A Spectral Approach to Consecutive Pattern-Avoiding Permutations

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Abstract

We consider the problem of enumerating permutations in the symmetric group on n elements which avoid a given set of consecutive pattern S, and in particular computing asymptotics as n tends to infinity. We develop a general method which solves this enumeration problem using the spectral theory of integral operators on $L^2([0,1]^m)$, where the patterns in S has length $m+1$. Kreĭn and Rutman's generalization of the Perron–Frobenius theory of non-negative matrices plays a central role. Our methods give detailed asymptotic expansions and allow for explicit computation of leading terms in many cases. As a corollary to our results, we settle a conjecture of Warlimont on asymptotics for the number of permutations avoiding a consecutive pattern.

1 Introduction

In this paper, we study integral operators of the form

$$(Tf)(x_1,\ldots,x_m) = \int_0^1 \chi(t,x_1,\ldots,x_m)\cdot f(t,x_1,\ldots,x_{m-1})dt \tag{1.1}$$

and their application to enumerating permutations that avoid a consecutive pattern. Here χ is a real-valued function on $[0,1]^m$ which takes values in $[0,1]$ and is continuous away from a set of measure zero in $[0,1]^{m+1}$. As we will show, operators of this type arise naturally when counting permutations that avoid a consecutive pattern of length $m+1$.

To define the enumeration problem, let $\mathfrak{S}_n$ denote the symmetric group on n elements. For $\pi \in \mathfrak{S}_n$ we write $\pi = (\pi_1\pi_2\cdots\pi_n)$ where the π_k are the integers from 1 to n. For $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all distinct i and j, we denote by $\Pi(x)$ the unique permutation $\pi \in \mathfrak{S}_n$ with $\pi_i < \pi_j$ if and only if $x_i < x_j$. A *pattern* of length k is an element σ of $\mathfrak{S}_k$. If π is a permutation of length $n \geq k$, π avoids σ if $\Pi(\pi_j,\pi_{j+1},\ldots,\pi_{j+k-1}) \neq \sigma$ for all j with $1 \leq j \leq n-k+1$. More generally, if $S \subseteq \mathfrak{S}_k$ we say that π avoids S if π avoids each $\sigma \in S$. That is, S is the set of forbidden patterns.

For a subset S of $\mathfrak{S}_{m+1}$, denote by $\alpha_n(S)$ the number of permutations $\pi \in \mathfrak{S}_n$ that avoid S. Observe that for $n \leq m$ we have $\alpha_n(S) = n!$. Our goal is to compute asymptotics of $\alpha_n(S)$ as n tends to infinity. Throughout the paper we will assume that $m \geq 2$.

For $S \subseteq \mathfrak{S}_{m+1}$ we define a function χ on $[0,1]^{m+1}$ by setting $\chi(x) = 0$ if $x_i = x_j$ for some distinct indices i and j, and otherwise setting

$$\chi(x) = \begin{cases} 1 & \Pi(x) \notin S, \\ 0 & \Pi(x) \in S. \end{cases} \tag{1.2}$$

Finally, let T be the operator of the form (1.1). The standard inner product on $L^2([0,1]^m)$, is defined by

$$(f,g) = \int_{[0,1]^m} f(x)\cdot\overline{g(x)}dx,$$

and hence the L^2 norm is given by $\|f\| = \sqrt{(f, f)}$. Moreover, let $\mathbf{1}$ denote the constant function 1 on $[0, 1]^m$. It is straightforward to prove (see Proposition 2.10) that the formula

$$\frac{\alpha_n(S)}{n!} = \left(T^{n-m}(\mathbf{1}), \mathbf{1}\right) \tag{1.3}$$

holds for $n \geq m$. Note that the left-hand side is the probability of selecting a permutation $\pi \in \mathfrak{S}_n$ at random that avoids the set S.

The asymptotic behavior of powers of a bounded linear operator is determined by its spectrum. Recall that if A is a bounded linear operator from a Hilbert space to itself, the resolvent set of A is the set of all $z \in \mathbb{C}$ so that $(zI - A)^{-1}$ is also a bounded operator. The complement in $\mathbb{C}$ of the resolvent set is the spectrum of A, denoted $\sigma(A)$, and the spectral radius of A is given by

$$r(A) = \sup \{|\lambda| : \lambda \in \sigma(A)\}.$$

The peripheral spectrum of A is the intersection of $\sigma(A)$ and the circle of radius $r(A)$ in the complex plane. As we will see, the peripheral spectrum of the operator T consists at least of a real eigenvalue at $r(A)$, although this need not be the only eigenvalue in the peripheral spectrum. Finally, define the adjoint of an operator A to be the operator A^* that satisfies $(f, A^*(g)) = (A(f), g)$.

Using spectral theory, we obtain:

Theorem 1.1. *Let S be a set of forbidden patterns in $\mathfrak{S}_{m+1}$. Then the nonzero spectrum of the associated operator T consists of discrete eigenvalues of finite multiplicity which may accumulate only at 0. Furthermore, let r be a positive real number such that there is no eigenvalue of T with modulus r and let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of T greater in modulus than r. Assume that $\lambda_1, \ldots, \lambda_k$ are simple eigenvalues, with associated eigenfunctions φ_i and that the adjoint operator T^* has eigenfunctions ψ_i corresponding the eigenvalues λ_i. Then we have the expansion*

$$\alpha_n(S)/n! = \left(T^{n-m}(\mathbf{1}), \mathbf{1}\right) = \sum_{i=1}^{k} \frac{(\varphi_i, \mathbf{1}) \cdot \left(\mathbf{1}, \overline{\psi_i}\right)}{\left(\varphi_i, \overline{\psi_i}\right)} \cdot \lambda_j^{n-m} + O(r^n). \tag{1.4}$$

Moreover, when the operator T has a positive spectral radius, that is, $r(T) > 0$ then the spectral radius is an eigenvalue of the operator T.

Observe that when T has spectral radius 0, this result is not useful.

In applications the error term in equation (1.4) can be selected to be the modulus of the next eigenvalue, that is,

$$\alpha_n(S)/n! = \sum_{j=1}^{k} c_j \cdot \lambda_j^n + O(|\lambda_{k+1}|^n), \tag{1.5}$$

where the eigenvalues satisfy $|\lambda_1| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}|$ and λ_{k+1} is a simple eigenvalue. This follows from letting r be smaller than $|\lambda_{k+1}|$ given a few more terms in equation (1.4) and increasing the error term.

For certain sets of patterns, we can show that there is a unique largest eigenvalue with respect to modulus and that this eigenvalue is simple and positive. Thus we obtain the leading term of the asymptotics of $\alpha_n(S)/n!$. A sufficient condition for the peripheral spectrum of T to consist only of the simple eigenvalue $r(T)$ is as follows. If (X, μ) is a measure space and $f \in L^2(X, \mu)$ is a real-valued function, we will say that $f > 0$ if $f(x) > 0$ for almost every $x \in X$, and $f \geq 0$ if $f(x) \geq 0$ for almost every x. A bounded operator A on $L^2(X, \mu)$ is *positivity improving* if for any $f \geq 0$ different from 0 there is an integer k (possibly depending on f) so that $A^k f > 0$. Kreĭn and Rutman [15, Theorem 6.3] showed that, for such an operator, the spectral radius $r(A)$ is a simple eigenvalue $r(A)$ and all other eigenvalues λ are smaller than $r(A)$, that is, $|\lambda| < r(A)$. Furthermore, the associated eigenfunction φ is positive almost everywhere. Finally, the adjoint operator A^* also has a largest real, positive and simple eigenvalue at $r(A)$ and an almost everywhere positive eigenfunction ψ. We summarize this discussion in:

Theorem 1.2. *If the operator A is positivity improving then its largest eigenvalue is real, positive and simple.*

A sufficient condition for T to be positivity improving can be formulated in combinatorial terms as follows. If S is a set of patterns, let G_S be the graph with vertex set $\mathfrak{S}_m$ and a directed edge from π to σ if there is a permutation $\tau \in \mathfrak{S}_{m+1} \backslash S$ with $\Pi(\tau_1, \ldots, \tau_m) = \pi$ and $\Pi(\tau_2, \ldots, \tau_{m+1}) = \sigma$. The graph $G_\emptyset$ is known as the graph of overlapping permutations. Recall that a graph G is *strongly connected* if any vertex of G is connected to any other vertex by a directed path.

Theorem 1.3. *Suppose that the graph G_S is strongly connected and that the monotone permutations $12 \cdots (m+1)$ and $(m+1) \cdots 21$ do not belong to S. Then the operator T is positivity improving.*

As a corollary we have the following result, proving a conjecture of Warlimont [22]. See also Theorem 4.1 in [9].

Corollary 1.4. *Let S consists of a single permutation σ. Then the asymptotics of $\alpha_n(S)$ is given by $\alpha_n(S)/n! = c \cdot \lambda^n + O(r^n)$, where c, λ and r are positive constants such that $\lambda > r$.*

Proof. When the permutation σ differs from the two monotone permutations $12 \cdots (m+1)$ and $(m+1) \cdots 21$, Theorem 1.3 applies to the forbidden set $S = \{\sigma\}$. The two remaining cases are equivalent, and can be settled by Theorem 1.7. □

Another application of Theorem 1.3 is the following. Call a permutation π in $\mathfrak{S}_n$ *decomposable* if there exists an index i such that $1 \leq i \leq n-1$ and $\pi(1), \ldots, \pi(i) \leq i$ (equivalently $i+1 \leq \pi(i+1), \ldots, \pi(n)$). A permutation that is not decomposable is called *indecomposable*. These permutations are also known under the terms *connected* and *irreducible*.

Theorem 1.5. *Let S be a subset of $\mathfrak{S}_{m+1}$ such that each permutation in S is indecomposable and the monotone permutations $(m+1) \cdots 21$ do not belong to S. Then $\alpha_n(S)$ has the asymptotic expression $\alpha_n(S)/n! = c \cdot \lambda^n + O(r^n)$, where c, λ and r are positive constants such that $\lambda > r$.*

More generally we can characterize the spectrum of T in terms of another graph associated to S. To define it, let Δ_π denote the set of points $x = (x_1, \ldots, x_m) \in (0,1)^m$ with $x_i \neq x_j$ for $i \neq j$ and $\Pi(x) = \pi$. The graph H_S has vertex set $\cup_{\pi \in \mathfrak{S}_m} \Delta_\pi$ and directed edges from $(x_1, \ldots, x_m)$ to $(x_2, \ldots, x_{m+1})$ if $x_1 \neq x_{m+1}$ and $\Pi(x_1, \ldots, x_{m+1}) \notin S$. We show that if the graph H_S is strongly connected there is an upper bound on the length of the directed path connecting any two vertices. We define the *period* of a strongly connected graph G as follows. Fix a vertex v of G and, for k a non-negative integer, let X_k be the set of all vertices in G that can be reached from v in exactly k steps. The set Q of all k with $v \in X_k$ is a semigroup and generates a subgroup $d\mathbb{Z}$ of $\mathbb{Z}$. The integer d is the *period* of the graph G. Note that, if G is strongly connected, then G has period d for some positive integer d. Finally, a graph is called *ergodic* if it is strongly connected and has period 1.

Theorem 1.6. *Suppose that S is a set of forbidden patterns and that the graph H_S is strongly connected with period d. Then the operator T has positive spectral radius $r(T)$ and T has a simple eigenvalue $\lambda = r(T)$ with strictly positive eigenfunction. Moreover, the spectrum of T is invariant under multiplication by* $\exp(2 \cdot \pi \cdot i/d)$.

In case when the period is 1, we have the conclusion:

Theorem 1.7. *Suppose that S is a set of forbidden patterns such that the graph H_S is ergodic. Then the operator T is positivity improving. That is, the operator has a unique largest eigenvalue which is simple, real and positive and the associated eigenfunction is positive.*

For certain explicit patterns, we can compute the spectrum and eigenfunctions of T and obtain sharp asymptotic formulas for $\alpha_n(S)/n!$.

Example 1.8. When S is empty, directly we have $\alpha_n(S) = n!$ for all $n \geq 0$. The associated operator only has one non-zero eigenvalue, namely 1, with eigenfunction and adjoint eigenfunction $\mathbf{1}$. This is the only case we know where the number of eigenvalues is finite.

Example 1.9. If $S = \{123\}$ we show that the operator T has a trivial kernel and spectrum given by $\{\lambda_k\}_{k \in \mathbb{Z}}$ where

$$\lambda_k = \frac{\sqrt{3}}{2 \cdot \pi \cdot \left(k + \frac{1}{3}\right)}.$$

Furthermore all the eigenvalues are simple. We also compute the eigenfunctions of T and the adjoint operator T^* and obtain

$$\frac{\alpha_n(123)}{n!} = \exp\left(\frac{1}{2 \cdot \lambda_0}\right) \cdot \lambda_0^{n+1} + O\left(|\lambda_{-1}|^n\right) \tag{1.6}$$

where $\lambda_0 = r(T)$ and λ_{-1} is the next largest eigenvalue in modulus. For more terms in the asymptotic expansion see Theorem 5.4.

Example 1.10. If $S = \{213\}$, we show that the nonzero eigenvalues of the operator T are the roots of the equation

$$\operatorname{erf}\left(\frac{1}{\sqrt{2} \cdot \lambda}\right) = \sqrt{\frac{2}{\pi}}$$

which has the unique real root $\lambda_0 = 0.7839769312\ldots$. Moreover, λ_0 is the largest root in modulus of the equation. We then have

$$\frac{\alpha_n(213)}{n!} = \exp\left(\frac{1}{2 \cdot \lambda_0^2}\right) \cdot \lambda_0^{n+1} + O\left(|\lambda_1|^n\right) \tag{1.7}$$

where $\lambda_{1,2} = 0.2141426360\ldots \pm 0.2085807022\ldots \cdot i$ are the next two largest roots of the eigenvalue equation. See Section 6 for the calculations.

Example 1.11. If $S = \{123, 321\}$, the numbers $\alpha_n(S)$ are given by $\alpha_n(S) = 2E_n$ for $n \geq 2$ where E_n is the nth Euler number. In this case, we can use spectral methods to obtain the classical convergent expansion

$$\frac{E_n}{n!} = 2 \cdot \sum_{\substack{j \geq 1 \\ j \text{ odd}}} (-1)^{\frac{j-1}{2}(n+1)} \cdot \left(\frac{\pi \cdot j}{2}\right)^{-n-1}. \tag{1.8}$$

This formula was derived by Ehrenborg, Levin, and Readdy [8] (Corollary 4.2) by using Fourier series. In this case the spectrum of the operator T real and invariant under the reflection $\lambda \mapsto -\lambda$; in particular the two largest eigenvalues are $\pm 2/\pi$. Moreover, the matrix U introduced in Proposition 2.11, is similar to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for a cyclic permutation of order two.

