Counting independent sets in Riordan graphs*

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Abstract

The notion of a Riordan graph was introduced recently, and it is a far-reaching generalization of the well-known Pascal graphs and Toeplitz graphs. However, apart from a certain subclass of Toeplitz graphs, nothing was known on independent sets in Riordan graphs.

In this paper, we give exact enumeration and lower and upper bounds for the number of independent sets for various classes of Riordan graphs. Remarkably, we offer a variety of methods to solve the problems that range from the structural decomposition theorem to methods in combinatorics on words. Some of our results are valid for any graph.

Keywords: Riordan graph, Toeplitz graph, independent set, pattern avoiding sequence, Fibonacci number, Pell number, Hamiltonian path

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1 Introduction

An *independent set*, also known as a *stable set*, of a graph is a set of vertices in the graph, no two of which are adjacent. A *maximum independent set* for a given graph G is an independent set of largest possible size. This size is called the *independence number* of G, and is denoted by $\alpha(G)$, or simply by α if G is clear from the context. The problem of finding such a set is called the *maximum independent set problem*, and it is an NP-hard optimization problem (see [1] for a survey paper). Independent sets in graphs are studied extensively in the literature for various classes of graphs (see [16] for a short survey paper).

In this paper, we study the number of independent sets in "Riordan graphs" which have been introduced recently in [2, 3]. Riordan graphs have a number of interesting properties [2] and applications such as creating computer networks with certain desirable features and designing algorithms to compute values of graph invariants [7].

A *Riordan matrix* $L = [\ell_{ij}]_{i,j\geq 0}$ generated by two formal power series $g = \sum_{n=0}^{\infty} g_n z^n$ and $f = \sum_{n=1}^{\infty} f_n z^n$ in $\mathbb{Z}[[z]]$ is denoted as (g, f) and defined as an infinite lower triangular matrix whose j-th column generating function is gf^j, *i.e.* $\ell_{ij} = [z^i]gf^j$ where $[z^k]\sum_{n\geq 0} a_n z^n = a_k$. If $g_0 \neq 0$ and $f_1 \neq 0$ then the Riordan matrix is called *proper*. For example, $P = \left(\frac{1}{1-z}, \frac{z}{1-z}\right)$ is proper and $T = \left(z + z^2 + z^3, z\right)$ is not proper.

A simple graph G of order n is said to be a *Riordan graph* if the vertices of G can be labelled as $1, 2, \ldots, n$ so that its adjacency matrix $A(G)$ whose ith row corresponds to the vertex i can be expressed as

$$A(G) \equiv (zg, f)_n + (zg, f)_n^T \pmod 2 \tag{1}$$

for some generating functions g and f over $\mathbb{Z}$ where $(zg, f)_n$ is the $n \times n$ leading principle matrix of the Riordan matrix (zg, f). The Riordan graph G on n vertices with the adjacency matrix $A(G)$ given by (1) is denoted as $G = G_n(g, f)$. If we let $A(G) = (a_{i,j})_{1\leq i,j\leq n}$, then, for $i \geq j$,

$$a_{i,j} = a_{j,i} \equiv [z^{i-2}]gf^{j-1} \pmod 2$$

by (1). In particular, if $[z^0]g \equiv [z^1]f \equiv 1 \pmod 2$, then the graph $G_n(g, f)$ is called *proper*.

There are several naturally defined classes/families of Riordan graphs [2, 3]. In this paper, we focus on two types of Riordan graphs: the Appell type and the Bell type. A Riordan graph $G_n(g, z)$ is said to be *of the Appell*

n	1	2	3	4	5	6	7	8	9	10	11	12
$i(PG_n)$	2	3	4	6	7	12	15	23	24	46	60	98
$i(MG_n)$	2	3	4	7	9	13	17	26	29	48	55	95
$i(CG_n)$	2	3	4	7	8	14	21	35	36	60	81	134

Table 1: The number of independent sets for $1 \leq n \leq 12$

type. Riordan graphs of the Appell type are also known as *Toeplitz graphs*. Toeplitz graphs have been studied in [6, 15, 8, 12, 9].

The adjacency matrix $A(G_n) = (t_{i,j})_{1\leq i,j\leq n}$ of a Toeplitz graph G_n with n vertices is the $(0,1)$-symmetric matrix such that $ij \in E(G_n)$ with $j < i$ if and only if $t_{i-j+s,s} = t_{s,i-j+s} = 1$ for each $s = 1, \ldots, n-i+j$. Thus, one can see that a graph G is a Toeplitz graph $G_n(g, z)$ with n vertices for $g = z^{t_1-1} + z^{t_2-1} + \cdots + z^{t_k-1}$ if and only if there exist positive integers $t_1, t_2, \ldots, t_k \leq n-1$ such that $E(G) = \{ij \mid |i-j| = t_s,\ s = 1, \ldots, k\}$. In this context, we can represent a Toeplitz graph $G_n(g, z)$ for $g = z^{t_1-1} + z^{t_2-1} + \cdots + z^{t_k-1}$ as $T_n\langle t_1, t_2, \ldots, t_k\rangle$.

A Riordan graph $G_n(g, zg)$ is said to be *of the Bell type*. Moreover, a Riordan graph $G_n(g, zg)$ is called the *Pascal graph* and denoted by PG_n, the *Catalan graph* and denoted by CG_n, and the *Motzkin graph* and denoted by MG_n if $g = \frac{1}{1-z}$, $g = \frac{1-\sqrt{1-4z}}{2z}$, and $g = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$, respectively.

See Table 1 for our experimental computations for the numbers of independent sets in Pascal graphs, Motzkin graphs, and Catalan graphs with small numbers of vertices. These data can be used to measure the sharpness of the upper bounds obtained in this paper.

The following theorem regarding the adjacency matrices of Riordan graphs plays a key role throughout the paper. Given a positive integer n, we let

$$N_o := \{2i-1 \mid 1 \leq i \leq \lceil n/2 \rceil\} \quad \text{and} \quad N_e := \{2i \mid 1 \leq i \leq \lfloor n/2 \rfloor\}.$$

For a given graph G with n vertices labeled by $1, \ldots, n$, we denote by $\langle V_o \rangle$ and $\langle V_e \rangle$ the subgraphs of G induced by N_o and N_e, respectively.

