Avoidance of boxed mesh patterns on permutations

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Abstract

We introduce the notion of a boxed mesh pattern and study avoidance of these patterns on permutations. We prove that the celebrated former Stanley-Wilf conjecture is not true for all but eleven boxed mesh patterns; for seven out of the eleven patterns the former conjecture is true, while we do not know the answer for the remaining four (length-four) patterns. Moreover, we prove that an analogue of a well-known theorem of Erdős and Szekeres does not hold for boxed mesh patterns of lengths larger than 2. Finally, we discuss enumeration of permutations avoiding simultaneously two or more length-three boxed mesh patterns, where we meet generalized Catalan numbers.

Keywords: boxed mesh pattern, enumeration, Stanley-WIlf conjecture, Erdős-Szekeres theorem, generalized Catalan numbers

1 Introduction

Permutations in this paper are presented in *one-line notation*. An occurrence of a "*classical*" *pattern* p in a permutation π is defined as a subsequence in π (of the same length as p) whose letters are in the same relative order as those in p. For example, the permutation 31425 has three occurrences of the pattern 123, namely the subsequences 345, 145, and 125. *Vincular patterns* allow the requirement that some adjacent letters in a pattern must also be adjacent in the permutation. We indicate this requirement by underlining the letters that must be adjacent. For example, if the pattern 2<u>31</u> occurs in a permutation π, then the letters in π that correspond to 3 and 1 are adjacent. For instance, the permutation 516423 has only one occurrence of the pattern 2<u>31</u>, namely the subsequence 564, whereas the pattern 231 occurs, in addition, as the subsequences 562 and 563.

Bivincular patterns are a generalization of vincular patterns, when letters of a pattern, as well as its positions, may be required to be consecutive. We refer to [3] for a comprehensive source of results on just discussed patterns.

In this paper, S_n denotes the number of permutations of length n and $s_n(p)$ is the number of permutations of length n avoiding a pattern p.

The notion of a *mesh pattern* was introduced by Brändén and Claesson [1] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns. These patterns are a generalization of bivincular patterns, and they were studied in a series of papers [2, 4, 5, 6, 9].

The notion of a mesh pattern can be best described using the permutation diagrams (for a more detailed description, we refer to [1, 9]). For example, the diagrams in Figure 3, after ignoring the shaded areas and paying attention to the height of dots (circles) while going through them from left-to-right, correspond to the permutations 3142 and 1342, respectively. A mesh pattern is the diagram corresponding to a permutation where some of squares determined by the grid are shaded. There are three mesh patterns in Figure 1 and two mesh patterns in Figure 3.

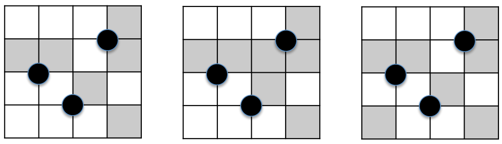

Figure 1: Three mesh patterns.

We say that a mesh pattern p of length k occurs in a permutation π if the permutation diagram of π contains k circles whose order is the same as that of the permutation diagram of p, that is, π contains a subsequence that is order-isomorphic to p, and, additionally, no element of π can be present in a shaded area determined by p and the corresponding elements of π in the subsequence. For example, the three circled elements in the permutation 82536174 in Figure 2 are an occurrence of the leftmost mesh pattern in Figure 1, as demonstrated by the diagram to the right in Figure 2 (no of the permutation elements fall into the shaded area determined by the mesh pattern). These (circled) elements are not an occurrence of the middle pattern in Figure 1 because of the element 6 in the permutation; they are not an occurrence of the rightmost pattern in Figure 1 because of the element 2 in the permutation. One can see, using the diagram in Figure 1, that the subsequence (actually, the factor) 536 in the permutation 82536174 is an occurrence of the leftmost and the middle, but not the righmost mesh patterns in Figure 1.

We will be interested in mesh patterns like the one to the right in Figure 3. In such patterns all but the boundary squares are shaded. We call these patterns *boxed mesh*

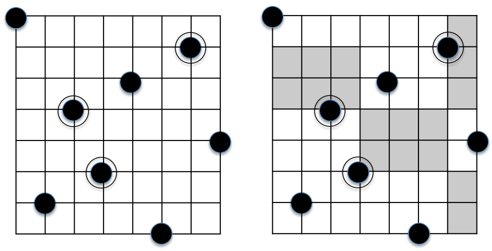

Figure 2: An example of an occurrence of a mesh pattern.

patterns or *boxed patterns* and we denote a boxed pattern by a rectangle containing the corresponding *underlying permutation*.