Example 1.12. Let $S = \{123, 231, 312\}$. In this case the asymptotic expansion converges and we can conclude that the number of such permutations in $\mathfrak{S}_n$ is given by $n \cdot E_{n-1}$, a result due to Kitaev and Mansour [13]. See Section 7.

It is easy to find examples of patterns S for which $\rho(T) = 0$.

Example 1.13. Let $S = \{132, 231\}$. An S-avoiding permutation has no peaks (viewed as the graph of a function from $\{1, \ldots, n\}$ to itself) and it is easy to see that $\alpha_n(S) = 2^{n-1}$. However, observe that the proportion $\alpha_n(S)/n!$ is subexponential. It is straightforward to verify that the operator T has no non-zero eigenvalues. Also note that the graph G_S is not strongly connected.

Example 1.14. Let $S = \{123, 213, 231, 321\}$. The directed graph G_S is strongly connected but the monotone permutations 123 and 321 are excluded. In this case $\alpha_n(S) = 2$ for all $n \geq 2$.

We close our introduction by a brief overview on the subject of pattern avoidance in permutations (for more details we refer to [3]). The "classical" definition of a pattern is slightly different than one provided above. We say that a permutation π avoids a pattern σ if π does not contain a *subsequence* which is order-isomorphic to σ. The study of such patterns originated in theoretical computer science by Knuth [14]. However, the first systematic study was done by Simon and Schmidt [18], who completely classified the avoidance of patterns of length three. Since then several hundred papers related to the field have been published.

One of the most important results in the subject is the proof by Marcus and Tardos [16] of the so-called Stanley–Wilf conjecture related to the asymptotic behavior of the number of permutations that avoid a given pattern. It states that for any permutation σ there exists a constant c (depending on σ) such that the number of the permutations of length n that avoid σ is less than c^n.

In this paper we also study asymptotic behavior of permutations avoiding patterns, but we consider *consecutive* patterns, occurrences of which correspond to (contiguous) factors, rather than subsequences, anywhere in permutations. Simultaneous avoidance of consecutive patterns of length 3 is studied in [12] using direct combinatorial arguments. Other approaches to study consecutive patterns were introduced recently which include considering increasing binary trees [10] by Elizalde and Noy, symmetric functions [17] by Mendes and Remmel, homological algebra [5] by Dotsenko and Knoroshkin, and graphs of pattern overlaps [1] by Avgustinovich and Kitaev.

In [10] asymptotics for the following consecutive patterns is given: 123, 132, 1342, 1234 and 1243. These results are obtained by representation of permutations as increasing binary trees, then using symbolic methods followed by solving certain linear differential equations with polynomial coefficients to get corresponding exponential generating functions, and, finally, using the following result (see [11, Theorem IV.7] for a discussion).

Theorem 1.15. *If $f(z)$ is analytic at 0 and R is the modulus of a singularity nearest to the origin in the sense that*

$$R = \sup\{r \geq 0 : f \text{ is analytic in } |z| < r\},$$

then the coefficient $f_n = [z^n]f(z)$ satisfies

$$\limsup_{n\to\infty} |f_n|^{1/n} = 1/R.$$

This differ from our method, as we obtain detailed asymptotic expansions that allows for explicit computation of leading coefficients in many cases. As special cases of our results, we get more detailed asymptotics for some of the results of Elizalde and Noy [10]. Also note that the associated generating function is related to our operator T by the identity

$$\sum_{n\geq 0} \alpha_n(S) \cdot \frac{z^n}{n!} = 1 + \cdots + z^{m-1} + z^m \cdot \left((I - zT)^{-1}(\mathbf{1}), \mathbf{1}\right).$$

From this identity it is clear that the radius of convergence of the generating function is determined by the spectrum of T.

The underlying principle that makes our method work is that one can pick a permutation in $\mathfrak{S}_n$ at random with uniform distribution, by picking a point $(x_1, \ldots, x_n)$ in the unit cube $[0,1]^n$ and applying the function Π. This method has been used in [8] to obtain quadratic inequalities for the descent set statistics and in [7] to enumerate alternating 2 by n arrays.

2 The operator T

We now begin our study of the operator T.

2.1 Connection with pattern avoidance

Recall that S is a collection of forbidden patterns of length $m+1$, that is, S is a subset of $\mathfrak{S}_{m+1}$. The function χ is defined on the unit cube $[0,1]^{m+1}$ by

$$\chi(x) = \begin{cases} 1 & \Pi(x) \notin S, \\ 0 & \Pi(x) \in S, \end{cases}$$

and the operator T on $L^2([0,1]^m)$ by

$$(Tf)(x_1, \ldots, x_m) = \int_0^1 \chi(t, x_1, \ldots, x_m) \cdot f(t, x_1, \ldots, x_{m-1}) dt.$$

For $n \geq m$, define χ_n on the n-dimensional cube $[0,1]^n$ by

$$\chi_n(x_1, \ldots, x_n) = \prod_{j=1}^{n-m} \chi(x_j, \ldots, x_{m+j}). \tag{2.1}$$

This allows us to express powers of our operator T. If $k < m$ then

$$\left(T^k f\right)(x) = \int_{[0,1]^k} \chi_{m+k}(t_1, \ldots, t_k, x_1, \ldots, x_m) \cdot f(t_1, \ldots, t_k, x_1, \ldots, x_{m-k}) dt_1 \cdots dt_k \tag{2.2}$$

while if $k \geq m$,

$$\left(T^k f\right)(x) = \int_{[0,1]^k} \chi_{m+k}(t_1, \ldots, t_k, x_1, \ldots, x_m) \cdot f(t_1, \ldots, t_m) dt_1 \cdots dt_k \tag{2.3}$$

Proposition 2.1. *The number of S-avoiding permutations in $\mathfrak{S}_n$ for $n \geq m$ is given by $\alpha_n(S) = n! \cdot (T^{n-m}(\mathbf{1}), \mathbf{1})$.*

Proof. Equation (2.1) allows us to conclude that

$$\chi_n(x) = \begin{cases} 1 & \text{if } \Pi(x) \text{ avoids } S, \\ 0 & \text{if } \Pi(x) \text{ does not avoid } S. \end{cases}$$

Hence by integrating over the n-dimensional cube

$$\int_{[0,1]^n} \chi_n(x) dx = \frac{\alpha_n(S)}{n!},$$

since each simplex in the standard triangulation of $[0,1]^n$ corresponds to a permutation $\pi \in \mathfrak{S}_n$ and each such simplex has volume $1/n!$. Using this observation and the identities (2.2) and (2.3), we can rewrite the integral as $(T^{n-m}(\mathbf{1}), \mathbf{1})$. □

Lemma 2.2. *The operator T is a bounded operator with norm at most* 1*.*

Proof. Apply the Cauchy–Schwarz inequality to equation (1.1) in the variable t to obtain

$$\begin{aligned} & \left| \int_0^1 \chi(t, x_1, \ldots, x_m) \cdot f(t, x_1, \ldots, x_{m-1}) dt \right|^2 \\ \leq \; & \int_0^1 |\chi(t, x_1, \ldots, x_m)|^2 dt \cdot \int_0^1 |f(t, x_1, \ldots, x_{m-1})|^2 dt \\ \leq \; & \int_0^1 |f(t, x_1, \ldots, x_{m-1})|^2 dt. \end{aligned}$$

Now integrating over the variables $x_1, \ldots, x_m$, we obtain that $\|T(f)\|^2 \leq \|f\|^2$ proving the bound. □

Lemma 2.3. *The adjoint operator T^* is given by*

$$(T^* f)(x) = \int_0^1 \chi(x_1, \ldots, x_m, u) f(x_2, \ldots, x_m, u) du.$$

Proof. Since χ is real-valued, we have that

$$\begin{aligned} (T(f), g) &= \int_{[0,1]^m} \int_0^1 \chi(t, x_1, \ldots, x_m) \cdot f(t, x_1, \ldots, x_{m-1}) dt \cdot \overline{g(x_1, \ldots, x_m)} dx_1 \cdots dx_m \\ &= \int_{[0,1]^m} f(t, x_1, \ldots, x_{m-1}) dt \cdot \overline{\int_0^1 \chi(t, x_1, \ldots, x_m) \cdot g(x_1, \ldots, x_m) dx_m} dt dx_1 \cdots dx_{m-1}, \end{aligned}$$

proving the lemma. □

The spectrum of an adjoint operator A^* is given by conjugate of the spectrum of A. That is, if λ is an eigenvalue of A, then $\overline{\lambda}$ is an eigenvalue of A^*. However, for our operator T, the complex eigenvalues come in conjugate pairs. This is proved using that χ is real-valued.

Lemma 2.4. *We have that $\overline{T(f)} = T(\overline{f})$ and $\overline{T^*(f)} = T^*(\overline{f})$. Hence if λ is an eigenvalue of T with eigenfunction φ, then $\overline{\lambda}$ is also eigenvalue of T with eigenfunction $\overline{\varphi}$. Similarly, if λ is an eigenvalue of T^* with eigenfunction ψ, then $\overline{\lambda}$ is also eigenvalue of T^* with eigenfunction $\overline{\psi}$.*

2.2 Eigenvalues and asymptotic expansion

Since the operator T is bounded, we can use spectral analysis in order to explore the operator T. We refer the reader to Dunford and Schwarz [6, Chapter VII] for a more detailed exposition.

As we defined in the introduction, the resolvent set $\varsigma(T)$ is the set of complex numbers z such the operator $(zI - T)^{-1}$ exists as a bounded operator. The *spectrum* $\sigma(T)$ of T is the complement of the resolvent set $\varsigma(T)$.

The *index* of a complex number λ is the smallest non-negative integer ν such that the equation $(\lambda I - T)^{\nu+1} f = 0$ implies $(\lambda I - T)^{\nu} f = 0$ for all functions f. Informally speaking, for operators on a finite-dimensional vector spaces, the index of an eigenvalue is the size of the largest Jordan block associated with that eigenvalue. A point λ in the spectrum is called a pole T of *order* ν if the function $R(z; T)$ has a pole at λ of order ν. Theorem 18 in [6, Section VII.3] states that the order of a pole λ is equal to its index.

Define the operator $E(\lambda)$ by the integral

$$E(\lambda) = \frac{1}{2\pi i} \cdot \oint_C \frac{1}{zI - T} dz,$$

where C is a positive oriented closed curve in the complex plane only containing the eigenvalue λ from the spectrum $\sigma(T)$. It follows from [6, Section VII.3] that $E(\lambda)$ is a projection.

Lemma 2.5. *The operator T^m is compact.*

Proof. The operator T^m has the form

$$T^m(f) = \int_{[0,1]^m} \chi_{2m}(t_1, \ldots, t_m, x_1, \ldots, x_m) \cdot f(t_1, \ldots, t_m) dt_1 \cdots dt_m.$$

Since χ_{2m} is a bounded function we conclude that T^m is a Hilbert–Schmidt operator, and hence a compact operator. □

Using Theorems 5 and 6 in [6, Section VII.4] we conclude:

Theorem 2.6. *The spectrum of T is at most denumerable and has no point of accumulation in the complex plane except possibly 0. Every non-zero number λ in $\sigma(T)$ is a pole of T and has finite positive index. For such a number λ the projection $E(\lambda)$ has a non-zero finite dimensional range and it is given by $\{f \in L^2([0,1]^m) \, : \, (\lambda I - T)^{\nu} f = 0\}$, where ν is the order of the pole λ.*

Recall that an eigenvalue λ is simple if the range of $E(\lambda)$ is one-dimensional. That is, the eigenvalue equation $\lambda\varphi = T\varphi$ has a unique solution up to a scalar multiple and the generalized eigenvalue equation $\lambda f = Tf + \varphi$ has no solution.

Lemma 2.7. *Let λ be a simple eigenvalue of the operator T with associated eigenfunction φ. Let ψ be the eigenfunction of the adjoint operator T with eigenvalue λ. Then the projection $E(\lambda)$ is given by*

$$E(\lambda)(f) = \frac{(f, \overline{\psi})}{(\varphi, \overline{\psi})} \cdot \varphi.$$

Proof. Since the eigenvalue λ is simple, the range of the projection $E(\lambda)$ is one-dimensional and spanned by the eigenfunction φ. Since the projection is continuous we may assume that it has the form $E(\lambda)(f) = (f, \alpha) \cdot \varphi$ for some function α. It is straightforward to observe that since $E(\lambda)$ is a projection, that is, $E(\lambda)^2 = E(\lambda)$, we have that $(\varphi, \alpha) = 1$.

Now the adjoint operator $E(\lambda)^*$ is given by $E(\lambda)^*(g) = (g, \varphi) \cdot \alpha$ since

$$(g, E(\lambda)(f)) = (g, (f, \alpha) \cdot \varphi) = (\alpha, f) \cdot (g, \varphi) = ((g, \varphi) \cdot \alpha, f) .$$

Hence α belongs to the range of the adjoint operator, that is, it is multiple of the eigenfunction $\overline{\psi}$. In fact, we observe that α is given by $\overline{(\varphi, \overline{\psi})}^{-1} \cdot \overline{\psi}$. □

Proof of Theorem 1.1. By analytic functional calculus we can evaluate the operator T^{n-m} by integrating in the complex plane; see Theorem 6(c) in [6, Section VII.3]. We have

$$T^{n-m} = \frac{1}{2\pi i} \cdot \oint_{|z|=R} \frac{z^{n-m}}{zI - T} dz,$$

where R is greater than the spectral radius of T and we orient the circle in positive orientation.