Theorem 1.1 (Riordan Graph Decomposition, [2]). *Let $G_n = G_n(g, f)$ be a Riordan graph with $[z^1]f = 1$. Then,*

(i) *the induced subgraph $\langle V_o \rangle$ is a Riordan graph of order $\lceil n/2 \rceil$ given by $G_{\lceil n/2 \rceil}(g'(\sqrt{z}), f(z))$;*

(ii) *the induced subgraph $\langle V_e \rangle$ is a Riordan graph of order $\lfloor n/2 \rfloor$ given by $G_{\lfloor n/2 \rfloor}\left(\left(\frac{gf}{z}\right)'(\sqrt{z}), f(z)\right)$;*

(iii) *For the adjacency matrix $A(G_n)$, suppose that*

$$A(G_n) = P^T \begin{bmatrix} X & B \\ B^T & Y \end{bmatrix} P \tag{2}$$

where $P = \left[e_1 \mid e_3 \mid \cdots \mid e_{2\lceil n/2\rceil-1} \mid e_2 \mid e_4 \mid \cdots \mid e_{2\lfloor n/2\rfloor}\right]^T$ is the $n \times n$ permutation matrix and e_i is the elementary column vector with the i-th entry equal 1 and the other entries equal 0. Then, the matrix B representing the adjacency of a vertex in V_o and a vertex in V_e can be expressed as the sum of two Riordan matrices as follows:

$$B \equiv (z \cdot (gf)'(\sqrt{z}), f(z))_{\lceil n/2\rceil \times \lfloor n/2\rfloor} + ((zg)'(\sqrt{z}), f(z))^T_{\lfloor n/2\rfloor \times \lceil n/2\rceil} \pmod 2.$$

Let $G_n = G_n(g, f)$ be a proper Riordan graph. If $\langle V_o\rangle \cong G_{\lceil n/2\rceil}(g, f)$ and $\langle V_e\rangle$ is a null graph, then G_n is said to be *isomorphically odd decomposable* and is abbreviated as *io-decomposable*.

In the rest of this paper, we let $[n] := \{1, 2, \ldots, n\}$.

This paper is organized as follows. In Section 2, we study the number of independent sets in a Toeplitz graph which is a Riordan graphs of the Appell type. We give a lower bound and an upper bound for the number of independent sets in a Toeplitz graph and provide the exact number of independent sets in a chordal Toeplitz graph in terms of the Fibonacci numbers. In Section 3, we study the number of independent sets in a Riordan graph of the Bell type. We give upper bounds for various Riordan graphs of the Bell type in terms of the Pell numbers. We also find the independence number and the number of maximal independent sets in an io-decomposable Riodan graph of the Bell type. Then, we give a lower bound for the number of independent sets in an io-decomposable Riordan graph through a decomposition of graphs. Finally, in Section 4 we discuss directions of further research.

2 Independent sets in Toeplitz graphs

In [13], independent sets on *path-schemes* are considered. As a matter of fact, the class of path-schemes is precisely the class of Toeplitz graphs. Explicit enumeration of independent sets in terms of generating functions is obtained in [13] for a subclass of Toeplitz graphs defined by the notion of a *well-based sequence* (see [19] for enumerative properties of such sequences). The combinatorics on words approach used in the enumeration relies on the methods developed in [10] to count pattern-avoiding words. In fact, a result in [13] gives an upper bound on the number of independent sets for any Toeplitz graph, which was not mentioned in [13].

Suppose that $k \geq 2$ and $\mathcal{A} = \{A_1, \ldots, A_k\}$ is a set of words of the form $A_i = 1\underbrace{0\ldots0}_{a_i-1}1$, where $a_i \geq 1$, and $a_i < a_j$ if $i < j$. Moreover, we assume that for any $i > 1$ and $A_i \in \mathcal{A}$, if we replace any number of 0's in A_i by 1's, then we obtain a word A'_i that contains the word $A_j \in \mathcal{A}$ as a factor for some $j < i$. In this case, we call $\mathcal{A}$ a *well-based set*, and we call the sequence of a_i's associated with $\mathcal{A}$ a *well-based sequence*. It is known that any well-based set must contain the word 11 [13].

Theorem 2.1. *([13]) Let $t_1, t_2, \ldots, t_k$ be a well-based sequence with $t_1 = 1$ where $\{t_1, t_2, \ldots, t_k\}$ is a subset of $[n]$. Let $c(x) = 1 + \sum_{i=1}^{k} x^{t_i}$. Then the generating function for the number of independent sets in $T_n\langle t_1, t_2, \ldots, t_k\rangle$ with the vertex set $V = [n]$ is given by*

$$G(x) = \frac{c(x)}{(1-x)c(x) - x}.$$

The above theorem can be restated to give a lower bound for the number of independent sets in any Toeplitz graph.

Theorem 2.2. *Let $G_n = T_n\langle t_1, t_2, \ldots, t_k\rangle$ be a Toeplitz graph with n vertices, and $A_n = \{t_1, t_2, \ldots, t_k\}$. Further, let $B_n = \{b_1, b_2, \ldots, b_t\}$ be a minimal, possibly empty subset of $[n]$ such that $A_n \cap B_n = \emptyset$ and the elements of $A_n \cup B_n$ form a well-based sequence. Finally, let $c(x) = 1 + \sum_{i=1}^{k} x^{t_i} + \sum_{j=1}^{t} x^{b_j}$. Then,*

$$i(G_n) \geq [x^n]\frac{c(x)}{(1-x)c(x) - x},$$

where the equality holds if and only if $B_n = \emptyset$, that is, the elements of A_n form a well-based sequence.

Proof. Let $s_1, \ldots, s_{t+k}$ be an increasing sequence formed by the element in $A_n \cup B_n$ and $G'_n := T_n\langle s_1, \ldots, s_{t+k}\rangle$. Since G_n is a spanning subgraph of G'_n, we have $i(G_n) \geq i(G'_n)$ where the equality holds if and only if $B_n = \emptyset$. Since the elements of $A_n \cup B_n$ form a well-based sequence, Theorem 2.1 can be applied to the Toeplitz graph G'_n to conclude that

$$\sum_{i \geq 0} i(G'_n)x^i = \frac{c(x)}{(1-x)c(x) - x}$$

and the desired result follows. $\square$

Let $F_k(n)$ be the *n-th k-generalized Fibonacci number* defined by

$$F_k(1) = \cdots = F_k(k) = 1 \text{ and } F_k(n) = F_k(n-1) + F_k(n-k) \text{ for } n > k.$$

In particular, when $k = 2$, $F_k(n)$ is the *n-th Fibonacci number* $F(n)$.