It is straightforward to see from definitions, but is rather useful, that a boxed pattern occurs in a permutation π if and only if there is a sequence of removals of the minimum or maximum or leftmost or rightmost elements in π that brings to the pattern's underlying permutation. For example, the subsequence 3674 in the permutation 82536174 presented in Figure 2 is the occurrence of the mesh pattern $\boxed{1342}$ (pictured to the right in Figure 3), and the sequence of removals discussed above consists of the elements 1, 2, 5 and 8. This property shows that the language of boxed patterns avoiding permutations is *factorial*, that is, removing one of the aforementioned elements in a permutation belonging to such a language gives a permutation inside the same language. Note that our boxed patterns can be seen as a generalization of consecutive patterns, where only removals of elements from the left and right sides in a permutation are allowed (in a consecutive pattern entire columns between the elements are shaded).

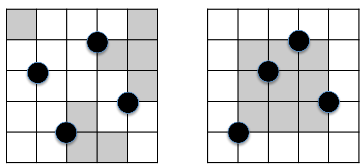

Figure 3: A mesh pattern and the boxed mesh pattern $\boxed{1342}$.

It is easy to see that avoidance of the three boxed patterns of lengths 1 or 2 is equivalent to avoidance of the corresponding classical patterns. For example, if there is an occurrence of the classical pattern 21 in a permutation π, then it is easy to see that there must be an occurrence of the consecutive pattern $\underline{21}$ (a descent) in π, which is an occurrence of the boxed pattern $\boxed{21}$; the converse is straightforward. What is more surprising is that avoidance of $\boxed{132}$ is equivalent to avoidance of the (classical) pattern 132. One can provide here the following argument similar to a known proof of equivalence, in the sense of avoidance of the patterns 132 and $1\underline{32}$.

Clearly if we have an occurrence of the pattern $\boxed{132}$ then we have an occurrence of the pattern 132. Vice versa, suppose xyz is an occurrence of the pattern 132 in a permutation π presented schematically in Figure 4. If there are no elements in the regions I, II, III and IV, then xyz is an occurrence of $\boxed{132}$ and we are done. On the other hand, if, say, III is non-empty, then we can pick any element x' in III, say, the maximum element in III, and $x'yz$ will still be an occurrence of the pattern 132 in π where x', y and z stay "closer" to each other. Similarly, we can consider cases of non-empty I, II and IV, repeating this procedure, if needed, until we get an occurrence of $\boxed{132}$.

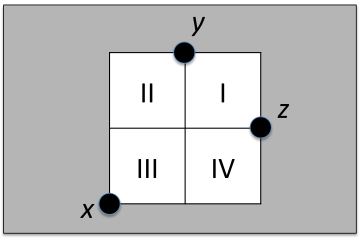

Figure 4: An occurrence of the pattern 132 in a permutation.

On the other hand, $s_n(123) \neq s_n(\boxed{123})$ for $n \geq 4$, thus 123 and $\boxed{123}$ are not equivalent in the sense of avoidance. The minimal permutation that avoids $\boxed{123}$ but contains 123 is 1324 in Figure 5.

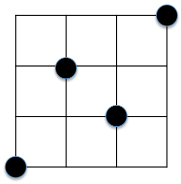

Figure 5: The only permutation of length 4 that avoids $\boxed{123}$ but contains 123.

It is straightforward to see that a permutation π avoids a boxed pattern p if and only if $i(\pi)$ avoids $i(p)$, $r(\pi)$ avoids $r(p)$ and $c(\pi)$ avoids $c(p)$, where i, r and c are (usual group theoretical) inverse, reverse and complement, respectively. For example, $i(2431) = 4132$, $r(41352) = 25314$ and $c(2143) = 3412$; applying any of these bijections, called *trivial bijections*, to a boxed pattern means applying the bijection to the underlying permutation and keeping the rectangle around it.

Proposition 1. *Except for the patterns* $\boxed{1}$, $\boxed{12}$, $\boxed{21}$, $\boxed{132}$, $\boxed{213}$, $\boxed{231}$ *and* $\boxed{312}$, *avoidance of a boxed pattern is never equivalent to avoidance of the corresponding classical pattern.*

Proof. The cases of the pattern $\boxed{123}$ and the other patterns of length at most 3 were discussed above. So, let p be a boxed pattern of length $k \geq 4$. We would be done if we would be able to prove that the number of $(k+1)$-permutations avoiding p and $(k+1)$-permutations avoiding the underlying permutation as a classical pattern are different. To this end, it is enough to construct one permutation of length $k+1$ that avoids p but contains the classical pattern corresponding to p.