Let σ be the set $\{\lambda_1, \ldots, \lambda_k\}$ and let $E(\sigma)$ denotes the sum of the projections $E(\lambda_1) + \cdots + E(\lambda_k)$. By Theorem 22 in [6, Section VII.3] and that the eigenvalues $\lambda_1, \ldots, \lambda_k$ are simple, we have that

$$T^{n-m} \cdot E(\sigma) = \sum_{i=1}^{k} E(\lambda_i) \cdot \lambda_i^{n-m}.$$

We can estimate the operator $T^{n-m} \cdot (I - E(\sigma))$ by shrinking the path of integration to a circle of radius r

$$T^{n-m} \cdot (I - E(\sigma)) = \frac{1}{2\pi i} \cdot \oint_{|z|=r} \frac{z^{n-m}}{zI - T} dz.$$

We bound this integral by

$$\begin{aligned}
\left\| T^{n-m} \cdot (I - E(\sigma)) \right\| &= \left\| \frac{1}{2\pi i} \cdot \oint_{|z|=r} \frac{z^{n-m}}{zI - T} dz \right\| \\
&\leq \frac{1}{2\pi} \cdot \oint_{|z|=r} \left\| \frac{1}{zI - T} \right\| dz \cdot r^{n-m} \\
&\leq \sup_{|z|=r} \left\| (zI - T)^{-1} \right\| \cdot r^{n-m} \\
&= O\left(r^n\right),
\end{aligned}$$

where the last equality follows from that the supremum does not depend on n. Hence the inner product $(T^{n-m} \cdot (I - E(\sigma))\mathbf{1}, \mathbf{1})$ is also bounded by $O(r^n)$. Thus we conclude that

$$\begin{aligned}
\left(T^{n-m}\mathbf{1}, \mathbf{1}\right) &= \left(T^{n-m} E(\sigma)\mathbf{1}, \mathbf{1}\right) + \left(T^{n-m} \cdot (I - E(\sigma))\mathbf{1}, \mathbf{1}\right) \\
&= \sum_{i=1}^{k} (E(\lambda_i)\mathbf{1}, \mathbf{1}) \cdot \lambda_i^{n-m} + O(r^n) \\
&= \sum_{i=1}^{k} \frac{(\varphi_i, \mathbf{1}) \cdot \left(\mathbf{1}, \overline{\psi_i}\right)}{\left(\varphi_i, \overline{\psi_i}\right)} \cdot \lambda_i^{n-m} + O(r^n).
\end{aligned}$$

□

If the eigenvalue λ is not real then $\overline{\lambda}$ is also an eigenvalue. The two terms corresponding to these two eigenvalues can be combined as follows. Let $\lambda = r \cdot e^{i\cdot\theta}$ and $(\varphi, \mathbf{1}) \cdot \left(\mathbf{1}, \overline{\psi}\right) / \left(\varphi, \overline{\psi}\right) = s \cdot e^{i\cdot\beta}$. Then we have

$$\begin{aligned} \frac{(\varphi, \mathbf{1}) \cdot \left(\mathbf{1}, \overline{\psi}\right)}{\left(\varphi, \overline{\psi}\right)} \cdot \lambda^{n-m} + \frac{(\overline{\varphi}, \mathbf{1}) \cdot (\mathbf{1}, \psi)}{(\overline{\varphi}, \psi)} \cdot \overline{\lambda}^{n-m} &= 2 \cdot \Re\left(s \cdot e^{i\cdot\beta} \cdot r^{n-m} \cdot e^{i\cdot(n-m)\cdot\theta}\right) \\ &= 2 \cdot s \cdot r^{n-m} \cdot \cos(\beta + (n-m)\cdot\theta). \end{aligned}$$

2.3 Bounds on the norm and spectral radius

Proposition 2.8. *Let T be an operator of the form (1.1) and suppose that T has a positive spectral radius $r(T)$. Then T has a positive eigenvalue $\lambda = r(T)$ with non-negative eigenfunction φ. Furthermore, the eigenfunction φ is almost everywhere positive. Similarly, the adjoint operator T^* also has $\lambda = r(T)$ as an eigenvalue with an almost everywhere positive eigenfunction ψ.*

Proof. Let K be the cone of non-negative functions in $L^2([0,1]^m)$. Since T preserves this cone and T has nonzero spectral radius, it follows from Theorem 6.1 of [15] and the fact that the cone of positive functions is self-dual that T and T^* both have $\lambda = r(T)$ as an eigenvalue with at least one strictly positive eigenfunction. □

For an example of an eigenfunction φ corresponding to the largest eigenvalue, taking the value 0 on a set of measure 0, see Proposition 7.2 where $123, 231, 312$-avoiding permutations are discussed.

Lemma 2.9. *For $k \geq m$ the norm of the operator T^k is bounded above by $\sqrt{a_{m+k}(S)/(m+k)!}$.*

Proof. Applying the Cauchy–Schwarz inequality to equation (2.3) in the variables t_1 through t_k, we obtain

$$\begin{aligned} &\left| \int_{[0,1]^k} \chi_{m+k}(t_1, \ldots, t_k, x_1, \ldots, x_m) \cdot f(t_1, \ldots, t_m) dt \right|^2 \\ \leq\ & \int_{[0,1]^k} |\chi_{m+k}(t_1, \ldots, t_k, x_1, \ldots, x_m)|^2 dt \cdot \int_{[0,1]^k} |f(t_1, \ldots, t_m)|^2 dt \\ =\ & \int_{[0,1]^k} \chi_{m+k}(t_1, \ldots, t_k, x_1, \ldots, x_m) dt \cdot \int_{[0,1]^m} |f(t_1, \ldots, t_m)|^2 dt. \end{aligned}$$

Now integrating the variables $x_1, \ldots, x_m$ over $[0,1]^m$ we have $\|T^k(f)\|^2 \leq \alpha_{m+k}(S)/(m+k)! \cdot \|f\|^2$, proving that $\|T^k\| \leq \sqrt{\alpha_{m+k}(S)/(m+k)!}$. □

Proposition 2.10. *For a nonempty set of forbidden patterns S we have that spectral radius of the operator T is less than 1, that is, $r(T) < 1$.*

Proof. We have $\alpha_{2m}(S) < (2\cdot m)!$ implying that $\|T^m\| < 1$. The inequality $\|T^{m\cdot n}\| \leq \|T^m\|^n$ implies that $r(T^m) = \lim_{n \longrightarrow \infty} \|T^{m\cdot n}\|^{1/n} < 1$ strictly. The result follows by taking the mth root. □

Proposition 2.11. *Let T be an operator of the form (1.1) and suppose that $\rho = r(T) > 0$. Then the operator T admits a decomposition of the form*

$$T = \rho \cdot U + W$$

where $UW = WU = 0$, U has finite-dimensional range, U maps the interior of the cone of positive functions into itself, the operator W has spectral radius less than ρ, and the eigenvalues of U are roots of unity including 1.

Proof. The orthogonal decomposition follows from Theorem 8.1 of [15] applied to the operator $A = \rho^{-1} \cdot T$. □

In particular, the leading behavior of powers T^n is determined by the spectral radius $r(T)$ and the finite-rank operator U.

Theorem 2.12. *Let S be a set of forbidden patterns. Then spectral radius of the operator T is given by*

$$r(T) = \lim_{n \longrightarrow \infty} \left(\frac{\alpha_n(S)}{n!} \right)^{1/n}.$$

Proof. Suppose first that $r(T) = 0$. From the inequality $|(T^n \mathbf{1}, \mathbf{1})| \leq \|T^n\|$ we immediately conclude that $(\alpha_n(S)/n!)^{1/n}$ tends to 0 as n goes to infinity. If $r(T) > 0$, then by Proposition 2.11, we have

$$\frac{\alpha_n(S)}{n!} = r(T)^n \left(U^n \mathbf{1}, \mathbf{1} \right) + \left(W^n \mathbf{1}, \mathbf{1} \right)$$

where the second term obeys the estimate

$$|(W^n \mathbf{1}, \mathbf{1})| \leq (r(T) - \varepsilon)^n$$

for some $\varepsilon > 0$ and all sufficiently large n. Moreover, $(U^n \mathbf{1}, \mathbf{1})$ is periodic in n and strictly positive since U maps the interior of the cone of positive functions into itself. Thus $(U^n \mathbf{1}, \mathbf{1})$ is both bounded above and below by strictly positive constants. It follows that $\lim_{n \longrightarrow \infty} (U^n \mathbf{1}, \mathbf{1})^{1/n} = 1$ so that $\lim_{n \longrightarrow \infty} (\alpha_n(S)/n!)^{1/n} = r(T)$ as claimed. □

This result extends Theorem 4.1 of [9], where consecutive patterns consisting of a single permutation were considered. Moreover, Theorem 2.12 characterizes the limit in terms of a spectral quantity which can be computed in many cases of interest by solving the eigenvalue problem for the integral operator T.

3 Associated graphs

3.1 The directed graph H_S

In this section we study the spectrum of T using the infinite graph H_S described in the introduction. Recall that Δ_π denotes the open subset of $(0,1)^m$ with $x_i \neq x_j$ for $i \neq j$ and $x_i < x_j$ if and only if $\pi(i) < \pi(j)$, and let

$$X = \bigcup_{\pi \in \mathfrak{S}_m} \Delta_\pi.$$

Thus the complement of X consists of those points x with $x_i = x_j$ for at least one pair of distinct indices i and j and hence is a set of measure zero. The graph H_S has vertex set X. Recall that the directed edges of H_S connected points x and y in X with $x_{j+1} = y_j$ for $1 \leq j \leq m-1$, $x_1 \neq y_m$, and $\Pi(x_1, \ldots, x_m, y_m) \notin S$. It follows from the definition that $\chi(t, x_1, \ldots, x_m) = 1$ if and only if there is a directed edge from $(t, x_1, \ldots, x_{m-1})$ to $(x_1, \ldots, x_m)$. That is, the function χ encodes the edge information of the graph H_S.

The next lemma connects the graph H_S to mapping properties of the operator T. It will be used to show that the operator is positivity improving.

Lemma 3.1. *Suppose that $x, y \in X$ and that there is a directed path from x to y of length $k \geq m$. Suppose further that f is a non-negative continuous function such that f is non-zero in a neighborhood of x. Then $(T^k f)(y) > 0$.*

Proof. Assume that the directed path is

$$x = (x_1, \ldots, x_m) \longrightarrow (x_2, \ldots, x_{m+1}) \longrightarrow \cdots \longrightarrow (x_{k+1}, \ldots, x_{k+m}) = y.$$

Let ε be the minimum of the following finite set

$$\{|x_i - x_j| : 1 \leq i < j \leq k+m,\ j - i \leq m\} \cup \{x_i, 1 - x_i : 1 \leq i \leq k+m\}.$$

Observe that $\varepsilon > 0$ by the definition of X. Let $\delta = \varepsilon/3$. For $s_i \in [x_i - \delta, x_i + \delta]$, $1 \leq i \leq k+m$, we have that

$$(s_1, \ldots, s_m) \longrightarrow (s_2, \ldots, s_{m+1}) \longrightarrow \cdots \longrightarrow (s_{k+1}, \ldots, s_{k+m})$$

is also a directed path in H_S. It follows that $\chi_{k+m}(s_1, \ldots, s_{k+m}) = 1$ for all such s. Using (2.3) we may estimate

$$\begin{aligned} \left(T^k f\right)(y) &\geq \int_{x_1-\delta}^{x_1+\delta} \cdots \int_{x_k-\delta}^{x_k+\delta} f(t_1, \ldots, t_m) dt_1 \cdots dt_k \\ &= (2\delta)^{k-m} \int_{x_1-\delta}^{x_1+\delta} \cdots \int_{x_m-\delta}^{x_m+\delta} f(t_1, \ldots, t_m) dt_1 \cdots dt_m \\ &> 0, \end{aligned}$$

where in the last step we have used the positivity of f in a neighborhood of $(x_1, \ldots, x_m)$. □

Proposition 3.2. *Suppose that H_S is strongly connected with period d. Then there is a decomposition*

$$X = \bigcup_{i=0}^{d-1} Y_i$$

of X into disjoint sets Y_i with the property that $T : L^2(Y_i) \longrightarrow L^2(Y_{i+1})$, where $Y_d = Y_0$.

Proof. Pick a base vertex v of H_S. Let X_k be the set of all vertices in H_S that can be reached from v in k steps, and let Q be the subset of the non-negative integers defined by

$$Q = \{k \ :\ v \in X_k\}.$$

Then Q is a semigroup under addition and generates a subgroup of the integers $\mathbb{Z}$. A subgroup of $\mathbb{Z}$ has the form $d\mathbb{Z}$ for some positive integer d; in this case, d is the period of the graph H_S.

Now define

$$Y_i = \bigcup_{j:\ j \equiv i \bmod d} X_j.$$

Observe that every directed edge in the graph H_S goes from some Y_i to the next Y_{i+1} (with addition modulo d). Also, observe that the sets Y_i are pairwise disjoint.

We claim that each Y_i is open. To see this, suppose that $y \in Y_i$. Pick a path from some vertex x to the vertex y having length greater than $2m$. We can perturb this path in a small neighborhood of l y using a variant of the argument used at the beginning of Lemma 3.1 and conclude that Y_i is open.

Now pick a permutation $\pi \in \mathfrak{S}_m$. Since the sets $Y_0, \ldots, Y_{d-1}$ are pairwise disjoint we have that $\Delta_\pi = \Delta_\pi \cap X = \cup_{i=0}^{d-1}(\Delta_\pi \cap Y_i)$. Note Δ_π is a connected set and $\Delta_\pi \cap Y_i$ are all open. A connected set can only be the disjoint union of one open set and hence there exists a unique index i such that $\Delta_\pi \subseteq Y_i$. Hence each set Y_i is the disjoint union of the sets Δ_π.

Finally, suppose that f is a continuous function on $[0,1]^m$ with support in the set Y_i. We claim that Tf is supported in the next set Y_{i+1}. To see this, note that $\chi(t, x_1, \ldots, x_m) = 1$ if and only if there is a directed edge from $(t, x_1, \ldots, x_{m-1})$ to $(x_1, \ldots, x_m)$. Assuming that $f(t, x_1, \ldots, x_{m-1})$ is non-zero where $(t, x_1, \ldots, x_{m-1})$ belongs to the set Y_i. Since f is continuous function, f is supported in a neighborhood of the point $(t, x_1, \ldots, x_{m-1})$. By applying the definition of the operator T, we have the function Tf supported in a neighborhood of the point $(x_1, \ldots, x_m)$ in the next set Y_{i+1}. That is, Tf is supported in Y_{i+1}. □

The next lemma is a straightforward consequence of the definition of the graph H_S and hence the proof is omitted.

Lemma 3.3. *Suppose that $\alpha : (0,1) \longrightarrow (0,1)$ is a strictly increasing function. If*

$$(x_1, \ldots, x_m) \longrightarrow (x_2, \ldots, x_{m+1}) \longrightarrow \cdots \longrightarrow (x_{k+1}, \ldots, x_{k+m})$$

is a directed path in H_S, then so is

$$(\alpha(x_1), \ldots, \alpha(x_m)) \longrightarrow (\alpha(x_2), \ldots, \alpha(x_{m+1})) \longrightarrow \cdots \longrightarrow (\alpha(x_{k+1}), \ldots, \alpha(x_{k+m})) \,.$$

We next show that there is an upper bound on the length of the directed path between any two vertices in the case when the graph H_S is strongly connected.

Proposition 3.4. *Suppose that H_S is strongly connected with period d. Then there is a positive integer N, a multiple of d, such that for any two points x and z in the same component Y_i there is a path from x to z in the graph H_S of length N. Especially, between any two vertices in the graph H_S there is a directed path of length at most $N + d - 1$.*

Proof. For two vertices x and y of the graph H, let $D(x, y)$ denote the length of the shortest path from x to y. Observe that this is not a distance since it is not symmetrical in general.