Lemma 2.3 ([13]). *$F_k(n+k)$ counts the number of independent sets in $T_n\langle 1, 2, \ldots, k-1\rangle$.*

Now, we characterize the Riordan graphs G_n satisfying $i(G_n) \le F_k(n)$.

Theorem 2.4. *Let $k \ge 3$ and $G_n^{(k)} = G_n(g, f)$ be a Riordan graph such that $[z^i]g \equiv 1$ for each $i = 0, \ldots, k-2$, $[z^1]f \equiv 1$ and $[z^j]f \equiv 0 \pmod 2$ for each $j = 2, \ldots, k-1$. Then,*

$$i(G_n^{(k)}) \le F_k(n+k)$$

and the equality holds if and only if $G_n^{(k)} = T_n\langle 1, 2, \ldots, k-1\rangle$.

Proof. It is easy to see that if a graph H is a spanning subgraph of G, then $i(G) \le i(H)$ and the equality holds if and only if $G = H$. For each $k \ge 3$, let g and f be generating functions over $\mathbb{Z}$ such that

$$[z^i]g \equiv 1 \pmod 2 \quad \text{for each } i = 0, \ldots, k-2,$$

$$[z^1]f \equiv 1 \pmod 2 \text{ and } [z^j]f \equiv 0 \pmod 2 \quad \text{for each } j = 2, \ldots, k-1.$$

Then, we consider a Riordan graph

$$G_n^{(k)} = G_n(g, f).$$

By definition, $G_n(\sum_{i=0}^{k-2} z^i, z) = T_n\langle 1, 2, \ldots, k-1\rangle$. By Lemma 2.3,

$$i(T_n\langle 1, 2, \ldots, k-1\rangle) = F_k(n+k).$$

As a matter of fact, $T_n\langle 1, 2, \ldots, k-1\rangle$ is a spanning subgraph of $G_n^{(k)}$. Indeed, take two adjacent vertices u and v in $T_n\langle 1, 2, \ldots, k-1\rangle$. Without loss of generality, we assume that $u > v$. Then, $1 \le u - v \le k-1$. To see the adjacency of u and v in $G_n^{(k)}$, we compute $[z^{u-2}]gf^{v-1}$. By the definitions,

$$g = 1 + z + \cdots + z^{k-2} + O(z^{k-1}) \quad \text{and} \quad f = z + O(z^k). \tag{3}$$

By (3), the minimum degree of the nonzero term in f^{v-1} is $v-1$ and the second minimum degree of the nonzero term in f^{v-1} is at least $v+k-2$, if it exists. However, the term z^{u-v-k} does not exist in g since $u - v \le k-1$. Since $0 \le u-v-1 \le k-2$, we obtain $[z^{u-v-1}]g = 1$. Thus, $[z^{u-2}]gf^{v-1} = 1$, which is obtained by multiplying z^{v-1} in f^{v-1} by z^{u-v-1} in g. Therefore, u and v are adjacent in $G_n^{(k)}$ which completes the proof. $\square$

Even though in our paper we normally obtain lower and/or upper bounds for the number of independent sets in question, in this section, we find the *exact* number of independent sets in Toeplitz graphs related to chordal graphs. A *chordal graph* is a simple graph in which every graph cycle of length four and greater has a cycle chord.

Lemma 2.5 ([14]). *For $k \geq 1$, let $G_n := T_n\langle t_1, t_2, \ldots, t_k\rangle$ be a Toeplitz graph of order $n \geq t_k + t_{k-1} + 1$. Then, G_n is chordal if and only if $t_j = jt_1$ for each $j = 1, \ldots, k$.*

Lemma 2.6 ([14]). *For $k, t \geq 1$, let $G_n = T_n\langle t, 2t, \ldots, kt\rangle$ be a Toeplitz graph of order $n \geq (2k-1)t + 1$. Then, every connected component H_i, $i = 1, \ldots, t$, of G_n is isomorphic to the graph $T_{\lfloor (n-i)/t \rfloor + 1}\langle 1, 2, \ldots, k\rangle$, and the vertex set of H_i is*

$$V(H_i) = \{v \in [n] \mid v = i + st,\ s = 0, 1, \ldots\}.$$

For each i, we can easily check that any $k+1$ consecutive vertices in $T_{\lfloor (n-i)/t \rfloor + 1}\langle 1, 2, \ldots, k\rangle$ is a clique, so as H_i.

The following theorem follows immediately from Lemmas 2.5, 2.6 and 2.3.

Theorem 2.7. *For $k \geq 1$, let $G_n = T_n\langle t, 2t, \ldots, kt\rangle$ be a Toeplitz graph of order $n \geq (2k-1)t + 1$. Then,*

$$i(G_n) = \prod_{j=1}^{k} F_{k+1}\left(\left\lfloor \frac{n-j}{t} \right\rfloor + k + 2\right).$$

An independent set of a graph G is a clique in its complement graph $\overline{G}$. Therefore, the number of independent sets in G is the number of cliques in $\overline{G}$. In addition, it is well-known that the complement of a Toeplitz graph is also a Toeplitz graph. In the following theorem, we count the number of cliques in a chordal Toeplitz graph to give the number of independent sets in a Toeplitz graph.

Theorem 2.8. *For $k \geq 1$, let $G_n = T_n\langle t, 2t, \ldots, kt\rangle$ be a Toeplitz graph of order $n \geq (2k-1)t + 1$. Then, the number of cliques in G_n is*

$$(n - (k-1)t)\, 2^k.$$

Proof. By Lemmas 2.6, G_n has t components $H_1, \ldots, H_t$ and each component is isomorphic to $H_i := T_{\lfloor (n-i)/t \rfloor + 1}\langle 1, 2, \ldots, k\rangle$ for $i = 1, \ldots, t$. Let $\mathcal{K}_i$ be the collection of cliques in H_i. Since $n \geq (2k-1)t + 1$, we obtain