Consider the four rightmost letters of p. We can assume that these letters end with an ascent; if not, we can apply the complement operation and run the same arguments. So, there are 12 possible endings of p, and for each of them, in Figure 6, we will show, using a square, a position to place one more element so that the resulting permutation will avoid p but will contain the corresponding classical pattern.

Figure 6 presents a place (not necessarily unique) to put a new element in each case. We will explain in detail two cases there, the top-leftmost one and the top-rightmost one; explanations for the other cases are similar. Note that in each case, our pattern p may have elements to the left of a 4×4 matrix under consideration, but not above, below or to the right. We will get use of the following approach: to show that p does not occur in a $(k+1)$-permutation, we need to make sure that removing the maximum or the minimum or the leftmost or the rightmost element we will not get the underlying permutation of p.

For the top-leftmost possibility (represented by the pattern 3214) if none of the four elements pictured is the element to be removed, then after removing one element, the obtained permutation will end on the pattern 2314, which is different from the pattern 3214, p is supposed to end on. On the other hand, if the topmost element was removed, the obtained permutation will be ending on the pattern 4231, still different from 3214; thus we do not get the underlying permutation of p. If the minimum element was removed (in the case when this element is minimum in p), then the obtained permutation ends on 3124. Finally, if the leftmost element was removed (which can only happen if p is of length 4) then we obtain the permutation 2314, still different from 3214.

For the top-rightmost case (represented by the pattern 2413), if none of the four elements pictured is the element to be removed, then after removing one element, the obtained permutation will end on the pattern 4213, which is different from the pattern 2413, and thus we do not get p. Removing the topmost, rightmost, bottommost and leftmost elements, one gets the patterns 2314, 2431, 1423 and 4213, respectively, none of which is the pattern 2413, and thus the whole permutation of length k cannot be p.

$\square$

Note that for all patterns but $\boxed{2143}$, $\boxed{3142}$, $\boxed{2413}$ and $\boxed{3412}$, we have an alternative proof of Proposition 1 by comparing Theorems 1, 2 and 3 discussed below.

The rest of the paper is organized as follows. In Section 2 we show that the former Stanley-Wilf conjecture is not true for all but eleven boxed mesh patterns (four length-four patterns remain unsolved). In Section 3 we prove that an analogue of a well-known theorem of Erdős and Szekeres does not hold for boxed mesh patterns of lengths larger than 2. In Section 4 we discuss enumeration of permutations avoiding simultaneously two or more length-three boxed mesh patterns. Finally, in Section 5 we discuss a few directions of further research.

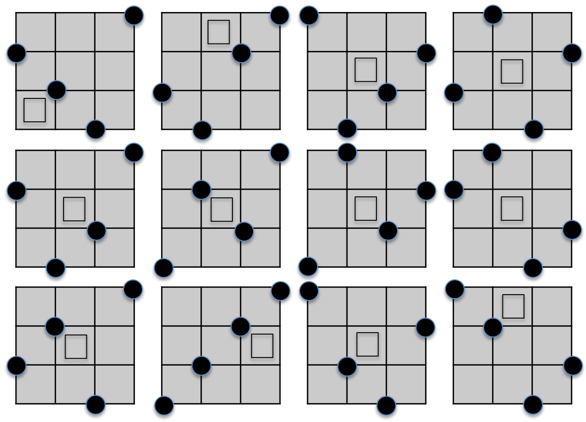

Figure 6: All possible endings with an ascent of a boxed k-pattern.

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2 The Stanley-Wilf conjecture and boxed mesh patterns

The former *Stanley-Wilf conjecture* (answered in affirmative in [7]) can be stated in the following form.

Theorem 1. *([7]) For any classical pattern $p \in S_k$, the limit $\lim_{n\to\infty}(s_n(p))^{\frac{1}{n}}$ exists and is finite.*

This conjecture is not true for vincular patterns, in particular, for consecutive patterns (see, e.g., [3]). Note that the Stanley-Wilf conjecture holds for the boxed pattern $\boxed{132}$ (and the equivalent to it patterns modulo trivial bijections) since its avoidance is equivalent to the avoidance of the classical pattern 132 (the same holds for boxed patterns of length 1 and 2). However, we will show that this conjecture is *not* true for the pattern $\boxed{123}$ (when we deal with a factorial growth) and we will generalize our argument to show that the Stanley-Wilf conjecture is *not* true for any boxed pattern of length larger than 3 except for the patterns $\boxed{2143}$, $\boxed{3142}$, $\boxed{2413}$ and $\boxed{3412}$ for which we do not know the answer. By trivial bijections, all of the four unknown cases are equivalent in the sense of avoidance.