For each permutation $\pi \in \mathfrak{S}_m$ pick a point x_π in Δ_π, such that all its coordinates are greater than $1/2$. Similarly, for all $0 \leq i \leq d-1$ pick a point y_i in Y_i, such that all its coordinates are less than $1/2$. Let K denote the maximum of the finite set

$$\{D(x_\pi, y_i) \, : \, \Delta_\pi \subseteq Y_i\} \ \cup \ \{D(y_i, x_\tau) \, : \, \Delta_\tau \subseteq Y_i\}.$$

Note that K is a multiple of the period d. Let Q_i denote the semigroup

$$Q_i = \{k \ : \ \text{ there is a path from } y_i \text{ to } y_i \text{ of exactly length } k\} \,.$$

Note that $Q_i \subseteq d \cdot \mathbb{N}$. Furthermore, there is no multiple e of the period d such that $Q_i \subseteq e \cdot \mathbb{N}$. Hence there exists a positive integer m_i such that $d \cdot \mathbb{N} + m_i \subseteq Q_i$. That is, there is path from y_i to y_i of any length which is a multiple of d and greater than or equal to m_i. Let M be the maximum of m_0 through m_{d-1}. Note again that M is a multiple of d.

Let $N = 2 \cdot K + M$. Pick two permutations π and τ such that $\Delta_\pi, \Delta_\tau \subseteq Y_i$. We claim that there is a path from the point x_π to the point x_τ of length N. We find this path by picking three paths. First, choose a path p_1 from x_π to y_i of length at most K. Second, choose a path p_3 from y_i to x_τ of length at most K. Finally, pick a path p_2 from y_i to y_i of length $M + 2 \cdot K - D(x_\pi, y_i) - D(y_i, x_\tau)$ which is at least M. By concatenating the three paths p_1, p_2 and p_3, the result follows.

Now let show that there is a path from any point x in Y_i to any other point z in Y_i of length N. Let π and τ be the permutations $\pi = \Pi(x)$ and $\tau = \Pi(z)$. We do so using that we did choose the coordinates of x_π and x_τ greater than the coordinates of y_i. Namely, let $\varepsilon > 0$ be the smallest coordinate of the two points x and z. Let y be the point $\varepsilon \cdot y_i$. Now there is a monotone function α such that $\alpha(x_\pi) = x$ and $\alpha(y_i) = y$. Similarly, there is a monotone function β such that $\beta(y_i) = y$ and $\beta(x_\tau) = z$. Concatenating the three paths $\alpha(p_1)$, $\varepsilon \cdot p_2$ and $\beta(p_3)$ we obtain a path from x to z of length N. □

Proof of Theorem 1.6. It follows from Proposition 3.2 that $T^d : L^2(Y_i) \longrightarrow L^2(Y_i)$. We will denote by A the restriction of T^d to $L^2(Y_0)$. Choosing a positive integer p with $p \cdot d \geq m$ we see that A^p is a compact operator from $L^2(Y_0)$ to itself. The operator A^p has discrete spectrum which may accumulate only at 0 so the same is true for A. We claim that, if φ is an eigenfunction of A with nonzero eigenvalue λ^d, then $T^i\varphi$ is a nonzero eigenfunction of T^d with eigenvalue λ^d and support in Y_i. The only nontrivial part of this claim is that $T^i\varphi \neq 0$. To see this, note that $T^{d-i}(T^i\varphi) = \lambda^d \cdot \varphi$ so $T^i\varphi$ cannot be zero since $\lambda \neq 0$ and φ is a nontrivial eigenfunction.

Suppose now that $\lambda^d \in \sigma(A)$, let φ be an eigenfunction of A corresponding to the eigenvalue λ^d, let ω be a dth root of unity, and let

$$\psi = \varphi + \frac{1}{\lambda \cdot \omega} \cdot T\varphi + \cdots + \left(\frac{1}{\lambda \cdot \omega}\right)^{d-1} \cdot T^{d-1}\varphi.$$

The function ψ is nonzero because the right-hand terms are nonzero and have disjoint supports. A direct computation shows that $T\psi = \omega \cdot \lambda \cdot \psi$ so the spectrum of T contains all of the numbers $\lambda \cdot \omega$ where $\lambda^d \in \sigma(A)$. On the other hand, any eigenvalue μ of T gives rise to an eigenvalue μ^d of T^d, so the nonzero spectrum of T consists exactly of the numbers $\omega \cdot \lambda$ where ω is a dth root of unity and λ^d is an eigenvalue of T^d.

By combining Lemma 3.1 and Proposition 3.4 we have that for a non-negative, but non-zero, function f in $L^2(Y_0)$ that $T^N(f) = A^{N/d}(f)$ is a positive function. Hence the operator A is positivity improving and by Kreĭn and Rutman [15, Theorem 6.3] the operator A has a positive spectral radius $r(A)$. Furthermore, the spectral radius $r(A)$ is a simple eigenvalue, all other eigenvalues are smaller in modulus and the eigenfunction φ corresponding to $r(A)$ is positive. Hence the spectral radius of T, $r(T) = \sqrt[n]{r(A)}$, is positive and a simple eigenvalue of T. Finally, the positive function φ is also an eigenfunction of T corresponding to the eigenvalue $r(T)$. □

3.2 The directed graph G_S

We examine relations between the infinite graph H_S and the finite graph G_S. We need the following lemmas,

Lemma 3.5. *Let $x = (x_1, \ldots, x_m)$ be a vertex in H_S such that $\Pi(x_1, \ldots, x_m) = \pi$. Furthermore assume there is an edge in G_S labeled τ leaving the vertex π. Then there exists x_{m+1} in $(0, 1)$ such that $\Pi(x_1, \ldots, x_{m+1}) = \tau$. That is, there is an edge in H_S leaving the vertex x.*

Proof. Observe that $\tau(m+1)$ is bigger than exactly $\tau(m+1)-1$ of the numbers $\tau(1), \ldots, \tau(m)$. Hence pick x_{m+1} such that it is bigger than exactly $\tau(m+1) - 1$ of the numbers $x_1, \ldots, x_m$. □

Iterating this lemma we obtain:

Lemma 3.6. *Given a directed path from π to σ in the graph G_S. Let x be a vertex in H_S such that $\Pi(x) = \pi$. Then this directed path can be lifted to a directed path in H_S that ends with a vertex y such that $\Pi(y) = \sigma$.*

Now we give a sufficient condition for H_S to be strongly connected in terms of the graph G_S.

Proposition 3.7. *Let $S \subseteq \mathfrak{S}_{m+1}$, suppose that G_S is strongly connected, and suppose that the two monotone permutations $12\cdots(m+1)$ and $(m+1)\cdots 21$ do not belong to the set S. Then the graph H_S is ergodic.*

Proof. We first prove that H_S is strongly connected. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ be two vertices of H_S, and let $\pi = \Pi(x)$ and $\sigma = \Pi(y)$. Since G_S is strongly connected we can find a directed path from π to $m\cdots 21$ in G_S. This directed path lifts to a directed path from x to $z = (z_1, \ldots, z_m)$ in H_S, where $z_1 > z_2 > \cdots > z_m$.

For any directed graph G define the reverse graph G^* to the graph G where we reverse the direction of each edge. In the reverse graph G_S^* we have a directed path from σ to $12\cdots m$. Lifting this directed path to a directed path in H_S^* and then reversing the path, we obtain a directed path from $w = (w_1, \ldots, w_m)$ to y in H_S, where $w_1 < \cdots < w_m$.

Finally, there is a directed path from $m\cdots 21$ to $12\cdots m$ in G_S. Hence we know that there is a directed path from $u = (u_1, \ldots, u_m)$ to $v = (v_1, \ldots, v_m)$ in H_S, where $u_1 > \cdots > u_m$ and $v_1 < \cdots < v_m$.

Choose α so that $0 < \alpha < \min(z_m, w_1)$. We then have a directed path from $\alpha \cdot u$ to $\alpha \cdot v$. Now, there is a directed path from z to $\alpha \cdot u$ of length m, namely

$$z = (z_1, \ldots, z_m) \longrightarrow (z_2, \ldots, z_m, \alpha u_1) \longrightarrow \cdots \longrightarrow (z_m, \alpha u_1, \ldots, \alpha u_{m-1}) \longrightarrow (\alpha u_1, \ldots, \alpha u_m) = \alpha \cdot u$$

using the fact that $(m+1)\cdots 21$ is not forbidden. We can now concatenate these five directed paths to obtain a path from x via z, via $\alpha \cdot u$, via $\alpha \cdot v$, via w, to y.

Since G_S is strongly connected and has $m!$ vertices, an upper bound on the length of the directed paths in G_S chosen above is $m! - 1$. Hence, the path that we have constructed has length at most $3(m! - 1) + 2m$.

To observe that H_S has period 1 note that we can construct a directed path from the vertex x to the vertex y that has length one more than the above construction. Namely, the path from π to $m\cdots 21$ can be extended by adding the loop $(m+1)\cdots 21$ at the end. Now by concatenating these two paths with a path from y to x we obtain two cycles whose lengths differ by one. Since the greatest common divisor of two consecutive integers is one, the graph H_S is ergodic. ☐

Recall that a permutation π in $\mathfrak{S}_n$ is *indecomposable* if there is no index i such that $1 \leq i \leq n-1$ and $\pi(1), \ldots, \pi(i) \leq i$. Otherwise the permutation is decomposable.

Proposition 3.8. *Let $S \subseteq \mathfrak{S}_{m+1}$ such that each permutation $\tau \in S$ is indecomposable. Then the graph G_S is strongly connected.*

Proof. Given two vertices $\pi = (\pi_1, \ldots, \pi_m)$ and $\sigma = (\sigma_1, \ldots, \sigma_m)$ of G_S. Since the integers $\pi_1, \ldots, \pi_m$, $\sigma_1 + m, \ldots, \sigma_m + m$ are distinct, the following path (described by its edges) is well defined:

$$\Pi(\pi_1, \ldots, \pi_m, \sigma_1 + m), \Pi(\pi_2, \ldots, \pi_m, \sigma_1 + m, \sigma_2 + m), \ldots, \Pi(\pi_m, \sigma_1 + m, \ldots, \sigma_m + m).$$

This path goes from the vertex π to vertex σ. Note that every edge τ' on the path is decomposable. Hence this path avoids the forbidden indecomposable edges of S. ☐

Proof of Theorem 1.5. This follows directly from Theorems 1.2 and 1.7 and Propositions 3.7 and 3.8. □

Proof of Corollary 1.4. Let τ be the single permutation in the set S. If τ is one of the two monotone permutations the result will follow from descent pattern avoidance, see Theorem 4.2. If τ is indecomposable the result follows from Theorem 1.5. Finally if τ is decomposable apply Theorem 1.5 to the upside down permutation $(m+2-\tau(1), \ldots, m+2-\tau(m+1))$ which is not decomposable. □

Note that there are examples of patterns S so that H_S does not have cycles even though the graph G_S has cycles.

Example 3.9. $S = \{312, 321\}$. In this case the graph G_S has a cycle. However, the graph H_S does not have a cycle and hence is not strongly connected. We observe this by noting that for a directed edge $(x, y) \longrightarrow (y, z)$ in H_S we have that $x < \max(y, z)$. Hence none of x_i's in a k-cycle $(x_1, x_2) \longrightarrow (x_2, x_3) \longrightarrow \cdots \longrightarrow (x_k, x_1) \longrightarrow (x_1, x_2)$ can be the largest.

Via the classical bijection $\pi \longmapsto \hat{\pi}$ (see [19, Section 1.3]) one obtains that the number $\{312, 321\}$-avoiding permutations are in bijection with involutions, that is, permutations π such that $\pi^2 = \mathrm{id}$. This was first observed by Claesson [4]. It follows that the generating function is $\exp(z + z^2/2)$ and the asymptotic is $1/\sqrt{2} \cdot \exp(-1/4) \cdot (n/e)^{n/2} \cdot \exp(\sqrt{n})$.

On the other hand, if G_S does not have a cycle, then neither does H_S.

Lemma 3.10. *Let $S \subseteq \mathfrak{S}_{m+1}$ and suppose that G_S has a directed cycle that contains the two vertices $12\cdots m$ and $m\cdots 21$. Moreover, assume that the two monotone permutations $12\cdots(m+1)$ and $(m+1)\cdots 21$ do not belong to the set S. Then the graph H_S contains a directed cycle.*

Proof. Pick a vertex $x = (x_1, \ldots, x_m)$ such that $\Pi(x) = 12\cdots m$. The directed path from the vertex $12\cdots m$ to the vertex $m\cdots 21$ can be lifted to a path in H_S from the vertex x to a vertex $y = (y_1, \ldots, y_m)$ where $\Pi(y) = m\cdots 21$. Similarly, pick a vertex $z = (z_1, \ldots, z_m)$ such that $\Pi(z) = m\cdots 21$. The directed path from the vertex $m\cdots 21$ to the vertex $12\cdots m$ can be lifted to a path in H_S from the vertex z to a vertex $w = (w_1, \ldots, w_m)$ where $\Pi(w) = 12\cdots m$.

Using the two monotone functions $\alpha, \beta : (0,1) \longrightarrow (0,1)$ defined by $\alpha(x) = (x+1)/2$ and $\beta(x) = x/2$, we have two directed paths: one from $\alpha(x)$ to $\alpha(y)$ and one from $\beta(z)$ to $\beta(w)$.

Using that $(m+1)\cdots 21$ is an edge in G_S we have the following path from $\alpha(y)$ to $\beta(z)$, namely

$$\alpha(y) = (\alpha(y_1), \ldots, \alpha(y_m)) \longrightarrow (\alpha(y_2), \ldots, \alpha(y_m), \beta(z_1)) \longrightarrow \cdots \longrightarrow (\beta(z_1), \ldots, \beta(z_m)) = \beta(z).$$

Similarly, we have the directed path

$$\beta(w) = (\beta(w_1), \ldots, \beta(w_m)) \longrightarrow (\beta(w_2), \ldots, \beta(w_m), \alpha(x_1)) \longrightarrow \cdots \longrightarrow (\alpha(x_1), \ldots, \alpha(x_m)) = \alpha(x).$$

Concatenate these four directed paths to obtain a directed cycle in H_S. □

4 Computational techniques

In this section we discuss descent pattern avoidance which is a special case of pattern avoidance. First we introduce an analogue of the de Bruijn graph D_U, which has the advantage that it is smaller than the graph G_S. Moreover, if the graph D_U is ergodic so is the graph $H_{S(U)}$ and we obtain that the associated operator is positivity improving. Second, for descent pattern avoidance we obtain that the eigenfunctions has a simplified form. Finally, in the last subsection we consider pattern avoidance that has symmetry. In these cases we show that we can obtain the adjoint eigenfunctions from the eigenfunctions.