$|V(H_i)| = \lfloor (n-i)/t \rfloor + 1 \geq k+1$. Now, we partition $\mathcal{K}_i$ into subsets $W_1, \ldots, W_{|V(H_i)|-k}$ where

$$W_1 = \{K \mid K \subset [k+1]\} \quad \text{and} \quad W_j = \{K \cup \{k+j\} \mid K \subset \{j, \ldots, k+j-1\}\}$$

for $2 \leq j \leq |V(H_i)| - k$. Since any $k+1$ consecutive vertices of H_i form a clique, $|W_1| = 2^{k+1}$ and $|W_j| = 2^k$ for $2 \leq j \leq |V(H_i)| - k$. Therefore,

$$|\mathcal{K}_i| = \sum_{j=1}^{|V(H_i)|-k} |W_j| = (|V(H_j)| - k + 1)\, 2^k.$$

Since $\sum_{i=1}^{t} |V(H_i)| = n$, the number of cliques in G_n is

$$\sum_{i=1}^{t} |\mathcal{K}_i| = \sum_{i=1}^{t} (|V(H_i)| - k + 1) 2^k = (n - (k-1)t)\, 2^k$$

which completes the proof. □

The following result follows immediately from Theorem 2.8 and Lemma 2.5.

Corollary 2.9. *Let G_n be a Toeplitz graph with $\overline{G}_n := T_n\langle t, 2t, \ldots, kt\rangle$ for some positive integer t. Then,*

$$i(G_n) = (n - (k-1)t)\, 2^k.$$

3 Independent sets of the Bell type Riordan graphs

In Section 2, we introduced the notion of association of binary words to give a lower bound for the number of independent sets in a Toeplitz graph. In this section, we continue to utilize association of binary words to study the number of independent sets in Riordan graphs of the Bell type.

Let G be a graph with vertices $1, \ldots, n$. We can associate a subset of vertices in a graph G on n vertices with a binary word $x_1 x_2 \cdots x_n$, where $x_i = 1$ if the vertex i is chosen to the subset, and $x_i = 0$ otherwise. We say that a binary word is *good* if it corresponds to an independent set in G, and the word is *bad* otherwise. An independent set of G corresponds to a good word. Thus, the number of good words $x_1 x_2 \cdots x_n$ equals the number of independent sets in G.

A *factor* in a word $w = w_1 w_2 \cdots w_n$ is a subword of the form $w_i w_{i+1} \cdots w_j$. Given two binary sequences X and W, we say that X is W-*avoiding* if W

is not a factor of X. For example, a binary sequence 1010 is 11-avoiding whereas 1100 is not 11-avoiding. It is shown in [2, Theorem 3.1] that a proper Riordan graph has a Hamiltonian path. In what follows, we assume that the vertices of a proper Riordan graph $G_n(g, f)$ are labelled by $1, \ldots, n$ so that one of the Hamiltonian paths in $G_n(g, f)$ is $1 \to 2 \to \cdots \to n$. Thus, a vertex set S of a proper Riordan graph is an independent set only if the binary sequence associated with S is 11-avoiding. Hence, given a proper Riordan graph $G_n(g, f)$, an upper bound for the number of 11-avoiding good binary words $x_1 x_2 \cdots x_n$ is an upper bound for the number of independent sets in $G_n(g, f)$. Thus, in the arguments below, we estimate the number of 11-avoiding good words.

Recall that the n-th Fibonacci number $F(n)$ is defined recursively as follows: $F(0) = F(1) = 1$, $F(n) = F(n-1) + F(n-2)$. The following fact is well-known and is easy to prove.

Lemma 3.1. *The number of* 11*-avoiding binary words of length* $n \geq 0$ *is given by* $F(n+1)$*.*

Theorem 3.2. *If a graph* G *with* n *vertices has a Hamiltonian path* $1 \to \cdots \to n$*, then* $i(G) \leq F(n+1)$*. The equality holds for the path graph of length* $n-1$*.*

Proof. The first claim follows from Lemma 3.1 because any good word must avoid 11, or else two adjacent vertices on the Hamiltonian path $1 \to 2 \to \cdots \to n$ will be chosen to an independent set, which is impossible. The second claim follows from the fact that if G_n is a path graph with length $n-1$, then G_n has no edges apart from those on the Hamiltonian path, and thus any 11-avoiding binary word corresponds to an independent set in G_n. $\square$

Since a proper Riordan graph has a Hamiltonian path $1 \to 2 \to \cdots \to n$, we have the following corollary.

Corollary 3.3. *For a proper Riordan graph* G_n *on* n *vertices,* $i(G_n) \leq F(n+1)$ *with the equality holding if and only if* $G_n = G_n(1, z)$*.*

In fact, any graph we deal with in this section has a Hamiltonian path, since our concern will be in proper Riordan graphs, so we can assume that any graph on n vertices in this section has the Hamiltonian path $1 \to 2 \to \cdots \to n$.

We shall give an upper bound for the number of independent sets in an io-decomposable Riordan graph of the Bell type in terms of the Pell numbers. The *n-th Pell number*, denoted by P_n, is defined by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2},$$

and it is known that

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

and the respective generating function is given by

$$P(z) = \sum_{n\geq 0} P_n z^n = \frac{z}{1-2z-z^2}. \tag{4}$$

Let Δ_n and $\tilde{\Delta}_n$ denote the graphs with n vertices labeled by $1, 2, \ldots, n$, and the edge sets given by

$$E(\Delta_n) = \{i(i+1) \mid 1 \leq i < n\} \cup \{(2i-1)(2i+1) \mid 1 \leq i \leq \lfloor (n-1)/2 \rfloor\}$$

and

$$E(\tilde{\Delta}_n) = \{i(i+1) \mid 1 \leq i < n\} \cup \{(2i)(2i+2) \mid 1 \leq i \leq \lfloor (n-2)/2 \rfloor\},$$

respectively. One can easily see that $\Delta_{2n} \cong \tilde{\Delta}_{2n}$ and $\Delta_{2n+1} \ncong \tilde{\Delta}_{2n+1}$ for $n \geq 1$. We denote $i(\Delta_n)$ and $i(\tilde{\Delta}_n)$ by δ_n and $\tilde{\delta}_n$, respectively. We set $\delta_0 = \tilde{\delta}_0 = 1$.