Theorem 2. *We have $s_n(\boxed{123}) > \left(\lfloor \frac{n}{2} \rfloor\right)!$.*

Proof. Take any permutation of length n and replace each element x of it by $(x+1)x$ raising existing elements larger than x in the permutation by 1. For example, if the chosen permutation was 3142, then the resulting permutation will be 65218743. It is not difficult to see that such permutations avoid the pattern $\boxed{123}$. Indeed, if an element plays the role of the internal element in an occurrence of $\boxed{123}$. then its duplicated "sibling" will bring us to a contradiction (this cannot be an occurrence of $\boxed{123}$). The choice of the original permutation (before duplication) was arbitrary, and thus the result follows. $\square$

A direct corollary to Theorem 2 is that if a boxed pattern of length at least 4 contains three consecutive elements that are consecutive in value, then the Stanley-Wilf conjecture is not true for this pattern.

Theorem 3. *Let* $\boxed{p}$ *be any boxed pattern of length* $k \geq 4$ *which does not belong to the set* $\{\boxed{2143}, \boxed{3142}, \boxed{2413}, \boxed{3412}\}$. *Then* $s_n(\boxed{p}) > \left(\lfloor \frac{n}{2} \rfloor\right)!$.

Proof. We call a permutation $p = p_1 p_2 \cdots p_k$ *good* if there exists i, $1 < i < k$, such that $1 < p_i < k$ and the pattern built by $p_{i-1}p_ip_{i+1}$ is not monotone, that is, is different from 123 and 321. For example, the permutation $p = 51342$ is good since $1 < p_4 = 4 < 5$ and $p_3 = 3 < p_4 = 4 > p_5 = 2$. Let us consider two cases.

Case 1: p **is good.** There exists $xyz = p_{i-1}p_ip_{i+1}$ not forming a monotone pattern such that $1 < y < k$. Suppose xyz forms the pattern 213. Consider a length n permutation and substitute in it each element t by $t(t+1)$ to obtain the length $2n$ permutation π. For example, if we start with the permutation 231 then $\pi = 345612$. We will now show that π avoids the pattern $\boxed{p}$.

Suppose this is not true and π contains an occurrence of $\boxed{p}$. Then in this occurrence, the letter corresponding to y belongs to a pair $t(t+1)$. Suppose y corresponds to $t+1$. Since y is not the minimum element of p, we have that the letter t is located inside the box corresponding to the occurrence of $\boxed{p}$ (this box is a rectangle containing only elements of the occurrence), which is a contradiction with the fact that x stays next to y in $\boxed{p}$ and $x > y$. Thus y must correspond to t and $t+1$ is located inside the box corresponding to the occurrence of $\boxed{p}$. Since z is next to y in $\boxed{p}$, z must correspond to $t+1$. We get a contradiction with $y < x < z$ (x cannot exist). Thus π avoids $\boxed{p}$, and since we started with an arbitrary permutation of length n to obtain π, we see that $s_n(\boxed{p}) > \left(\lfloor \frac{n}{2} \rfloor\right)!$ as desired.

The case when xyz forms the pattern 231 (resp., 312, 132) follows from the already considered case by applying to $\boxed{p}$ the operation of complement (resp., reverse, reverse and complement) and getting a pattern avoidance of which gives the same number of permutations.

Case 2: p **is not good.** It is not difficult to see that in this case either $p = p'mp''Mp'''$ or $p = q'Mq''mq'''$, where m and M are the minimum and maximum elements in p, respectively, p', p''' and q'' are decreasing sequences, and q', q''' and p'' are increasing sequences. If p contains a monotone sequence of at least three consecutive elements, then whatever avoids $\boxed{123}$ or $\boxed{321}$ (depending on whether the monotone sequence in p is increasing or decreasing, respectively) will avoid $\boxed{p}$, and we are done by Theorem 2. If p does not contain a monotone sequence of at least three elements, then $p \in \{2143, 3142, 2413, 3412\}$. The theorem is proved.