4.1 Descent pattern avoidance

The descent set of a permutation π in the symmetric group on n elements is the subset of $\{1, \ldots, n-1\}$, given by $\{i : \pi_i > \pi_{i+1}\}$. An equivalent notion is the descent word, defined as follows. The descent word of the permutation π is the word $u(\pi) = u_1 \cdots u_{n-1}$ where $u_i = a$ if $\pi_i < \pi_{i+1}$ and $u_i = b$ otherwise.

Let U be a collection of ab-words of length m. The permutation π avoids the set U if there is no consecutive subword of the descent word of π contained in the collection U.

Descent pattern avoidance is a special case of consecutive pattern avoidance. For instance, permutations avoiding the word aab is the permutations avoiding the set $S = \{1243, 1342, 2341\}$, since these three permutations are the permutations with descent word aab. More formally, for U a subset of $\{a, b\}^m$ define $S(U) \subseteq \mathfrak{S}_{m+1}$ by

$$S(U) = \{\pi \in \mathfrak{S}_{m+1} \ : \ u(\pi) \in U\}.$$

Then the set of permutations avoiding the descent words in U is the set of permutations avoiding $S(U)$.

For U a subset of $\{a, b\}^m$ define the associated *de Bruijn graph* D_U by letting the vertex set be $\{a, b\}^{m-1}$. For $x, y \in \{a, b\}$ and $u \in \{a, b\}^{m-2}$ such that $xuy \notin U$ let there be a directed edge from xu to uy. When the set U is empty, the graph D_U is the classical de Bruijn graph D_{m-1}.

Lemma 4.1. *Let U be a subset of $\{a, b\}^m$. If there is a cycle c of length $N \geq n+1$ that do not consists only of the loop a^m or not only of the loop b^m, then the cycle c can be lifted to a cycle of length N in the graph $H_{S(U)}$.*

Proof. Let c be the cycle

$$v_1 v_2 \cdots v_{m-1} \longrightarrow v_2 v_3 \cdots v_m \longrightarrow \cdots \longrightarrow v_N v_1 \cdots v_{m-2} \longrightarrow v_1 v_2 \cdots v_{m-1},$$

where each v_i is either a or b. We would like to pick N real numbers $x_1, \ldots, x_N$ in the interval $(0, 1)$ such that

$$\begin{array}{lll} x_i < x_{i+1} & \text{if} & v_i = a, \\ x_i > x_{i+1} & \text{if} & v_i = b, \end{array} \tag{4.1}$$

where all the indices are modulo N. Since all the letters v_1 through v_N are not the same, we may assume that $v_{N-1} = a$ and $v_N = b$. Pick x_1 arbitrarily. Pick x_2 through x_{N-1} such that inequality (4.1) is satisfied. Finally, pick x_N in the interval $(\max(x_{N-1}, x_1), 1)$. Now in the graph $H_{S(U)}$ we have the cycle

$$(x_1, x_2, \ldots, x_m) \longrightarrow (x_2, x_3, \ldots, x_{m+1}) \longrightarrow \cdots \longrightarrow (x_N, x_1, \ldots, x_{m-1}) \longrightarrow (x_1, x_2, \ldots, x_m).$$

□

Theorem 4.2. *Let U be a subset of $\{a, b\}^m$. If the de Bruijn graph D_U is strongly connected, then the graph $H_{S(U)}$ is also strongly connected. Furthermore, the de Bruijn graph D_U has the same period as the graph $H_{S(U)}$.*

Proof. Note that the graph D_U has the directed edge $a^{m-1}b$ since otherwise it would not be strongly connected. Given two vertices x and y in $H_{S(U)}$. To prove that $H_{S(U)}$ is strongly connected it is enough to find a directed path from x to y.

We can find a path from $u(\Pi(x))$ to a^{m-1} in the graph D_U that consists of at least $m+1$ edges. Similarly to the lifting lemma, Lemma 3.6, we can lift this path to a path in $H_{S(U)}$ that starts at the vertex x and ends, say, in the vertex $z = (z_1, \ldots, z_m)$. Note that $z_1 < \cdots < z_m$. Moreover, we can find a path from $a^{m-2}b$ to $u(\Pi(y))$ in D_U that has length at least $m+1$. Lift this path to a path that ends in the vertex y and begins at $w = (w_1, \ldots, w_m)$, where $w_1 < \cdots < w_{m-1} > w_m$.

Let $v_i = \max(z_i, w_{i-1})$ for $2 \leq i \leq m$. Observe that we have the string of inequalities $z_1 < v_2 < \cdots < v_m > w_m$. We can now concatenate these two paths as follows. Replace each occurrence of z_i and w_{i-1} by v_i for $2 \leq i \leq m$ in each of the two paths. Then we may connect the vertex $(z_1, v_2, \ldots, v_m)$ with the vertex $(v_2, \ldots, v_m, w_m)$ via the edge that goes across the edge with descent word $a^{m-1}b$. Thus the graph $H_{S(U)}$ is strongly connected.

Since there is a graph homomorphism from $H_{S(U)}$ to D_U we know that the period of D_U divides the period of $H_{S(U)}$. To see that the periods are equal, pick a vertex w of D_U that differs from a^{m-1} and b^{m-1}. Then any cycle of length greater than $m+1$ through the vertex w in D_U lifts to a cycle of the same length in $H_{S(U)}$. Hence the greatest common divisor of lengths of cycles through w is a multiple of the greatest common divisor of the cycle lengths of $H_{S(U)}$. Hence the two periods are equal.

Note that this argument only works when $m \geq 3$ since there is no such vertex w in the $m = 2$ case. But the remaining $m = 2$ case is straightforward to check. $\square$

4.2 Invariant subspace for descent pattern avoidance

For an ab-word u of length $m-1$ define the descent polytope P_u to be the subset of the unit cube $[0,1]^m$ corresponding to all vectors with descent word u. That is,

$$P_u = \{(x_1, \ldots, x_m) \in [0,1]^m \ : \ x_i \leq x_{i+1} \text{ if } u_i = a \text{ and } x_i \geq x_{i+1} \text{ if } u_i = b\}.$$

Observe that the m-dimensional unit cube is the union of the 2^{m-1} descent polytopes P_u. Now the operator T corresponding to the descent pattern avoidance of the set U has the following form. For an ab-word u of length $m-2$ and $y \in \{a, b\}$ we have

$$\begin{aligned} T(f)|_{P_{uy}} &= \int_0^{x_1} \chi(auy) \cdot f(t, x_1, \ldots, x_{m-1})|_{P_{au}} \, dt \\ &+ \int_{x_1}^1 \chi(buy) \cdot f(t, x_1, \ldots, x_{m-1})|_{P_{bu}} \, dt, \end{aligned} \tag{4.2}$$

where by abuse of notation we let $\chi(w) = 1$ if w does not belong to the set U and $\chi(w) = 0$ otherwise.

Proposition 4.3. *Let T be the operator associated with a descent pattern avoidance and k is an integer such that $0 \leq k \leq m-1$. Let u be an ab-word of length $m-1$. Then the function $T^k(f)$ restricted to the descent polytope P_u only depends on the variables x_1 through x_{m-k}.*

Proof. Proof by induction on k. When $k = 0$ there is nothing to prove. When $1 \leq k \leq m-1$, we know by induction that the restriction of $T^{k-1}(f)$ only depends on $x_1, \ldots, x_{m-k+1}$. By the shift of variables in the right hand side of equation (4.2), we obtain that $T^k(f)$ does not depend on the variable x_{m-k+1}, completing the induction. $\square$

Corollary 4.4. *Let T be the operator associated with a descent pattern avoidance and let φ be an eigenfunction associated with a non-zero eigenvalue λ. Then the eigenfunction restricted to each descent polytope P_u only depends on the variable x_1.*

Proof. Since $\lambda^{m-1} \cdot \varphi = T^{m-1}(\varphi)$ the eigenfunction has the required form. □

Corollary 4.5. *Let T be the operator associated with a descent pattern avoidance and let φ be an eigenfunction associated with a non-zero eigenvalue λ. Assume that f is a generalized eigenfunction, that is, it satisfies the equation $\lambda \cdot f = T(f) + \varphi$. Then the function f restricted to each descent polytope P_u only depends on the variable x_1.*

Proof. By induction on k. Assume that $1 < k \leq m$ and that f restricted to each descent polytope only depends on the variables x_1 through x_k. Then $T(f) + \varphi$ only depends on x_1 through x_{k-1} showing that $\lambda \cdot f$ only depends on $x_1, \ldots, x_{k-1}$. □

Let V be the subspace of $L^2([0,1]^m)$ consisting of all functions f that only depend on the variable x_1 when restricted to each of the descent polytopes P_u. Observe that the subspace V is invariant under the operator T. That is, the operator T restricts to the subspace V. Moreover the constant function $\mathbf{1}$ belongs to V. Hence to understand the behavior of $T^n(\mathbf{1})$ it is enough to study this restricted operator.

In order to describe the subspace V more explicitly define for an ab-word u of length $m-1$ the polynomial $h(u; x_1)$ as follows:

$$h(u; x_1) = \int_{(x_1, x_2, \ldots, x_m) \in P_u} 1 dx_2 \cdots dx_m.$$

These polynomials were first introduced and studied in [8], with different notation.

Let p be a vector $(p_u(x_1))_{u \in \{a,b\}^{m-1}}$. That is, the vector p consists of one-variable functions in the variable x_1 and is indexed by ab-words of length $m-1$. Consider the function f on $[0,1]^m$ defined by

$$f(x_1, \ldots, x_m)|_{P_u} = p_u(x_1)$$

for all ab-words u of length $m-1$. Observe that the function f belongs to $L^2([0,1]^m)$, and hence to the invariant subspace V, if and only if

$$\int_0^1 h(u; x_1) \cdot |p_u(x_1)|^2 \, dx_1 < \infty$$

for all ab-words u of length $m-1$. For two functions f and g in the subspace V, corresponding to the two vectors $(p_u(x_1))_{u \in \{a,b\}^{m-1}}$ and $(q_u(x_1))_{u \in \{a,b\}^{m-1}}$, the inner product is given by

$$(f, g) = \sum_{u \in \{a,b\}^{m-1}} \int_0^1 h(u; x_1) \cdot p_u(x_1) \cdot \overline{q_u(x_1)} dx_1.$$

We end this section by a structural result about the subspace V.

Proposition 4.6. *The invariant subspace V is isometrically isomorphic to the Hilbert space*

$$L^2\left([0,1]\right)^{2^{m-1}}.$$

Proof. The isomorphism of the Hilbert spaces $V \longrightarrow L^2([0,1])^{2^{m-1}}$ is given by

$$(p_u(x_1))_{u \in \{a,b\}^{m-1}} \longmapsto \left(\sqrt{h(u; x_1)} \cdot p_u(x_1)\right)_{u \in \{a,b\}^{m-1}}.$$

□

4.3 Symmetries

Let J and R be the following two involutions on the space $L^2([0,1]^m)$:

$$\begin{aligned}(Jf)(x_1, x_2, \ldots, x_m) &= f(1-x_m, \ldots, 1-x_2, 1-x_1),\\ (Rf)(x_1, x_2, \ldots, x_m) &= f(x_m, \ldots, x_2, x_1).\end{aligned}$$

Observe that both J and R are self adjoint operators.

Lemma 4.7. *Assume that χ has the symmetry*

$$\chi(x_1, x_2, \ldots, x_m, x_{m+1}) = \chi(1-x_{m+1}, 1-x_m, \ldots, 1-x_2, 1-x_1).$$

Then the adjoint of the associated operator T is given by $T^* = JTJ$. Moreover, if φ is an eigenfunction of the operator T with eigenvalue λ then $\psi = J\varphi$ is an eigenfunction of the adjoint T^* with the eigenvalue λ. Furthermore, we have the equality $\left(\mathbf{1}, \overline{\psi}\right) = (\varphi, \mathbf{1})$.

Proof. We have that

$$\begin{aligned}JTJf(x_1, x_2, \ldots, x_m) &= JTf(1-x_m, \ldots, 1-x_2, 1-x_1)\\ &= J\int_0^1 \chi(t, x_1, \ldots, x_m) \cdot f(1-x_{m-1}, \ldots, 1-x_1, 1-t)dt\\ &= \int_0^1 \chi(t, 1-x_m, \ldots, 1-x_1) \cdot f(x_2, \ldots, x_m, 1-t)dt\\ &= \int_0^1 \chi(1-t, 1-x_m, \ldots, 1-x_1) \cdot f(x_2, \ldots, x_m, t)dt\\ &= \int_0^1 \chi(x_1, \ldots, x_m, t) \cdot f(x_2, \ldots, x_m, t)dt\\ &= T^* f(x_1, \ldots, x_{m-1}, x_m).\end{aligned}$$

For the second statement consider the following line of equalities $T^*J\varphi = JTJJ\varphi = JT\varphi = \lambda \cdot J\varphi$. Lastly, $\left(\mathbf{1}, \overline{\psi}\right) = \left(\mathbf{1}, \overline{J\varphi}\right) = (\mathbf{1}, J\overline{\varphi}) = (J\mathbf{1}, \overline{\varphi}) = (\mathbf{1}, \overline{\varphi}) = (\varphi, \mathbf{1})$. □

Similarly to Lemma 4.7 we have the next lemma. Its proof is similar to the previous proof and hence omitted.

Lemma 4.8. *Assume that χ has the symmetry*

$$\chi(x_1, x_2, \ldots, x_m, x_{m+1}) = \chi(x_{m+1}, x_m, \ldots, x_2, x_1).$$

Then we have that the adjoint of the associated operator T is given by $T^* = RTR$. Moreover, if φ is an eigenfunction of the operator T with eigenvalue λ then $\psi = R\varphi$ is an eigenfunction of the adjoint T^* with the eigenvalue λ. Furthermore, we have the equality $\left(\mathbf{1}, \overline{\psi}\right) = (\varphi, \mathbf{1})$.

5 123-Avoiding permutations

A 123-avoiding permutation is a permutation $\pi \in \mathfrak{S}_n$ with no index j so that $\pi_j < \pi_{j+1} < \pi_{j+2}$, where $1 \leq j \leq n-2$. Let $\alpha_n(123)$ denote the number of 123-avoiding permutations in $\mathfrak{S}_n$.

5.1 Eigenvalues and eigenfunctions

Since 123-avoiding permutations can be viewed as permutations with no double descents Corollary 4.4 allows us to recast the problem of finding eigenfunctions in two variables into finding two one-variable functions.