Lemma 3.4. *For $n \geq 1$, we have*

$$\delta_n = \begin{cases} 2P_{(n+1)/2} & \textit{if } n \textit{ is odd,} \\ P_{n/2} + P_{n/2+1} & \textit{otherwise;} \end{cases} \tag{5}$$

and

$$\tilde{\delta}_n = \begin{cases} P_{(n-1)/2} + 2P_{(n+1)/2} & \textit{if } n \textit{ is odd,} \\ P_{n/2} + P_{n/2+1} & \textit{otherwise.} \end{cases} \tag{6}$$

Proof. Let B_n be the collection of binary words $x_1x_2\cdots x_n$ such that

$$\text{both } x_1x_2\cdots x_n \text{ and } x_1x_3\cdots x_{2\lceil n/2\rceil-1} \text{ are 11-avoiding.} \tag{7}$$

Since Δ_n and the subgraph of Δ_n induced by $\{1, 3, \ldots, 2\lceil n/2\rceil - 1\}$ are Hamiltonian, $\delta_n \leq |B_n|$. In addition, since the set of edges of Δ_n can be partitioned into the set of edges on the path $1 \to 2 \to \cdots \to n$ and the set of edges on the path $1 \to 3 \to \cdots \to (2\lceil n/2 \rceil - 1)$, $|B_n| \le \delta_n$ and so

$$|B_n| = \delta_n. \tag{8}$$

By definition, every binary word in B_n is of the following form

$$x_1 x_2 \cdots x_{n-1} 0 \ \text{ or } \ x_1 x_2 \cdots x_{n-2} 01.$$

(†) In particular, if n is odd, then every word in B_n is of the form

$$x_1 x_2 \cdots x_n 0 \ \text{ or } \ x_1 x_2 \cdots x_{n-3} 001$$

by the second condition in (7). Thus, for any positive integer m,

$$\delta_{2m} = \delta_{2m-1} + \delta_{2m-2} \ \text{ and } \ \delta_{2m+1} = \delta_{2m} + \delta_{2m-2} \tag{9}$$

with the initial values $\delta_0 = 1, \delta_1 = 2$ and $\delta_2 = 3$. Using (9), one can obtain the following generating function

$$\sum_{n \ge 0} \delta_n z^n = \frac{1 + 2z + z^2}{1 - 2z^2 - z^4}. \tag{10}$$

Hence, by (4) and (10), we obtain (5).

Let $\tilde{B}_n$ be the collection of binary words $x_1 x_2 \cdots x_n$ such that

$$\text{both } x_1 x_2 \cdots x_n \text{ and } x_2 x_4 \cdots x_{2\lfloor n/2 \rfloor} \text{ are 11-avoiding.} \tag{11}$$

By applying an argument similar to the one used for B_n, one can show that

$$|\tilde{B}_n| = \tilde{\delta}_n \tag{12}$$

whose generating function is

$$\sum_{n \ge 0} \tilde{\delta}_n z^n = \frac{1 + 2z + z^2 + z^3}{1 - 2z^2 - z^4}.$$

Therefore, by (4), we obtain (6). Hence the proof is complete. □

Let G_n be an io-decomposable Riordan graph of the Bell type with n vertices where $2^k < n \le 2^{k+1}$ for an integer $k \ge 2$. Then, G_n and $\langle V_o \rangle$ are proper, so G_n contains the path $1 \to 2 \to \ldots \to n$ and the path $1 \to 3 \to \ldots \to (2\lceil n/2 \rceil - 1)$ as a subgraph. Thus, Δ_n is a spanning subgraph of G_n and so

$$i(G_n) \le \delta_n. \tag{13}$$

The following theorem improves the upper bound (13).

Theorem 3.5. *Let G_n be an io-decomposable Riordan graph of the Bell type on n vertices. If $2^k < n \le 2^{k+1}$ for an integer $k \ge 2$, then*

$$i(G_n) \le \delta_n - (\delta_{2^k-2} - 1)\tilde{\delta}_{n-2^k-3} - \tilde{\delta}_{n-2^k-1} \sum_{i=1}^{k-1} (\delta_{2^i-2} - 1)\tilde{\delta}_{2^i-3}\alpha_{i+1} \quad (14)$$

where $\tilde{\delta}_m = 1$ for $m < 0$ and $\alpha_{i+1} = \begin{cases} 1 & \text{if } i = k-1 \\ \prod_{j=i+1}^{k-1} \tilde{\delta}_{2^j-1} & \text{otherwise.} \end{cases}$

Proof. It is known in the paper [2] that, for a positive integer n, an integer k satisfying $2^k < n \le 2^{k+1}$, and an io-decomposable Riordan graph of the Bell type G_n, we obtain the statement that

($\star$) for a vertex $u \in C := \{1, 2, 3, 5, 9, \ldots, 2^k + 1\}$ and a vertex $v \in V(G_n)$ less than u, u is adjacent to v.

Let $X = x_1x_2\cdots x_n$ be a binary word satisfying (7). By ($\star$), X is a bad word if

(‡) there are at least two indices i and j with $i < j$ such that $x_i = x_j = 1$ and $j \in C$.

Now, we consider the set A of bad words satisfying (7) and (‡). For each $i \in C$, let

$$A_i = \{X \in A \mid x_i = 1, x_j = 0 \text{ if } i < j \text{ and } j \in C\}. \quad (15)$$

Then, A_1 and A_2 are empty sets and $\{A_i \mid i \in C \setminus \{1, 2\}\}$ is a collection of disjoint subsets of A.

By (†), we have the following statements:

- a bad word in A_{2^k+1} is of the following form

$$x_1 \cdots x_{2^k-2}00x_{2^k+1}00x_{2^k+4} \cdots x_n;$$

- a bad word in A_{2^i+1} for some $i = 2, 3, \ldots, k-1$ is of the following form

$$x_1 \cdots x_{2^i-2}00x_{2^i+1}00x_{2^i+4} \cdots x_{2^{i+1}}0x_{2^{i+1}+2} \cdots x_{2^k}0x_{2^k+2} \cdots x_n.$$

Let S be a subgraph of Δ_n induced by the vertex set $\{s, \ldots, s+t\} \subseteq V(\Delta_n)$ for some positive integers $s \le n$ and $t \le n - s$. Then S is isomorphic to Δ_{t+1} and $\tilde{\Delta}_{t+1}$ if s is odd and even, respectively. Thus, by (8) and (12), we obtain the following statements.

(i) The number of words $x_1 \cdots x_{2^i-2}$ with at least one 1 satisfying (7) is $\delta_{2^i-2} - 1$ for each $2 \le i \le k$.