□

3 Boxed mesh patterns and a theorem of Erdős and Szekeres

The well-known theorem of Erdős and Szekeres (e.g., see [3]) states that any sequence of $m\ell+1$ real numbers has either an increasing subsequence of length $m+1$ or a decreasing subsequence of length $\ell+1$. In particular, the increasing and decreasing patterns are unavoidable on permutations. A natural question is if a similar theorem holds in case when one or two of the patterns are allowed to be boxed. Clearly, if one of the monotone patterns is of length at most 2, these patterns are unavoidable (the length of the permutations avoiding them will be bounded). It turns out that otherwise (when the monotone patterns are of length at least 3), the patterns are avoidable as shown by the following proposition (dealing with length 3; larger lengths give weaker restrictions and will be given automatically).

Proposition 2. *For $n \geq 0$, the sequence $s_n(\boxed{123}, \boxed{321})$ is $1,1,2,4,6,4,4,4,4,4,\ldots$. For $n \geq 0$, the sequence $s_n(\boxed{123}, 321) = s_n(\boxed{321}, 123)$ is $1,1,2,4,5,2,2,2,2,2,\ldots$.*

Proof. The proposition is easy to prove by looking at generation of all permutations avoiding the patterns $\boxed{123}$ and $\boxed{321}$ simultaneously: we insert new elements to the right of permutations – see Figure 7. The only valid permutations will be of the forms $132547698\ldots$, $(n-1)n(n-3)(n-2)(n-5)(n-4)\ldots$, $21436587\ldots$ and $n(n-2)(n-1)(n-4)(n-3)\ldots$. If instead of $\boxed{321}$ we have 321, then the only valid forms of permutations will be $132547698\ldots$ and $21436587\ldots$. The case, of $s_n(\boxed{321}, 123)$ is given by applying the reverse operation to $s_n(\boxed{123}, 321)$. □

Looking at the structures in Figure 7, we can see that for $n \geq 5$ and any non-monotone pattern p of length 3, $s_n(\boxed{123}, \boxed{321}, p) = 2$. Also, $s_n(\boxed{123}, 321, 132) = s_n(\boxed{123}, 321, 213) = 0$ and $s_n(\boxed{123}, 321, 231) = s_n(\boxed{123}, 321, 312) = 2$. The cases of avoidance of more than three patterns involving the monotone (boxed) patters are easy to enumerate, again based on Figure 7, and we do not state them here.

4 Multi-avoidance of length-three patterns involving boxed patterns

It has been a popular direction in the permutation patterns literature to consider 132-avoiding permutations subject to extra restrictions (see [3] for an overview over the corresponding results). It turns out that simultaneous avoidance of $\boxed{123}$ and 132 is equivalent to the known simultaneous avoidance of 123 and 132. The reason for that is that whenever we meet subpermutations like in Figure 5 avoiding $\boxed{123}$ but containing 123, we will be forced to have an occurrence of the forbidden pattern 132. In either case, here is an argument showing that $s_n(132, \boxed{123}) = 2^{n-1}$.

Theorem 4. $s_n(132, \boxed{123}) = s_n(132, 123) = 2^{n-1}$.

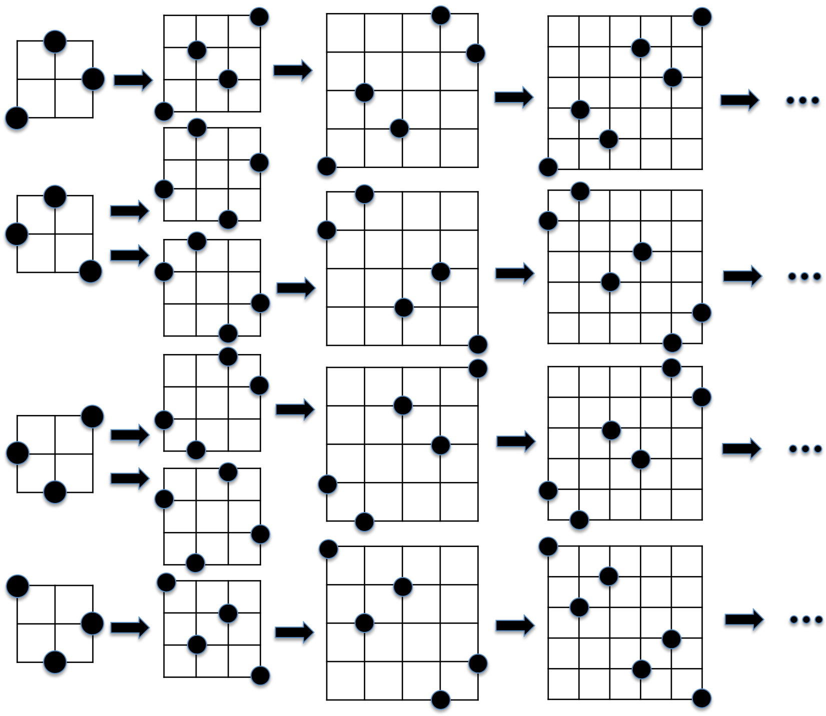

Figure 7: Generation of permutations avoiding $\boxed{123}$ and $\boxed{321}$.