Proposition 5.1. *The eigenvalues of the operator T are given by*

$$\lambda_k = \frac{\sqrt{3}}{2 \cdot \pi \cdot \left(k + \frac{1}{3}\right)}, \tag{5.1}$$

where $k \in \mathbb{Z}$ and the associated eigenfunctions are given by

$$\varphi_k = \exp\left(-\frac{x}{2 \cdot \lambda}\right) \cdot \begin{cases} \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} \cdot \frac{x}{\lambda}\right) & \text{if } 0 \leq x \leq y \leq 1, \\ \sin\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \cdot \frac{x}{\lambda}\right) & \text{if } 0 \leq y \leq x \leq 1. \end{cases} \tag{5.2}$$

Proof. Avoiding the pattern 123 is equivalent to avoiding the descent set pattern aa. Hence Corollary 4.4. states that the eigenfunctions φ can be written as

$$\varphi = \begin{cases} p(x) & \text{if } 0 \leq x \leq y \leq 1, \\ q(x) & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Then the defining equations for eigenvalues and eigenfunctions reduces to the integral system:

$$\lambda \cdot p(x) = \int_x^1 q(t)dt, \tag{5.3}$$

$$\lambda \cdot q(x) = \int_0^x p(t)dt + \int_x^1 q(t)dt. \tag{5.4}$$

First, differentiating with respect to x, we obtain the first-order system

$$\lambda \cdot p'(x) = -q(x), \tag{5.5}$$

$$\lambda \cdot q'(x) = p(x) - q(x). \tag{5.6}$$

These equations have only the trivial solution if $\lambda = 0$, so $\lambda = 0$ is not an eigenvalue. If $\lambda \neq 0$ then the first-order system (5.5)–(5.6) implies the second-order equation

$$\lambda^2 \cdot p''(x) + \lambda \cdot p'(x) + p(x) = 0.$$

This equation has the general solution

$$p(x) = A \cdot \exp\left(\frac{\omega}{\lambda} \cdot x\right) + B \cdot \exp\left(\frac{\omega^2}{\lambda} \cdot x\right), \tag{5.7}$$

where $\omega = \exp\left(\frac{2 \cdot \pi \cdot i}{3}\right)$. That is, ω satisfies the relation $\omega^2 + \omega + 1 = 0$. Moreover, equation (5.5) implies that

$$q(x) = -\omega \cdot A \cdot \exp\left(\frac{\omega}{\lambda} \cdot x\right) - \omega^2 \cdot B \cdot \exp\left(\frac{\omega^2}{\lambda} \cdot x\right). \tag{5.8}$$

Setting $x = 0$ and $x = 1$ in equations (5.3) and (5.4) and using that $\lambda \neq 0$ we obtain the boundary conditions:

$$p(0) = q(0), \tag{5.9}$$
$$p(1) = 0. \tag{5.10}$$

Substituting the expressions for $p(x)$ and $q(x)$ from equations (5.5) and (5.6) into boundary condition (5.9) we obtain $A + B = -\omega \cdot A - \omega^2 \cdot B$. This is equivalent to $\omega \cdot A + B = 0$. Hence we may set $A = 1/2 \cdot \exp\left(\frac{\pi \cdot i}{6}\right)$ and $B = \overline{A} = 1/2 \cdot \exp\left(-\frac{\pi \cdot i}{6}\right)$. Substituting equation (5.5) into the second boundary condition (5.10) implies that

$$A \cdot \exp\left(\frac{\omega}{\lambda}\right) = -B \cdot \exp\left(\frac{\omega^2}{\lambda}\right) = \omega \cdot A \cdot \exp\left(\frac{\omega^2}{\lambda}\right).$$

Cancelling A on both sides and taking the logarithm gives

$$\frac{\omega}{\lambda} = \frac{2 \cdot \pi \cdot i}{3} + \frac{\omega^2}{\lambda} + 2 \cdot \pi \cdot i \cdot k,$$

where k is an integer. Since $\omega - \omega^2 = \sqrt{3} \cdot i$ we obtain expression (5.1). Moreover $p(x)$ is given by

$$\begin{aligned} p(x) &= \frac{1}{2} \cdot \exp\left(\frac{\pi \cdot i}{6}\right) \cdot \exp\left(\omega \cdot \frac{x}{\lambda}\right) + \frac{1}{2} \cdot \exp\left(-\frac{\pi \cdot i}{6}\right) \cdot \exp\left(\omega^2 \cdot \frac{x}{\lambda}\right) \\ &= \frac{\exp\left(-\frac{x}{2 \cdot \lambda}\right)}{2} \cdot \left(\exp\left(\frac{\pi \cdot i}{6} + \frac{\sqrt{3}}{2} \cdot i \cdot \frac{x}{\lambda}\right) + \exp\left(-\frac{\pi \cdot i}{6} - \frac{\sqrt{3}}{2} \cdot i \cdot \frac{x}{\lambda}\right)\right) \\ &= \exp\left(-\frac{x}{2 \cdot \lambda}\right) \cdot \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} \cdot \frac{x}{\lambda}\right). \end{aligned}$$

Now equation (5.5) implies the claimed expression for $q(x)$. □

Note that the eigenvalues are ordered by

$$\lambda_0 > -\lambda_{-1} > \lambda_1 > -\lambda_{-2} > \lambda_2 > -\lambda_{-3} > \lambda_3 > \cdots > 0.$$

Furthermore, the calculations in Proposition 5.1 showed that they all have a unique eigenfunction. It remains to show that they are simple.

Proposition 5.2. *The eigenvalues λ of the operator T are simple, that is, they have index* 1. *In other words, there is no function $f(x, y)$ such that $\lambda_k \cdot f = T(f) + \varphi_k$.*

Sketch of proof. Using Corollary 4.5 we can write the function f as

$$f(x, y) = \begin{cases} r(x) & \text{if } 0 \leq x \leq y \leq 1, \\ s(x) & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Then equation reduces to the integral system:

$$\begin{aligned} \lambda \cdot r(x) &= \int_x^1 s(t)dt + p(x), \\ \lambda \cdot s(x) &= \int_0^x r(t)dt + \int_x^1 s(t)dt + q(x). \end{aligned}$$

Note that we obtain the two boundary conditions $r(1) = 0$ and $r(0) = s(0)$. Next differentiating with respect to x, we have the first-order system:

$$\begin{aligned} \lambda \cdot r'(x) &= -s(x) + p'(x), \\ \lambda \cdot s'(x) &= r(x) - s(x) + q'(x). \end{aligned}$$

This system of differential equations can be solved as in the proof of Proposition 5.1. However, the solution does not satisfy the boundary conditions, completing the sketch.

By applying the involution J we obtain the adjoint eigenfunction

$$\psi_k = \exp\left(\frac{y-1}{2\cdot\lambda}\right) \cdot \begin{cases} \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} \cdot \frac{1-y}{\lambda}\right) & \text{if } 0 \le x \le y \le 1, \\ \sin\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \cdot \frac{1-y}{\lambda}\right) & \text{if } 0 \le y \le x \le 1. \end{cases} \tag{5.11}$$

Proposition 5.3. *Let λ be an eigenvalue of T with eigenfunctions φ and let ψ be the eigenfunction of the adjoint operator T^* with eigenvalue λ. Then the following identities hold:*

$$(\varphi, \mathbf{1}) = \left(\mathbf{1}, \overline{\psi}\right) = \frac{\sqrt{3}}{2} \cdot \lambda^2, \tag{5.12}$$

$$\left(\varphi, \overline{\psi}\right) = \frac{3}{4} \cdot (-1)^k \cdot \lambda \cdot \exp\left(-\frac{1}{2\cdot\lambda}\right). \tag{5.13}$$

In particular

$$\frac{(\varphi, \mathbf{1}) \cdot \left(\mathbf{1}, \overline{\psi}\right)}{\left(\varphi, \overline{\psi}\right)} = (-1)^k \cdot \lambda^3 \cdot \exp\left(\frac{1}{2\cdot\lambda}\right). \tag{5.14}$$

Proof. In the following calculations we use the facts that $\cos(\sqrt{3}/(2\cdot\lambda)) = (-1)^k/2$ and $\sin(\sqrt{3}/(2\cdot\lambda)) = (-1)^k\sqrt{3}/2$. We also use the expression for φ in equation (5.2). First, we note that

$$\begin{aligned} (\varphi, \mathbf{1}) &= \int_{0\le x\le y\le 1} p(x)dxdy + \int_{0\le y\le x\le 1} q(x)dxdy \\ &= \int_0^1 (1-x)p(x)\cdot dx + \int_0^1 x\cdot q(x)dx. \end{aligned}$$

Explicit computation shows that

$$\begin{aligned} \int_0^1 (1-x)\cdot p(x)dx &= \frac{\sqrt{3}}{2} \cdot \lambda^2 \cdot (1 - (-1)^k \cdot \exp(-1/(2\cdot\lambda))) \\ \int_0^1 x\cdot q(x)dx &= \frac{\sqrt{3}}{2} \cdot \lambda^2 \cdot (-1)^k \cdot \exp(-1/(2\cdot\lambda)) \end{aligned}$$

which shows (5.12). Next, using (5.2) and (5.11) and dropping subscripts as before, we have

$$\begin{aligned} \left(\varphi, \overline{\psi}\right) &= \int_{0\le x\le y\le 1} p(x)\cdot p(1-y)dxdy + \int_{0\le y\le x\le 1} q(x)\cdot q(1-y)dxdy \\ &= \int_0^1 \left(p(x)\cdot\int_x^1 p(1-y)dy + q(x)\cdot\int_0^x q(1-y)dy\right) dx. \end{aligned}$$

Carrying out the y integration and simplifying, we obtain

$$\left(\varphi, \overline{\psi}\right) = \frac{3}{4} \cdot (-1)^k \cdot \int_0^1 \lambda \cdot \exp\left(-\frac{1}{2\cdot\lambda}\right) dx$$

which gives (5.13). □

5.2 Asymptotics

The above computations show that all eigenvalues of T are simple and give the eigenvalues and the coefficients explicitly. We thus obtain the following expansion for $\left(T^{n-2}(\mathbf{1}), \mathbf{1}\right) = \alpha_n(123)/n!$ as an immediate consequence of Theorem 1.1, Propositions 5.1 and 5.3.

Theorem 5.4. *Let K be a non-negative integer. The number of 123-avoiding permutations satisfies the following asymptotic expansion*

$$\frac{\alpha_n(123)}{n!} = \sum_{|k| \leq K} (-1)^k \cdot \exp\left(\frac{1}{2 \cdot \lambda_k}\right) \cdot \lambda_k^{n+1} + O\left(r_{K+1}^n\right),$$

where λ_k is given by (5.1), $r_k = |\lambda_{-k}| = \sqrt{3}/\left(2 \cdot \pi \cdot \left(k - \frac{1}{3}\right)\right)$ and the sum contains $2K+1$ terms corresponding to the $2K+1$ largest eigenvalues.

6 213-Avoiding permutations

A 213-avoiding permutation is a permutation $\pi \in \mathfrak{S}_n$ which contains no sequence of the form

$$\pi_{j+1} < \pi_j < \pi_{j+2}$$

for any j with $1 \leq j \leq n-2$. We denote the number of 213-avoiding permutations of $\mathfrak{S}_n$ by $\alpha_n(213)$. Thus, S consists of the single permutation 213 and

$$\chi(x_1, x_2, x_3) = \begin{cases} 0 & \text{if } x_2 \leq x_1 \leq x_3, \\ 1 & \text{otherwise.} \end{cases}$$

By symmetry, the study of 213-avoiding permutations is equivalent to 132-avoiding permutations, 231-avoiding permutations and 312-avoiding permutations. However the case of 213-avoiding permutations gives the most straightforward equations.

6.1 Eigenvalues and eigenfunctions

In what follows, we will make use of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \cdot \int_0^x \exp(-t^2) dt \tag{6.1}$$

which extends to an entire function on $\mathbb{C}$, and the function

$$q(x) = \exp\left(-\frac{x^2}{2 \cdot \lambda^2}\right). \tag{6.2}$$

Let

$$f(x, y) = \begin{cases} p(x, y) & \text{if } 0 \leq x \leq y \leq 1, \\ q(x, y) & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Then

$$(Tf)(x, y) = \begin{cases} \int_0^x p(t, x) dt + \int_y^1 q(t, x) dt & \text{if } 0 \leq x \leq y \leq 1, \\ \int_0^x p(t, x) dt + \int_x^1 q(t, x) dt & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Now we characterize the nonzero eigenvalues and eigenfunctions.

Proposition 6.1. *The non-zero eigenvalues* λ *of the operator* T *satisfy the equation*

$$\operatorname{erf}\left(\frac{1}{\sqrt{2}\cdot\lambda}\right) = \sqrt{\frac{2}{\pi}} \tag{6.3}$$

and the corresponding eigenfunctions are

$$\varphi(x,y) = \begin{cases} q(x) - \frac{1}{\lambda}\cdot\int_x^y q(t)dt & \text{if } 0 \le x \le y \le 1, \\ q(x) & \text{if } 0 \le y \le x \le 1, \end{cases}$$

where $q(x)$ *is given by (6.2).*

Proof. The defining relations for the eigenfunctions are

$$\begin{aligned} \lambda\cdot p(x,y) &= \int_0^x p(t,x)dt + \int_y^1 q(t,x)dt, && (6.4)\\ \lambda\cdot q(x,y) &= \int_0^x p(t,x)dt + \int_x^1 q(t,x)dt. && (6.5) \end{aligned}$$

Now observe that in the right-hand side of equation (6.5) there is no dependency on the variable y. Hence we may replace $q(x,y)$ with $q(x)$. Now subtract equation (6.5) from equation (6.4)

$$\lambda\cdot(p(x,y) - q(x)) = -\int_x^y q(t)dt.$$

That is,

$$p(x,y) = q(x) - \frac{1}{\lambda}\cdot\int_x^y q(t)dt. \tag{6.6}$$

Substitute equation (6.6) into equation (6.5):

$$\begin{aligned} \lambda\cdot q(x) &= \int_0^x \left(q(t) - \frac{1}{\lambda}\cdot\int_t^x q(s)ds\right)dt + \int_x^1 q(t)dt \\ &= \int_0^1 q(t)dt - \frac{1}{\lambda}\cdot\int_0^x\int_t^x q(s)dsdt \\ &= \int_0^1 q(t)dt - \frac{1}{\lambda}\cdot\int_0^x\int_0^s q(s)dtds \\ &= \int_0^1 q(t)dt - \frac{1}{\lambda}\cdot\int_0^x s\cdot q(s)ds. \end{aligned}$$

Hence we have the following integral equation for $q(x)$

$$\lambda^2\cdot q(x) = \lambda\cdot\int_0^1 q(t)dt - \int_0^x s\cdot q(s)ds. \tag{6.7}$$

Differentiating once we have

$$\lambda^2\cdot q'(x) = -x\cdot q(x). \tag{6.8}$$

The solution to this differential equation is

$$q(x) = C\cdot\exp\left(-\frac{x^2}{2\cdot\lambda^2}\right). \tag{6.9}$$

By setting the constant C to be 1 we obtain a solution to equation (6.8). Now substitute this solution for $q(x)$ into the integral equation (6.7) and set $x = 0$:

$$\begin{aligned} \lambda^2 &= \lambda \cdot \int_0^1 \exp\left(-\frac{t^2}{2\cdot\lambda^2}\right) dt \\ &= \sqrt{2}\cdot\lambda^2\cdot\int_0^{1/(\sqrt{2}\cdot\lambda)} \exp\left(-u^2\right) du \\ &= \frac{\sqrt{\pi}\cdot\lambda^2}{\sqrt{2}} \cdot \operatorname{erf}(1/(\sqrt{2}\cdot\lambda)), \end{aligned}$$

where the substitution in the integral is $u = t/(\sqrt{2}\cdot\lambda)$. Hence the non-zero eigenvalues λ satisfy equation (6.3). $\square$

Proposition 6.2. *The eigenvalues λ of the operator T that have index greater than or equal to 2, satisfy the equation*

$$\exp\left(-\frac{1}{2\cdot\lambda^2}\right) = \lambda - 1. \tag{6.10}$$

In other words, if there is a function $f(x, y)$ such that $\lambda \cdot f = T(f) + \varphi$ this implies that λ satisfies equation (6.10).