(ii) The number of words $x_{2^k+4} \cdots x_n$ satisfying (7) is $\tilde{\delta}_{n-2^k-3}$ if $n-2^k-3 \ge 0$.

(iii) The number of words $x_{2^i+4} \cdots x_{2^{i+1}}$ satisfying (7) is $\tilde{\delta}_{2^i-3}$ for each $2 \le i \le k-1$.

(iv) The number of words $x_{2^{i+1}+2} \cdots x_{2^{i+2}}$ satisfying (7) is $\tilde{\delta}_{2^{i+1}-1}$ for each $2 \le i < k-1$.

(v) The number of words $x_{2^k+2} \cdots x_n$ satisfying (7) is $\tilde{\delta}_{n-2^k-1}$.

We let $\tilde{\delta}_{n-2^k-3} = 1$ if $2^k < n < 2^k - 3$. Then, by observations (i)–(v), we have

$$|A_{2^k+1}| = (\delta_{2^k-2} - 1)\tilde{\delta}_{n-2^k-3}, \quad |A_{2^{k-1}+1}| = \tilde{\delta}_{n-2^k-1}(\delta_{2^k-2} - 1)\tilde{\delta}_{2^{k-1}-3},$$

and

$$|A_{2^i+1}| = \tilde{\delta}_{n-2^k-1}(\delta_{2^i-2} - 1)\tilde{\delta}_{2^i-3} \prod_{j=i+1}^{k-1} \tilde{\delta}_{2^j-1}$$

for $2 \le i < k-1$. Hence we obtain the desired result. $\square$

Remark 3.6. *From the way we obtain inequality* (14)*, we can see the equality holds in* (14) *if G_n is a graph with n vertices labeled as $1, 2, \ldots, n$ and the edge set of G_n is given by*

$$E(G_n) = E(\Delta_n) \cup \{ij \mid 1 \le i < j \text{ and } j = 2, 3, 5, \ldots, 2^k + 1\}$$

for the integer k satisfying $2^k < n \le 2^{k+1}$. We have checked that the graph G_n is a Catalan graph CG_n for $1 \le n \le 5$, but it is not a Riordan graph for $n \ge 6$.

The Pascal graph $PG_n = G_n\left(\frac{1}{1-z}, \frac{z}{1-z}\right)$ is an io-decomposable Riordan graph of the Bell type and so it satisfies the upper bound given in Theorem 3.5. As a matter of fact, an upper bound for the number of independent sets in PG_n can be improved by utilizing some properties of PG_n.

Theorem 3.7. *Let $2^k < n \le 2^{k+1}$ and $k \ge 2$. Then,*

$$\begin{aligned} i(PG_n) \le &\delta_n + 1 + 2^{\lfloor n/2 \rfloor - 1} - (\delta_{2^k-2} - 1)\tilde{\delta}_{n-2^k-3} \\ &- \tilde{\delta}_{n-2^k-1}\left(2\prod_{i=1}^{k-1}\tilde{\delta}_{2^i-1} + \sum_{i=1}^{k-1}(\delta_{2^i-2} - 1)\tilde{\delta}_{2^i-3}\alpha_{i+1}\right) \end{aligned}$$

where $\tilde{\delta}_m = 1$ *for* $m < 0$ *and* $\alpha_{i+1} = \begin{cases} 1 & \text{if } i = k-1 \\ \prod_{j=i+1}^{k-1} \tilde{\delta}_{2^j-1} & \text{otherwise.} \end{cases}$

Proof. By the definition of PG_n, $ij \in E(PG_n)$ if and only if $[z^{i-2}]\frac{z^{j-1}}{(1-z)^j} \equiv 1 \pmod 2$ where $n \geq i > j \geq 1$. By substituting $j = 1$ and $j = 2$, we obtain the following two facts:

(P1) the vertex 1 in PG_n is adjacent to all other vertices;

(P2) the vertex 2 in PG_n is adjacent to all odd vertices for $n \geq 2$.

Let $C = \{1, 2, 3, 5, 9, \ldots, 2^k + 1\}$ and $B'_n = B_n \setminus (\cup_{i \in C} A_i)$ where B_n was defined in the proof of Lemma 3.4 and A_i was defined in (15). Consider the following two sets of words $X := x_1 x_2 \cdots x_n$:

$A'_1 := \{X \in B_n \mid x_1 = 1, x_j = 0 \text{ for each } j \in C \setminus \{1\}, x_l = 1 \text{ for some } l \notin C\}$;

$A'_2 := \{X \in B_n \mid x_2 = 1, x_j = 0 \text{ for each } j \in C \setminus \{2\}, x_l = 1 \text{ for some odd } l \notin C\}$.

Since $X \notin \cup_{i \in C} A_i$ if $X \in A'_1 \cup A'_2$, $A'_1 \cup A'_2$ is a subset of B'_n.

By (P1) and (P2), we can see that A'_1 and A'_2 are sets of bad words associated with PG_n. Now, we count the words in each of A'_1 and A'_2, and subtract them from $|B'_n|$ to improve the upper bound given in Theorem 3.5.

We count the number of words in A'_j for each $j = 1, 2$. Fix $j \in \{1, 2\}$. Take a word $X = x_1 x_2 \cdots x_n$ in A'_j. Then, by the definition of A'_j, $x_{2^i+1} = 0$ for each $1 \leq i \leq k$; $x_1 = 1$ and $x_2 = x_3 = 0$ if $j = 1$ and $x_1 = x_3 = 0$ and $x_2 = 1$ if $j = 2$. Therefore, in order to count the words in A'_j, it is sufficient to count the number of ways to determine the factors

$$x_4, x_6 x_7 x_8, \ldots, x_{2^{k-1}+2} x_{2^{k-1}+3} \cdots x_{2^k}, x_{2^k+2} x_{2^k+3} \cdots x_n$$

so that X satisfies (7). Take $i \in \{1, \ldots, k-1\}$. Since $X \in B_n$, X satisfies (7). Thus, the factor $x_{2^i+2} x_{2^i+3} \cdots x_{2^{i+1}}$ satisfies (11). Therefore, $X \in A'_j$ if and only if the factor $x_{2^i+2} x_{2^i+3} \cdots x_{2^{i+1}}$ satisfies for each $i \in \{1, \ldots, k-1\}$; $X \neq 100\ldots0$ for $j = 1$, and not all odd entries of X are 0 for $j = 2$.

