 - 6702n + 2835}{45360}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} s_0(n) &= 1, \\ s_1(n) &= \frac{n(n+2)}{3}, \\ s_2(n) &= \frac{n(5n^3 + 16n^2 - 68n + 47)}{90}, \text{ and} \\ s_3(n) &= \frac{n(35n^5 + 126n^4 - 340n^3 - 417n^2 + 656n - 60)}{5760}. \end{aligned}$$

## 4 Conclusion

In this paper, we have shown that one can find the generating functions for the distribution of the quadrant marked mesh patterns  $MMP(1, 0, 0, 0)$ ,  $MMP(0, 1, 0, 0)$ ,  $MMP(0, 0, 1, 0)$ , and  $MMP(0, 0, 0, 1)$  in both up-down and down-up permutations by proving simple recursions based on the position of the largest element in a permutation. As noted in Subsection 2.5, these simple type of recursions no longer hold for the distribution of the quadrant marked mesh patterns  $MMP(k, 0, 0, 0)$ ,  $MMP(0, k, 0, 0)$ ,  $MMP(0, 0, k, 0)$ , and  $MMP(0, 0, 0, k)$  in both up-down and down-up permutations when  $k \geq 2$ . However, our techniques can be used to study the distribution of other quadrant marked mesh patterns in up-down and down-up permutations. For example, in [8], we have proved similar recursions based on the position of the smallest element in a permutation to study the distribution of the quadrant marked mesh patterns  $MMP(1, 0, \emptyset, 0)$ ,  $MMP(0, 1, 0, \emptyset)$ ,  $MMP(\emptyset, 0, 1, 0)$ , and  $MMP(0, \emptyset, 0, 1)$  in both up-down and down-up permutations. In this case, the recursions are a bit more subtle and the corresponding generating functions are not always as

simple as the results of this paper. For example, if we let

$$A^{(1,0,\emptyset,0)}(x, t) = 1 + \sum_{n \geq 1} \frac{t^{2n}}{(2n)!} \sum_{\sigma \in UD_{2n}} x^{\text{mmp}^{(1,0,\emptyset,0)}(\sigma)} \text{ and}$$

$$B^{(1,0,\emptyset,0)}(x, t) = \sum_{n \geq 0} \frac{t^{2n+1}}{(2n+1)!} \sum_{\sigma \in UD_{2n+1}} x^{\text{mmp}^{(1,0,\emptyset,0)}(\sigma)},$$

then we can show that

$$A^{(1,0,\emptyset,0)}(t, x) = (\sec(t))^x,$$

$$B^{(1,0,\emptyset,0)}(t, x) = \frac{\sin(t) \cos(t)(1-x+\sec(t))}{x+(1-x)\cos(t)} \times$$

$$\left( (1-x) {}_2F_1\left(\frac{1}{2}, \frac{1+x}{2}; \frac{3}{2}; \sin(t^2)\right) + x {}_2F_1\left(\frac{1}{2}, \frac{2+x}{2}; \frac{3}{2}; \sin(t^2)\right) \right)$$

where  ${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$  and  $(x)_n = x(x-1)\cdots(x-n+1)$  if  $n \geq 1$  and  $(x)_0 = 1$ .

There are several directions for further research that are suggested by the results of this paper. First, one can study the distribution in up-down and down-up permutations of other quadrant marked meshed patterns  $MMP(a, b, c, d)$  in the case where  $a, b, c, d \in \{\emptyset, 1\}$ . More generally, one can study the distribution of quadrant marked mesh patterns on other classes of pattern-restricted permutations such as *2-stack-sortable permutations* or *vexillary permutations* (see [6] for definitions of these) and many other permutation classes having nice properties. Finally, we conjecture that the polynomials  $A_{2n}(x)$ ,  $B_{2n+1}(x)$ ,  $C_{2n}(x)$ , and  $D_{2n+1}(x)$  are unimodal for all  $n \geq 1$ . This is certainly true for small values of  $n$ .

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