Proof. The defining relations for the eigenfunctions are

$$\lambda\cdot r(x,y) = \int_0^x r(t,x)dt + \int_y^1 s(t,x)dt + p(x,y), \tag{6.11}$$

$$\lambda\cdot s(x,y) = \int_0^x r(t,x)dt + \int_x^1 s(t,x)dt + q(x). \tag{6.12}$$

Note that in the right-hand side of equation (6.12) there is no dependency on the variable y. Hence $s(x, y)$ is a function of x only, and we write $s(x)$ henceforth. Subtracting equation (6.12) from equation (6.11) and dividing by λ gives

$$\begin{aligned} r(x,y) &= s(x) - \frac{1}{\lambda}\cdot\int_x^y s(t)dt + \frac{1}{\lambda}\cdot(p(x,y) - q(x)) \\ &= s(x) - \frac{1}{\lambda^2}\cdot\int_x^y (\lambda\cdot s(t) + q(t))dt, \end{aligned}$$

where the last step is by equation (6.6). Substituting this expression into equation (6.12) yields the integral equation for $s(x)$:

$$\begin{aligned} \lambda\cdot s(x) &= \int_0^x \left(s(t) - \frac{1}{\lambda^2}\cdot\int_t^x (\lambda\cdot s(u) + q(u))du\right) dt + \int_x^1 s(t)dt + q(x) \\ &= -\frac{1}{\lambda^2}\cdot\int_0^x\int_t^x (\lambda\cdot s(u) + q(u))dudt + \int_0^1 s(t)dt + q(x) \\ &= -\frac{1}{\lambda^2}\cdot\int_0^x\int_0^u (\lambda\cdot s(u) + q(u))dtdu + \int_0^1 s(t)dt + q(x) \\ &= -\frac{1}{\lambda^2}\cdot\int_0^x u\cdot(\lambda\cdot s(u) + q(u))du + \int_0^1 s(t)dt + q(x) \\ &= -\frac{1}{\lambda}\cdot\int_0^x u\cdot s(u)du + \int_0^1 s(t)dt - \frac{1}{\lambda}\cdot\int_0^1 q(t)dt + 2\cdot q(x), \end{aligned} \tag{6.13}$$

where the last step is using equation (6.7). Differentiate with respect to x to obtain the first order differential equation in $s(x)$:

$$\lambda \cdot s'(x) = -\frac{1}{\lambda} \cdot x \cdot s(x) + 2 \cdot q'(x).$$

The general solution to this differential equation is

$$s(x) = -\frac{x^2}{\lambda^2} \cdot \exp\left(-\frac{x^2}{2 \cdot \lambda^2}\right) + C \cdot q(x),$$

where we used that $q(x) = \exp\left(-\frac{x^2}{2 \cdot \lambda^2}\right)$. Observe that the homogeneous part of this solution is expected. It corresponds to the homogeneous part of the equation $\lambda \cdot f = T(f) + \varphi$. Hence we may set $C = 0$ without loss of generality. Setting $x = 0$ in (6.13) yields

$$0 = \int_0^1 s(t)dt - \frac{1}{\lambda} \cdot \int_0^1 q(t)dt + 2 \cdot q(0).$$

Using that $q(0) = 1$, $\int_0^1 q(t)dt = \lambda$ and

$$\begin{aligned} \int_0^1 s(t)dt &= \exp\left(-\frac{1}{2 \cdot \lambda^2}\right) - \sqrt{\frac{\pi}{2}} \cdot \lambda \cdot \operatorname{erf}\left(\frac{1}{\sqrt{2} \cdot \lambda}\right) \\ &= \exp\left(-\frac{1}{2 \cdot \lambda^2}\right) - \lambda, \end{aligned}$$

we obtain the desired equation. □

For completeness we state:

Lemma 6.3. *A function in the kernel of the operator T has the form*

$$\varphi(x, y) = \begin{cases} p(x, y) & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{if } 0 \le y \le x \le 1, \end{cases}$$

where $p(x, y)$ satisfies $\int_0^x p(t, x)dt = 0$.

The adjoint operator T^* is given by

$$T^*(f(x, y)) = \begin{cases} \int_0^y q(y, u)du + \int_y^1 p(y, u)du & \text{if } 0 \le x \le y \le 1, \\ \int_0^y q(y, u)du + \int_y^x p(y, u)du & \text{if } 0 \le y \le x \le 1. \end{cases}$$

Proposition 6.4. *For a non-zero eigenvalues λ of the operator T the corresponding eigenfunction of the adjoint operator T^* is*

$$\psi(x, y) = \begin{cases} p^*(y) & \text{if} 0 \le x \le y \le 1, \\ p^*(y) - \frac{1}{\lambda} \cdot \int_x^1 p^*(u)du & \text{if} 0 \le y \le x \le 1, \end{cases}$$

where

$$p^*(y) = -2 \cdot y \cdot \exp\left(\frac{y^2}{2 \cdot \lambda^2}\right) + 2 \cdot \lambda + \sqrt{2 \cdot \pi} \cdot y \cdot \exp\left(\frac{y^2}{2 \cdot \lambda^2}\right) \cdot \operatorname{erf}\left(\frac{y}{\sqrt{2}\lambda}\right). \tag{6.14}$$

Proof. The defining relations for the eigenfunctions are

$$\lambda \cdot p^*(x,y) = \int_0^y q^*(y,u)du + \int_y^1 p^*(y,u)du, \tag{6.15}$$

$$\lambda \cdot q^*(x,y) = \int_0^y q^*(y,u)du + \int_y^x p^*(y,u)du. \tag{6.16}$$

Observe that there is no dependency on the variable x in equation (6.15). Thus we write $p^*(x,y) = p^*(y)$. Subtracting these two equations we have

$$\lambda \cdot (q^*(x,y) - p^*(y)) = -\int_x^1 p^*(u)du,$$

such that

$$q^*(x,y) = p^*(y) - \frac{1}{\lambda} \cdot \int_x^1 p^*(u)du. \tag{6.17}$$

Substituting this expression into equation (6.15) one obtains

$$\begin{aligned} \lambda \cdot p^*(y) &= \int_0^y \left(p^*(u) - \frac{1}{\lambda} \cdot \int_y^1 p^*(v)dv\right) du + \int_y^1 p^*(u)du \\ &= \int_0^1 p^*(u)du - \frac{1}{\lambda} \cdot \int_0^y \int_y^1 p^*(v)dvdu \\ &= \int_0^1 p^*(u)du - \frac{1}{\lambda} \cdot y \cdot \int_y^1 p^*(v)dv. \end{aligned}$$

That is, $p^*(y)$ satisfies the integral equation

$$\lambda^2 \cdot p^*(y) = \lambda \cdot \int_0^1 p^*(u)du - y \cdot \int_y^1 p^*(v)dv. \tag{6.18}$$

Differentiating this equation twice we obtain

$$\lambda^2 \cdot p^{*\prime}(y) = y \cdot p^*(y) - \int_y^1 p^*(v)dv, \tag{6.19}$$

$$\lambda^2 \cdot p^{*\prime\prime}(y) = y \cdot p^{*\prime}(y) + 2 \cdot p^*(y). \tag{6.20}$$

The solution of this differential equation is given by

$$p^*(y) = C_1 \cdot y \cdot \exp\left(\frac{y^2}{2 \cdot \lambda^2}\right) + C_2 \cdot \left[2 \cdot \lambda + \sqrt{2 \cdot \pi} \cdot y \cdot \exp\left(\frac{y^2}{2 \cdot \lambda^2}\right) \cdot \operatorname{erf}\left(\frac{y}{\sqrt{2} \cdot \lambda}\right)\right]. \tag{6.21}$$

Setting $y = 0$ in equations (6.18) and (6.19) we obtain $\lambda \cdot p^*(0) = \int_0^1 p^*(u)du = -\lambda^2 \cdot p'(0)$. Inserting this condition into the solution of the differential equation (6.21) we obtain $C_1 = -2 \cdot C_2$. Moreover setting $C_2 = 1$ we obtain equation (6.14). □

Lemma 6.5. *A function in the kernel of the adjoint operator T^* has the form*

$$\psi(x,y) = \begin{cases} 0 & \text{if } 0 \le x \le y \le 1, \\ q^*(x,y) & \text{if } 0 \le y \le x \le 1 \end{cases}$$

where $q^*(x,y)$ satisfies $\int_0^y q^*(y,u)du = 0$.

Proposition 6.6. *For a non-zero eigenvalue λ with eigenfunction φ and eigenfunction ψ of the adjoint operator T^*, we have*

$$\begin{aligned}(\varphi, \mathbf{1}) &= \lambda^2, \\ \left(\mathbf{1}, \overline{\psi}\right) &= 2 \cdot \lambda^3, \\ \left(\varphi, \overline{\psi}\right) &= 2 \cdot \lambda^2 \cdot \exp(-1/(2 \cdot \lambda^2)).\end{aligned}$$

In particular,

$$\frac{(\varphi, \mathbf{1}) \cdot \left(\mathbf{1}, \overline{\psi}\right)}{\left(\varphi, \overline{\psi}\right)} = \lambda^3 \cdot \exp(1/(2 \cdot \lambda^2)).$$

Proof. In the calculations that follows we will use the relations $\operatorname{erf}(1/(\sqrt{2} \cdot \lambda)) = \sqrt{2/\pi}$, $\int_0^1 q(x)dx = \lambda$ and $\int_0^1 p^*(y)dy = 2 \cdot \lambda^2$.

First the inner product between the eigenfunction and the constant function $\mathbf{1}$:

$$\begin{aligned}(\varphi, \mathbf{1}) &= \int_{[0,1]^2} q(x)dxdy - \frac{1}{\lambda} \cdot \int_{0 \le x \le y \le 1} \int_x^y q(t)dtdxdy \\ &= \int_0^1 q(x)dx - \frac{1}{\lambda} \cdot \int_0^1 t \cdot (1-t) \cdot q(t)dt \\ &= \lambda - \lambda + \frac{\sqrt{\pi}}{\sqrt{2}} \cdot \lambda^2 \cdot \operatorname{erf}(1/(\sqrt{2} \cdot \lambda)) \\ &= \lambda^2.\end{aligned}$$

Second, the inner product between the adjoint eigenfunction and the constant function $\mathbf{1}$. We have

$$\begin{aligned}\left(\mathbf{1}, \overline{\psi}\right) &= \int_{[0,1]^2} p^*(y)dxdy - \frac{1}{\lambda} \cdot \int_{0 \le y \le x \le 1} \int_x^1 p^*(u)dudxdy \\ &= \int_0^1 p^*(y)dy - \frac{1}{\lambda} \cdot \int_0^1 \frac{u^2}{2} \cdot p^*(u)du \\ &= \int_0^1 \left(1 - \frac{y^2}{2 \cdot \lambda}\right) \cdot p^*(y)dy \\ &= \int_0^1 \left(1 - \frac{y^2}{2 \cdot \lambda}\right) \cdot \left(-2 \cdot y \cdot \exp\left(\frac{y^2}{2 \cdot \lambda^2}\right) + 2 \cdot \lambda\right) dy \\ &+ \int_0^1 \left(1 - \frac{y^2}{2 \cdot \lambda}\right) \cdot \sqrt{2 \cdot \pi} \cdot y \cdot \exp\left(\frac{y^2}{2 \cdot \lambda^2}\right) \cdot \operatorname{erf}\left(\frac{y}{\sqrt{2} \cdot \lambda}\right) dy.\end{aligned}$$

The first integral is given by $I_1 = (\lambda - 2 \cdot \lambda^2 - 2 \cdot \lambda^3) \cdot \exp(1/(2 \cdot \lambda^2)) - 1/3 + 2 \cdot (\lambda + \lambda^2 + \lambda^3)$. The second integral we solve by integration by parts letting $f' = \left(1 - \frac{y^2}{2 \cdot \lambda}\right) \cdot \sqrt{2 \cdot \pi} \cdot y \cdot \exp\left(\frac{y^2}{2 \cdot \lambda^2}\right)$ and $g = \operatorname{erf}\left(\frac{y}{\sqrt{2} \cdot \lambda}\right)$. Then $\int_0^1 f'gdy = [fg]_0^1 - \int_0^1 fg'dy$ is given by:

$$\begin{aligned}I_2 &= \left[\frac{\sqrt{\pi}}{\sqrt{2}} \cdot \lambda \cdot (2 \cdot \lambda + 2 \cdot \lambda^2 - y^2) \cdot \exp(y^2/(2 \cdot \lambda^2)) \cdot \operatorname{erf}(y/(\sqrt{2} \cdot \lambda))\right]_0^1 \\ &- \int_0^1 (2 \cdot \lambda + 2 \cdot \lambda^2 - y^2)dy.\end{aligned}$$

Combining all the terms in the sum $I_1 + I_2$ we obtain $2 \cdot \lambda^3$. The third inner product is given by

$$
\begin{aligned}
(\varphi, \overline{\psi}) &= \int_{[0,1]^2} q(x) \cdot p^*(y) dx dy \\
&- \frac{1}{\lambda} \cdot \int_{0 \le x \le y \le 1} \int_x^y q(t) dt \cdot p^*(y) dx dy - \frac{1}{\lambda} \cdot \int_{0 \le y \le x \le 1} q(x) \cdot \int_x^1 p^*(u) du dx dy \\
&= \left(\int_0^1 q(x) dx \right) \cdot \left(\int_0^1 p^*(y) dy \right) - \frac{2}{\lambda} \cdot \int_{0 \le t \le y \le 1} t \cdot q(t) \cdot p^*(y) dt dy \\
&= 2 \cdot \lambda^3 - 2 \cdot \lambda \cdot \int_0^1 \left(1 - \exp(-y^2/(2 \cdot \lambda^2))\right) \cdot p^*(y) dy \\
&= -2 \cdot \lambda^3 + 2 \cdot \lambda \cdot \int_0^1 \exp(-y^2/(2 \cdot \lambda^2)) \cdot p^*(y) dy \\
&= -2 \cdot \lambda^3 + 4 \cdot \lambda \cdot \int_0^1 \left(-y + \exp(-y^2/(2 \cdot \lambda^2)) + \sqrt{\pi}/\sqrt{2} \cdot y \cdot \operatorname{erf}(y/(\sqrt{2} \cdot \lambda)) \right) dy \\
&= 2 \cdot \lambda^2 \cdot \exp(-1/(2 \cdot \lambda^2)).
\end{aligned}
$$

□

6.2 Asymptotics

We know that the largest root λ_0 of the eigenvalue equation (6.3) is real, positive and simple, since the associated operator T is positivity improving. However, to say a bit more about the eigenvalues consider the related equation $\operatorname{erf}(z) = \sqrt{2/\pi}$.