For each $i \in \{1, \ldots, k-1\}$, there are $\tilde{\delta}_{2^i-1}$ $x_{2^i+2} x_{2^i+3} \cdots x_{2^{i+1}}$ factors satisfying (11). In addition, there are $\tilde{\delta}_{n-2^k-1}$ $x_{2^k+2} x_{2^k+3} \cdots x_n$ factors satisfying (11). Thus,

$$|A'_1| = \tilde{\delta}_{n-2^k-1} \prod_{i=1}^{k-1} \tilde{\delta}_{2^i-1} - 1. \tag{16}$$

In particular, if $j = 2$, there are $2^{\lfloor \frac{n}{2} \rfloor - 1}$ words in B'_n such that $x_i = 1$ only if i is even. Thus,

$$|A'_2| = \tilde{\delta}_{n-2^k-1} \prod_{i=1}^{k-1} \tilde{\delta}_{2^i-1} \; - \; 2^{\lfloor \frac{n}{2} \rfloor - 1}. \tag{17}$$

By (16) and (17), we obtain the desired result. □

The tools which we have developed to utilize association of binary words can be used in finding the independence number and the number of *maximal* independent sets in an io-decomposable Riordan graph of the Bell type.

Theorem 3.8. *The independence number of an io-decomposable Riordan graphs is $\lfloor n/2 \rfloor$. In particular, the number of maximal independent sets in an io-decomposable Riordan graph is at most 2 if n is even and at most 4 if n is odd.*

Proof. Let G_n be an io-decomposable Riordan graph and $\alpha(G_n)$ denote the independence number of G_n. First we show that $\alpha(G_n) = \lfloor n/2 \rfloor$. Since $\langle V_e \rangle$ is a null graph, V_e is an independent set. Thus, $\alpha(G_n) \geq |V_e| = \lfloor n/2 \rfloor$. Since G_n is proper, a good binary word $x_1 x_2 \cdots x_n$ associated with G_n is 11-avoiding. It leads to $\alpha(G_n) \leq \lfloor n/2 \rfloor$. Thus, $\alpha(G_n) = \lfloor n/2 \rfloor$.

Let d be an odd integer with $3 \leq d < n$. Since $\langle V_o \rangle$ is proper, a subword $x_1 x_3 \cdots x_{2\lceil n/2 \rceil - 1}$ of a good binary word $x_1 x_2 \cdots x_n$ associated with G_n is 11-avoiding as well. Therefore, if an independent set I contains a vertex d, then I contains none of $d-2$, $d-1$, $d+1$ and so $|I| \leq \frac{n-3}{2} < \lfloor n/2 \rfloor$. Thus, there is no maximal independent set containing an odd integer in between 1 and n. Hence, the possible maximal independent sets are given by the following:

(i) V_e and $(V_e \setminus \{2\}) \cup \{1\}$ if n is even;

(ii) V_e, $(V_e \setminus \{2\}) \cup \{1\}$, $(V_e \setminus \{n-1\}) \cup \{n\}$ and $(V_e \setminus \{2, n-1\}) \cup \{1, n\}$ if $n \geq 5$ is odd,

which implies the desired result. □

Corollary 3.9. *The Pascal graph PG_n has a unique maximal independent set V_e if n is an even integer greater than 2 or $n = 2^k + 1$ for some integer $k \geq 2$.*

Proof. By Theorem 3.8, $\alpha(PG_n) = \lfloor \frac{n}{2} \rfloor$. Since $\lfloor \frac{n}{2} \rfloor > 1$ for $n \geq 4$, a maximal independent set of PG_n has the size greater than 1 if $n \geq 4$.

Assume that n is an even integer greater than 2 or $n = 2^k + 1$ for some integer $k \geq 2$. By the definition of Pascal graph, the vertex 1 is adjacent to all the other vertices in PG_n for all $n \geq 2$. Therefore, an independent set which contains 1 cannot have any other elements. If n is an even integer greater than 2, then a maximal independent set of PG_n does not contain 1 for $n \geq 4$ and so V_e is a unique maximal independent set of PG_n by (i) in the proof of Theorem 3.8.

Now, we assume that $n = 2^k + 1$ for some integer $k \geq 2$. It is known [2] that the vertex $2^k + 1$ in an io-decomposable Riordan graph $G_n(g, zg)$ is adjacent to all the other vertices if $n = 2^k + 1$. Therefore, an independent set which contains $2^k + 1$ cannot have any other elements. Since $k \geq 2$, $2^k + 1 \geq 4$ and so a maximal independent set of PG_n contains none of 1 and $2^k + 1$. Thus, V_e is a unique maximal independent set of PG_n by (ii) in the proof of Theorem 3.8. $\square$

Next, we give a lower bound for the number of independent sets in a graph through a decomposition of graphs. For a $(0, 1)$-matrix M, we denote by $\sigma_0(M)$ and $\sigma_1(M)$ the number of 0's in M and the number of 1's in M, respectively.

Theorem 3.10. *Let G be a graph with n vertices labeled by $1, 2, \dots, n$. Then*

$$i(G) \geq i(\langle V_o \rangle) + i(\langle V_e \rangle) + \lceil n/2 \rceil \, \lfloor n/2 \rfloor - |E(G)| + |E(\langle V_o \rangle)| + |E(\langle V_e \rangle)|.$$

Proof. Let A be the adjacency matrix of G such that the rows and columns $1, 2, \dots, n$ correspond to the vertices labeled in the order $1, 3, \dots, 2\lceil n/2 \rceil - 1, 2, 4, \dots, 2\lfloor n/2 \rfloor$. Then,

$$A = \begin{bmatrix} X & B \\ B^T & Y \end{bmatrix}$$

where X is the adjacency matrix of $\langle V_o \rangle$ and Y is the adjacency matrix of $\langle V_e \rangle$. We consider three types of independent sets in G as follows: the independent sets in $\langle V_o \rangle$; the independent sets in $\langle V_e \rangle$; the independent sets of size 2 formed by an element in V_o and an element in V_e. Note that an independent set of the third type corresponds to a 0 in B. Therefore,

$$i(G_n) \geq i(\langle V_o \rangle) + i(\langle V_e \rangle) + \sigma_0(B) \tag{18}$$

Since an edge between V_o and V_e corresponds to a 1 in B, we have

$$\sigma_1(B) = |E(G)| - |E(\langle V_o \rangle)| - |E(\langle V_e \rangle)|.$$

Since $\sigma_0(B) + \sigma_1(B) = \lceil n/2 \rceil \lfloor n/2 \rfloor$,

$$\sigma_0(B) = \lceil n/2 \rceil \, \lfloor n/2 \rfloor - |E(G)| + |E(\langle V_o \rangle)| + |E(\langle V_e \rangle)|.$$

By substituting this into (18), we obtain the desired result. □

The following result is a corollary of Theorem 3.10.