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Proof. Let $a_n = s_n(132, \boxed{123})$. Consider an $(n+1)$-permutation avoiding 132 and $\boxed{123}$. To avoid the pattern 132, everything to the left of $n+1$, if anything, must be larger than everything to the right of it. To the right of $n+1$ there are no extra restrictions but avoidance of the two patterns, which gives a_i possibilities to arrange those elements. Assuming there are elements to the left of $n+1$, everything to the left of n, if anything, must be larger than everything to the right of n to avoid the pattern 132. However, if there is at least one element to the left of n, the rightmost or the largest such element $(n-1)$, together with n and $n+1$, will form the pattern $\boxed{123}$. So, there is no element to the left of n. Arguing in a similar way, we see that all the elements to the left of $n+1$ must be decreasing. Thus, we have the recursion:

$$a_{n+1} = \sum_{i=0}^{n} a_i$$

with the initial condition $a_0 = 1$, from which we conclude that $a_n = 2^{n-1}$. $\square$

Theorem 5. *Let $a(n) = s_n(231, \boxed{123})$ and $a(n;i)$ be the number of $(231, \boxed{123})$-avoiding permutations of length n that end with letter i. Then $a(0) = a(1) = 1$, and for $n \geq 2$,*

$$a(n) = 1 + a(n-1) + \sum_{i=1}^{n-2} (n-i-1)a(n-2;i),$$

$i\backslash n$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	2	3	4	5
3	0	0	1	2	3	4
4	0	0	0	2	5	9
5	0	0	0	0	4	10
6	0	0	0	0	0	8

Table 1: Initial values for $a(n;i)$ in Theorem 5.

and

$$a(n;i) = a(n-1;i) + \sum_{j=1}^{i-2} a(n-2;j).$$

Additionally, $a(n;1) = 1$ *for* $n \geq 1$*,* $a(n;2) = 1$ *for* $n \geq 2$*, and* $a(1;2) = 0$*. The sequence* $a(n)$ *begins as* $1, 1, 2, 4, 8, 17, 37, 82, \ldots$*. Initial values of* $a(n;i)$ *are in Table 1.*

Proof. If our permutation ends with a descent, then the descent letters must be consecutive (in value) to avoid 231 pattern. Thus, to count all such permutations, we can take any $(231, \boxed{123})$-avoiding permutation of length $n-1$ and to create a descent at the end by adjoining to the right the letter one less than the leftmost letter (the letters larger than the new rightmost letter must be raised by 1). This explains the "$a(n-1)$ term". Finally, since our permutation cannot end with three letters in increasing order (in order to avoid $\boxed{123}$), the only cases to consider are when it ends with the pattern 213 or the pattern 312:

- If we end with the pattern 213, then the values corresponding to 2 and 1 must be consecutive, since otherwise, we have an occurrence of the pattern 231. Thus, we can pick any $(231, \boxed{123})$-avoiding permutation of length $n-2$ ending with i, add one more element to the right of it to form a descent (this element is i; "old" i will become $i+1$), and then in $(n-i-1)$ ways to pick a letter greater than $i+1$ to form a $(231, \boxed{123})$-avoiding n-permutation. This explains the term "$\sum_{i=1}^{n-2}(n-i-1)a(n-2;i)$".

- If our permutation ends with the pattern 312, then 1 in the pattern must correspond to 1 in the permutation, since if there is a letter x in the permutation to the left of 3 that is smaller than the next to last letter, then this letter, together with x and the rightmost letter will form the pattern $\boxed{123}$. To avoid 231, the letter in the permutation corresponding to 2 in the pattern must be 2, and all letters to the left of 1 must be in decreasing order. This explains the "1 term".