Since the error function is an increasing function on the real axis, the equation $\operatorname{erf}(z) = \sqrt{2/\pi}$ has a unique real root. The error function is an odd function hence we know by the strong version of the little Picard theorem that the equation $\operatorname{erf}(z) = \sqrt{2/\pi}$ has infinitely many roots. Moreover, the complex roots appear in conjugate pairs. To summarize this discussion we have: The eigenvalue equation has a unique real root which is positive and is the largest root. The remaining infinitely many roots are all complex and appear in conjugate pairs.

Numerically, we can approximate the roots of the eigenvalue equation (6.3). The unique real root is $\lambda_0 = 0.7839769312\ldots$. The next four largest roots are:

$$
\begin{aligned}
\lambda_{1,2} &= 0.2141426360\ldots \pm 0.2085807022\ldots \cdot i \\
\lambda_{3,4} &= -0.1677323922\ldots \pm 0.2418627350\ldots \cdot i
\end{aligned}
$$

Furthermore, these five roots do not satisfy equation (6.10) in Proposition 6.2, so they are simple eigenvalues. From Proposition 6.6 we conclude:

Theorem 6.7. *The number of* 213*-avoiding permutations satisfies*

$$
\frac{\alpha_n(213)}{n!} = \exp\left(\frac{1}{2 \cdot \lambda_0^2}\right) \cdot \lambda_0^{n+1} + O\left(|\lambda_1|^n\right)
$$

where $\lambda_0 = 0.7839769312\ldots$ *is the unique real root of the equation* $\operatorname{erf}\left(1/(\sqrt{2} \cdot \lambda)\right) = \sqrt{2/\pi}$*, and* λ_1 *is the next largest root and its modulus is given by* $|\lambda_1| = 0.298936411\ldots$*.*

By considering the next two conjugate roots λ_1 and λ_2 we obtain an approximation for the next real term in the expansion of $\alpha_n(213)/n!$:

$$
2 \cdot 1.158597034\ldots \cdot (0.2989364111\ldots)^{n+1} \cdot \cos\left(-5.593221320\ldots + (n+1) \cdot 0.7722415374\ldots\right).
$$

7 123,231,312-Avoiding permutations

Let us consider $123, 231, 312$-avoiding permutations. Now the set S is not a singleton, but has cardinality three, that is, S is given by the set $\{123, 231, 312\}$. The function χ is given by

$$\chi(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_3 \le x_2 \le x_1, \\ 1 & \text{if } x_2 \le x_1 \le x_3, \\ 1 & \text{if } x_1 \le x_3 \le x_2, \\ 0 & \text{otherwise.} \end{cases}$$

That is, the operator T is described by

$$(Tf)(x, y) = \begin{cases} \int_x^y f(t, x) dt & \text{if } 0 \le x \le y \le 1, \\ \int_0^y f(t, x) dt + \int_x^1 f(t, x) dt & \text{if } 0 \le y \le x \le 1. \end{cases}$$

7.1 Eigenvalues and eigenfunctions

For a real number t let $\{t\}$ denote the fractional part of the real number t, that is, $\{t\} = t - \lfloor t \rfloor$. Observe that the fractional part belongs to the interval $[0, 1)$.

Lemma 7.1. *The subspace W of $L^2([0, 1]^2)$ consisting of functions of the form*

$$f(x, y) = g(\{x - y\}) \quad \textit{for } (x, y) \in [0, 1]^2 \tag{7.1}$$

is invariant under the operator T.

Proof. There are two cases to consider. First, if $x \le y$, we have

$$T(g(\{x - y\})) = \int_x^y g(\{t - x\}) dt = \int_0^{y-x} g(s) ds.$$

Second, if $y \le x$ we have

$$\begin{aligned} T(g(\{x - y\})) &= \int_0^y g(\{t - x\}) dt + \int_x^1 g(\{t - x\}) dt \\ &= \int_{1-x}^{1+y-x} g(s) ds + \int_0^{1-x} g(s) ds = \int_0^{1+y-x} g(s) ds. \end{aligned}$$

These two cases can be summarized as the integral from 0 to $\{y - x\}$. However the upper bound of the integral can be written as $1 - \{x - y\}$, that is, the operator can be expressed as

$$T(g(\{x - y\})) = \int_0^{1-\{x-y\}} g(s) ds.$$

□

Since constant function **1** belongs to W, it is enough to consider eigenvalues and eigenfunctions in the subspace W.

Proposition 7.2. *The eigenvalues of the operator T restricted to the invariant subspace W are given by $\lambda_j = (-1)^{(j-1)/2} \cdot 2/(\pi \cdot j)$ where j is an odd positive integer and the corresponding eigenfunctions are described by*

$$\varphi(x, y) = \cos(\{x - y\}/\lambda).$$

Proof. The defining relations for the eigenfunctions are

$$\lambda \cdot g(t) = \int_0^{1-t} g(s)ds. \tag{7.2}$$

Observe that $\lambda = 0$ is not an eigenvalue. Differentiating equation (7.2) twice, we obtain

$$\lambda \cdot g'(t) = -g(1-t), \tag{7.3}$$
$$\lambda \cdot g''(t) = g'(1-t) = -1/\lambda \cdot g(t). \tag{7.4}$$

This second order equation has the general solution $A \cdot \cos(t/\lambda) + B \cdot \sin(t/\lambda)$. Setting $t = 1$ in (7.2) implies that $g(1) = 0$ and then setting $t = 0$ in (7.3) implies that $g'(0) = 0$. Hence $B = 0$ and the eigenfunction is $\cos(t/\lambda)$. For this function, equation (7.2) implies that $\cos(t/\lambda) = \sin((1-t)/\lambda)$. Setting $t = 0$ we obtain that the eigenvalues satisfy $\sin(1/\lambda) = 1$. They can be parametrized as in the proposition. □

Note that the eigenvalues tend to 0 as

$$\lambda_1 > -\lambda_3 > \lambda_5 > -\lambda_7 > \cdots .$$

Proposition 7.3. *The eigenvalues λ of the operator T restricted to the subspace W are simple. That is, there is no function $h(t)$ such that $\lambda \cdot h = T(h) + g$ where $g(t) = \cos(t/\lambda)$.*

Proof. The defining relations for the eigenfunctions are

$$\lambda \cdot h(t) = \int_0^{1-t} h(s)ds + g(t). \tag{7.5}$$

Differentiating this equation twice gives

$$\lambda \cdot h'(t) = -h(1-t) + g'(t), \tag{7.6}$$
$$\lambda \cdot h''(t) = h'(1-t) + g''(t). \tag{7.7}$$

Substituting (7.6) into (7.7) and using (7.4), we obtain the second order differential equation:

$$\lambda^2 \cdot h''(t) + h(t) = 2 \cdot \lambda \cdot g''(t).$$

The general solution is

$$h(t) = -\frac{1}{\lambda^2} \cdot t \cdot \sin(t/\lambda) + C_1 \cdot \cos(t/\lambda) + C_2 \cdot \sin(t/\lambda).$$

In the following we use that $\sin(1/\lambda) = 1$ and $\cos(1/\lambda) = 0$. Observe that setting $t = 1$ in (7.5) gives $\lambda \cdot h(1) = g(1)$, which implies $C_2 = 1/\lambda^2$. Similarly, setting $t = 0$ in (7.6) gives $\lambda \cdot h'(0) = -h(1) + g'(0)$ which implies $C_2 = 1/(2 \cdot \lambda^2)$, a different value for C_2. Hence there is no such a function h satisfying the integral equation (7.5). □

Observe that the function χ satisfies the symmetry in Lemma 4.7. Hence the adjoint eigenfunction ψ is given by $J\varphi$. However, since the value of $\varphi(x,y)$ only depends on the difference $x-y$, we have that $J\varphi = \varphi$.

Proposition 7.4. *For a non-zero eigenvalue λ with eigenfunction φ and eigenfunction $\psi = \varphi$ of the adjoint operator T^*, we have*

$$\begin{aligned} (\varphi, \mathbf{1}) &= \left(\mathbf{1}, \overline{\psi}\right) = \lambda, \\ \left(\varphi, \overline{\psi}\right) &= 1/2. \end{aligned}$$

In particular,

$$\frac{(\varphi, \mathbf{1}) \cdot \left(\mathbf{1}, \overline{\psi}\right)}{\left(\varphi, \overline{\psi}\right)} = 2 \cdot \lambda^2.$$

7.2 Asymptotics

The above computations show that all eigenvalues of the operator T are simple and given explicitly. We thus obtain the following asymptotic expansion for $\left(T^{n-2}(\mathbf{1}), \mathbf{1}\right) = \alpha_n(123, 231, 312)/n!$ as an immediate consequence of Theorem 1.1, Propositions 7.2 and 7.4:

$$\frac{\alpha_n(123, 231, 312)}{n!} = 2 \cdot \sum_{\substack{j=1 \\ j \text{ odd}}}^{2k+1} \lambda_j^n + O\left(|\lambda_{2k+3}|^n\right).$$

However, when we let k tend to infinity, observe that the right hand side converges. Hence we obtain the exact expression

$$\frac{\alpha_n(123, 231, 312)}{n!} = 2 \cdot \sum_{\substack{j \geq 1 \\ j \text{ odd}}} (-1)^{(j-1)/2 \cdot n} \cdot \left(\frac{2}{\pi \cdot j}\right)^n$$

for $n \geq 2$. However this is the expression for $E_{n-1}/(n-1)!$ where E_n denotes the nth Euler number; see equation (1.8). Thus we conclude that

Theorem 7.5. *The number of 123, 231, 312-avoiding permutations in $\mathfrak{S}_n$ is given by $n \cdot E_{n-1}$ for $n \geq 2$.*

This result is due to Kitaev and Mansour [13]. Note that the special form of the eigenfunctions and the invariant subspace are reflected in their proof that this class of permutations is invariant under the shift $\pi_1 \pi_2 \cdots \pi_n \longmapsto (\pi_1 + 1)(\pi_2 + 1) \cdots (\pi_n + 1)$, where the addition is modulo n.

8 Concluding remarks

It is straightforward to design a Viennot "pyramid" to compute the number α_n of S-avoiding permutations. For the original Viennot triangle, see [20, 21]. Let the entry $\alpha_n^{i_1,\ldots,i_m}$ of the pyramid be the number of permutations in the symmetric group on n elements, avoiding the set S and ending with the m entries $i_1, \ldots, i_m$. Then the entry $\alpha_n^{i_1,\ldots,i_m}$ is a sum of entries of the form $\alpha_{n-1}^{j,i_1,\ldots,i_{m-1}}$. This sum is a discrete analogue of the operator T. How far does this analogue between the discrete model and the continuous one go? Does the function $f_n = T^{n-m}(\mathbf{1})$ approximate the n-th level of the pyramid? More exactly, how well does the integer $\alpha_n^{i_1,\ldots,i_m}$ compare with $n! \cdot f_n(i_1/n, \ldots, i_m/n)$?

In the case of descent pattern avoidance, can one prove that T restricted to the invariant subspace V is compact? We have done so in the case of 123-avoiding permutations.

Consider the graph $G_\emptyset$ of overlapping permutations on the vertex set $\mathfrak{S}_m$. What is the smallest number of edges one has to remove in order to make the graph not strongly connected? Clearly one can remove m edges disconnecting the vertex $12\cdots m$. Is m the right answer? This would suggest that one can remove $m-1$ edges without making the directed graph disconnected.

A more general enumeration problem is as follows. For a function w on $\mathfrak{S}_{m+1}$ define the weight of a permutation $\pi = (\pi_1 \pi_2 \cdots \pi_n)$ by the product

$$\mathrm{wt}(\pi) = \prod_{k=1}^{n-m} w(\Pi(\pi_k, \ldots, \pi_{k+m})).$$

Now what can be said about the values and asymptotics of the sum

$$\alpha_n(w) = \sum_{\pi \in \mathfrak{S}_n} \mathrm{wt}(\pi)$$

as n tends to infinity. For instance, if w is a positive function we know by the result of Kreĭn and Rutman that

$$\alpha_n(w) \sim c \cdot \lambda^n \cdot n!.$$

An operator is called a *Volterra operator* if its spectral radius is 0. What can be said about Volterra operators of the form (1.1)? More specifically, what can be said about the asymptotic of $(T^n(\mathbf{1}), \mathbf{1})$? See Examples 1.13, 1.14 and 3.9 for different behaviors.

Are there examples of forbidden sets S such that the associated operator T has non-simple non-zero eigenvalues. This situation can also be analyzed using Theorem 22 in [6, Section VII.3]. When the operator T satisfies a symmetry condition as in Lemma 4.7 or in Lemma 4.8, we conjecture that all the eigenvalues have index 1.

In the case of 213-avoiding permutations are all the eigenvalues simple? In other words, are there a common non-zero root to the two equations (6.3) and (6.10)?

In this paper our object is to understand consecutive patterns avoidance. Generalized pattern avoidance was introduced by Babson and Steingrímsson [2]. Is there an analytic approach to obtain asymptotics for these classes of permutations? Lastly, it would be daring to ask for an analytic proof of the former Stanley–Wilf conjecture, recently proved in [16].

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