Corollary 3.11. *Let $G_n := G_n(g, f)$ be an io-decomposable Riordan graph with $n \geq 2$ vertices. Then,*

$$i(G_n) \geq i(G_{\lceil n/2 \rceil}) + 2^{\lfloor n/2 \rfloor} + \lceil n/2 \rceil \, \lfloor n/2 \rfloor - |E(G_n)| + |E(G_{\lceil n/2 \rceil})|.$$

Proof. By the definition of an io-decomposable Riordan graph with n vertices labeled by $1, \ldots, n$, $\langle V_o \rangle \cong G_{\lceil n/2 \rceil}$ and $E(\langle V_e \rangle) = \emptyset$, so the result follows. □

In what follows, we give a lower bound for the number of independent sets in io-decomposable Riordan graphs of the Bell type. We first present two properties of io-decomposable Riordan graphs of the Bell type.

Lemma 3.12 ([2])**.** *An io-decomposable Riordan graph $G_n(g, zg)$ is $(\lceil \log_2 n \rceil + 1)$-partite with the partitions $V_1, V_2, \ldots, V_{\lceil \log_2 n \rceil + 1}$ where*

$$V_j = \left\{ 2^{j-1} + 1 + (i-1)2^j \mid 1 \leq i \leq \left\lfloor \frac{n - 1 + 2^{j-1}}{2^j} \right\rfloor \right\}$$

for $1 \leq j \leq \lceil \log_2 n \rceil$ and $V_{\lceil \log_2 n \rceil + 1} = \{1\}$.

Lemma 3.13 ([2])**.** *Let $G_n(g, zg)$ be an io-decomposable Riordan graph. Then, the lines of the adjacency matrix of $G_n(g, zg)$ can be simultaneously permuted to have the matrix*

$$\begin{pmatrix} X & B \\ B^T & O \end{pmatrix}$$

where X is the adjacency matrix of $\langle V_o \rangle \cong G_{\lceil n/2 \rceil}(g(z), zg(z))$ and

$$B \equiv (zg, zg)_{\lceil n/2 \rceil, \lfloor n/2 \rfloor} + ((zg)'(\sqrt{z}), zg))^T_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \pmod 2. \tag{19}$$

Theorem 3.14. *Let G_n be an io-decomposable Riordan graph of the Bell type with $n \geq 2$ vertices. Then,*

$$i(G_n) \geq 2 - \lceil \log_2 n \rceil + \sum_{j=1}^{\lceil \log_2 n \rceil} \left(2^{\alpha_j} + \frac{\alpha_{j+1}^2 - \alpha_{j+1}}{2} \right)$$

where $\alpha_j = \left\lfloor \frac{n-1+2^{j-1}}{2^j} \right\rfloor$.

Proof. For each $1 \le j \le \lceil \log_2 n \rceil + 1$, the subgraph $G_n[V_j]$ induced by the vertex subset V_j defined in Lemma 3.12 is a null graph. In addition, $|V_j| = \alpha_j$ for each $1 \le j \le \lceil \log_2 n \rceil$ and $|V_{\lceil \log_2 n \rceil} + 1| = 1$. Thus

$$i(G_n[V_{\lceil \log_2 n \rceil + 1}]) = 2 \text{ and } i(G_n[V_j]) = 2^{\alpha_j} \tag{20}$$

for each $j = 1, \ldots, \lceil \log_2 n \rceil$.

Now, we count the 2-element independent sets each of which is formed by a vertex in V_i and a vertex in V_j for some $1 \le i < j \le \lceil \log_2 n \rceil$. By definition, $V_1 = \{2i \mid 1 \le i \le \lfloor \frac{n}{2} \rfloor\} = V_e$. Therefore, $\bigcup_{i=2}^{\lceil \log_2 n \rceil + 1} = V_o$ and so the number of independent sets $\{u, v\}$ for $u \in V_1$ and $v \in V_j$, $j > 1$ is the number of zeros in the matrix B given in Lemma 3.13. By (19), the number of zeros below the main diagonal in B is equals to the number of zeros below the main diagonal in $A(\langle V_o \rangle)$. Recall that $\langle V_2 \rangle$ is the null graph of order α_2. Since V_2 is a subgraph of V_o, there are at least $(\alpha_2^2 - \alpha_2)/2$ zeros below the main diagonal in $A(\langle V_o \rangle)$. Therefore, there are at least $(\alpha_2^2 - \alpha_2)/2$ independent sets $\{u, v\}$ for $u \in V_1$ and $v \in V(G_n) \backslash V_1$. Similarly, we can show that there are at least $(\alpha_{j+1}^2 - \alpha_{j+1})/2$ independent sets $\{u, v\}$ for $u \in V_j$ and $v \in \cup_{j < i \le n} V_i$ for each $2 \le j \le \lceil \log_2 n \rceil$. Hence, we have

$$i(G_n) \ge 2 - \lceil \log_2 n \rceil + \sum_{j=1}^{\lceil \log_2 n \rceil} \left(2^{\alpha_j} + \frac{\alpha_{j+1}^2 - \alpha_{j+1}}{2} \right)$$

where $\lceil \log_2 n \rceil$ is the number of the empty sets which overlapped in (20). $\square$

4 Directions of further research

This paper focuses on giving lower and upper bounds for the number of independent sets for various classes of Riordan graphs. Of course, the most challenging thing here is in finding exact enumeration in question, which does not seem to be feasible in the context due to the problem generality. In any case, there are other questions one can ask. For example, the Catalan graphs are io-decomposable of the Bell type, so the results of Theorem 3.5 can be applied to them. However, can we provide a more accurate upper bound, and some lower bound for this class of graphs?

In addition, Table 1 gives initial values for the number of independent sets for the Fibonacci graphs and Motzkin graphs. However, obtaining any lower/upper bounds, or exact enumeration, for the number of independent sets for these graphs in general remains an open problem.

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