The recursion for $a(n;i)$ can be obtained by similar considerations. □

Theorem 6. *The sequence* $(s_n(231, \boxed{\mathit{123}}))_{n\geq 0}$ *is counted by the* generalized Catalan numbers *which, up to a shift, are given by the sequence* A004148 *in the OEIS [8]. The generating function for* $s_n(231, \boxed{\mathit{123}})$ *is*

$$\frac{1 - x - x^2 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^3}.$$

Proof. Let again $a(n) = s_n(231, \boxed{123})$. Our strategy is to prove the following recursion

$$a(n+1) = a(n) + a(n-1) + \sum_{m=1}^{n-1} a(m-1)a(n-m-1) \tag{1}$$

with the initial conditions $a(0) = a(1) = 1$. Once this will be done, one can observe that (1) together with the initial conditions defines the sequence 1, 1, 2, 4, 8, 17, 37, ..., which is the sequence A004148 in [8] (with the initial values 1, 1, 1, 2, 4, 8, 17, 37, ...) shifted one position to the left. Indeed, the sequence A004148 is defined by the following recursion

$$b(n+1) = b(n) + \sum_{m=1}^{n-1} b(m)b(n-1-m) \tag{2}$$

with the initial condition $b(0) = 1$. This is now straightforward to use mathematical induction, with the base case $a(0) = b(1)$, to prove that $a(n+1) = b(n+2)$:

$$a(n{+}1) = a(n){+}a(n{-}1){+}\sum_{m=1}^{n-1} a(m{-}1)a(n{-}m{-}1) = b(n{+}1){+}b(n){+}\sum_{m=1}^{n-1} b(m)b((n{+}1){-}m{-}1)$$

$$= b(n+1) + \sum_{m=1}^{n} b(m)b((n+1) - m - 1) = b(n+2).$$

Once the equivalence of the recursions is established, we can take the generating function for the sequence A004148 in [8], subtract 1 from it and divide the result by x to take the shift into consideration.

Let us now prove (1). Note that to avoid the classical pattern 231, everything to the left of the largest element in a permutation must be smaller than everything to the right of it.

If the largest element, $n+1$, in a $(231, \boxed{123})$-avoiding permutation is the leftmost letter then taking any $(231, \boxed{123})$-avoiding permutation of length n and placing it to the right of $n+1$, we will obtain all such permutations. This explains the term $a(n)$ in (1). Now we can assume that there is at least one element to the left of $n+1$.

If there are at least two elements to the left of $n+1$, then the permutation to the left of $n+1$ must end with $(i+1)i$ for some i. Indeed, if these two letters would be an ascent, they, together with $n+1$ would form the pattern $\boxed{123}$; on the other hand, if they form a descent mz involving non-consecutive letters, there is a letter between them to the left of m and z, which, together with m and z would form the pattern 231. This observation

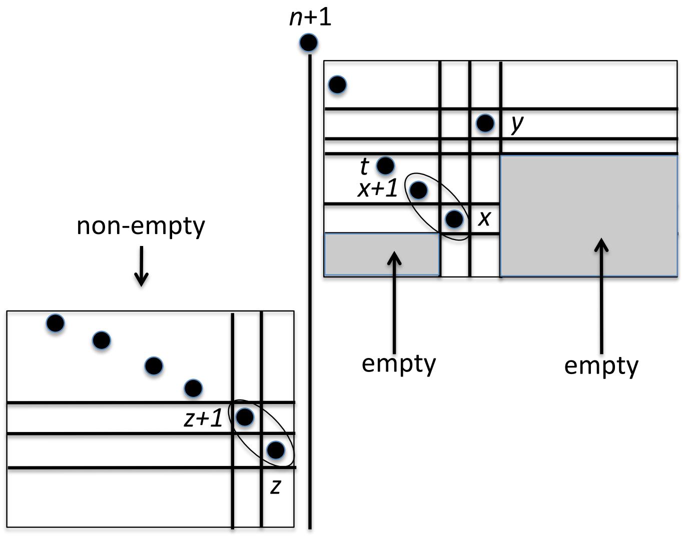

Figure 8: Structure of (231,$\boxed{123}$)-avoiding permutations.

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explains the left factors in the terms $a(m-1)a(n-m-1)$, $m=1,2,\ldots,n-1$, in (1), as well as the term $a(n-1)$ there which is responsible for the largest letter, $n+1$, to be the rightmost letter in a permutation. Indeed, to build a permutation of length m, $1 \leq m \leq n$, located to the left of $n+1$, we take any (possibly empty) (231,$\boxed{123}$)-avoiding permutation of length $m-1$, and insert a new element z from the right side which is one less than the old rightmost element (the elements larger than or equal to z in the "old" permutation will be increased by 1); in the case of the empty permutation (corresponding to the case $m=1$) we simply have 1 to the left of $n+1$. Note that everything larger than $z+1$ to the left of $z+1$ must be in decreasing order.

Explaining the right factors of the terms $a(m-1)a(n-m-1)$, $m=1,2,\ldots,n-1$, in (1) is a bit more involved, and we provide it now based on the schematic structure in Figure 8. Note that there is at least one element to the left of $n+1$; z is the rightmost such element. Either all elements to the right of $n+1$ are in decreasing order or there is at least one ascent in it. Let xy be the leftmost ascent. In particular, there is no elements between $n+1$ and x that are less than x. Note that there must be at least one element between x and y in value that is located between $n+1$ and x, since otherwise, zxy is an occurrence of the pattern $\boxed{123}$. Taking the maximum such element t we see, that to avoid the pattern 231, there is no element to the right of y that is less than t. Taking into account that the elements between x and $n+1$ are in decreasing order, we conclude that the elements between x and y located between $n+1$ and x must be consecutive decreasing elements. In particular, we always have $x+1$ next to x from the left side. Note that it must be that x is the minimum letter to the right of $n+1$.

Thus, to generate a good subpermutation of length $n-m \geq 1$ consisting of the largest letters to the right of $n+1$, we take any $(231, \boxed{123})$-avoiding permutation of length $n-m-1$, and if it is empty, we simply write 1, otherwise, we replace 1 by 21 raising all other letters by 1. Once the choice of the permutation to the right of $n+1$ is made, we can make this permutation to be build on the letters $\{m+1, m+2, \ldots, n\}$.

The last thing we need to justify is that no occurrence of $\boxed{123}$ begins to the left of $n+1$ and ends to the right of $n+1$. This will be guaranteed by $(z+1)z$ to the left and 21 to the right of $n+1$. That is, if one element to the left of $n+1$ is involved in an occurrence of $\boxed{123}$, then 1 and 2 guarantee that the internal boxes of any such 123 occurrence are not all empty; a contradiction. Similarly if one element to the right of $n+1$ is involved in an occurrence of $\boxed{123}$, then z and $z+1$ guarantee that the internal boxes of any such 123 occurrence are not all empty; a contradiction.

□

Note that Theorems 4 and 6, together with the trivial bijections give enumeration of all the cases of avoidance of one classical non-monotone length-three pattern and a monotone boxed length-three pattern. The multi-avoidance involving two monotone boxed length-three patterns is considered in the previous section. However, all cases of avoidance of two or more classical length-three patterns and a monotone boxed length-three pattern are trivial and we do not discuss them here in any detail.

5 Further research directions

One of the interesting questions we were not able to answer is whether or not the former Stanley-Wilf conjecture is true for the boxed patterns $\boxed{2143}$, $\boxed{3142}$, $\boxed{2413}$ and $\boxed{3412}$. Of course, because of the trivial bijections one only needs to answer the question for one of these patterns.

Another problem we were not able to solve is enumerating $\boxed{123}$-avoiding permutations, that is, finding (the generating function for) $s_n(\boxed{123})$. The corresponding sequence begins with 1, 1, 2, 5, 15, 51, and the next term being larger than 303, makes the sequence not to appear in the OEIS [8].

One should be able to strengthen Proposition 1 by showing that avoidance of a boxed pattern is never equivalent to avoidance of a classical pattern. This statement is clearly true for patterns of different lengths, and, by Proposition 1, it is true when a classical pattern is equal (actually, modulo trivial bijections) to the underlying permutation of the boxed pattern. However, we miss an argument why two permutations not obtainable from each other by the trivial bijections, where one of the permutations is boxed, cannot be equivalent in the sense of avoidance, which seems to be true intuitively.

Another open problem is related to the sequence $a(n)$ in Theorem 5. It is the sequence A004148 in the OEIS [8], and it has interesting interpretations, e.g., as the number of *Dyck paths* avoiding three consecutive up steps and three consecutive down steps, or *Motzkin paths* without peaks. Can one provide a "nice" bijective way to construct these objects from our permutations? This could be, for example, an elegant geometric way.

Other open problems include more cases of multi-avoidance of (boxed mesh) patterns to be solved. For example, can we enumerate some cases when one or two of monotone patterns are boxed, and at least one of them is of length more than 3?

Finally, one can fix a natural (maybe well-studied before) class of permutations, for example, *alternating permutations* or 2-*stack sortable permutations* and do boxed pattern avoidance on them, e.g., finding the number of such permutations avoiding the pattern $\boxed{123}$. These studies may give us known cardinalities establishing new links to already studied objects.

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