Patterns and their generalizations

Sergey Kitaev

Reykjavík University

Occurrences of the “classical" pattern 1-3-2 in 13524:

1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4

Occurrences of the “classical" pattern 1-3-2 in 13524:

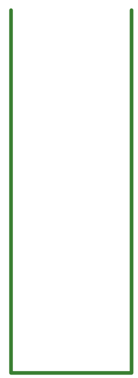
1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4

A generalized pattern is a pattern that allows the requirement that two adjacent letters in the pattern must be adjacent in the permutation.

Pattern	Occurrences in 13542
1-3-2	1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2
1-32	1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2
[1-3-2	1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2
132	1 3 5 4 2

Sorting with a stack

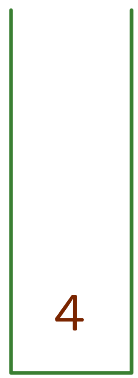
4 1 6 3 2 5



Sorting with a stack

1 6 3 2 5

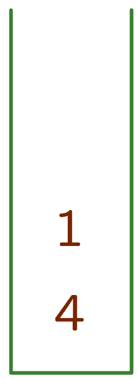
Numbers on stack
must increase
from top



Sorting with a stack

6 3 2 5

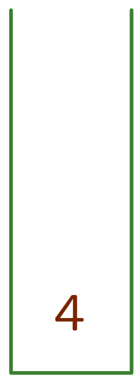
Numbers on stack
must increase
from top



Sorting with a stack

1 6 3 2 5

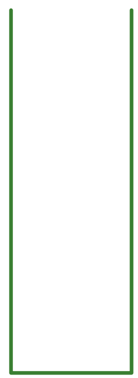
Numbers on stack
must increase
from top



Sorting with a stack

1 4

6 3 2 5



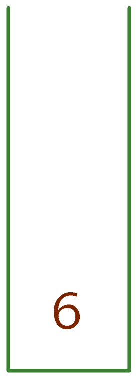
Numbers on stack
must increase
from top

Sorting with a stack

1 4

3 2 5

Numbers on stack
must increase
from top

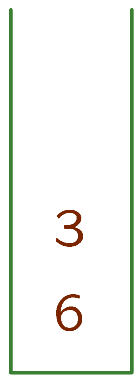


Sorting with a stack

1 4

2 5

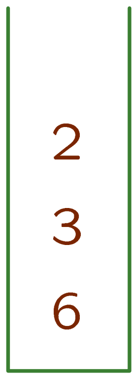
Numbers on stack
must increase
from top



Sorting with a stack

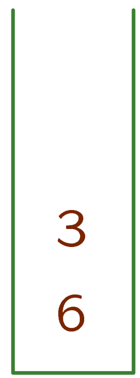
1 4 5

Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 5

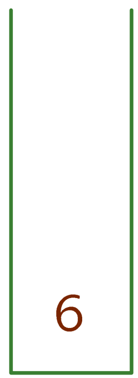


Numbers on stack
must increase
from top

Sorting with a stack

1 4 2 3 5

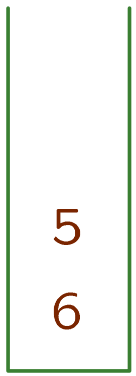
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3

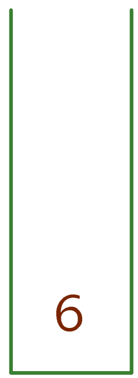
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3 5

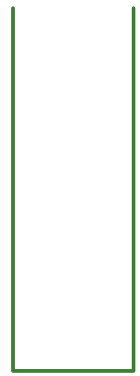
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3 5 6

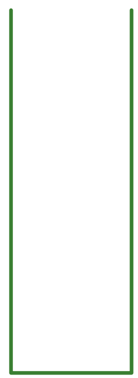
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3 5 6

Numbers on stack
must increase
from top

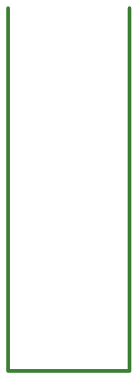


Sorting with a stack

1 4 2 3 5 6

4 1 6 3 2 5

Numbers on stack
must increase
from top

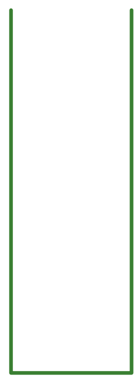


Sorting with a stack

1 4 2 3 5 6

4 1 6 3 2 5

Numbers on stack
must increase
from top

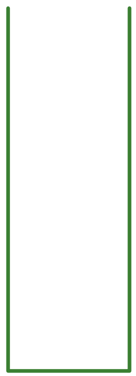


Sorting with a stack

1 4 2 3 5 6

4 1 6 3 2 5
2 3 1

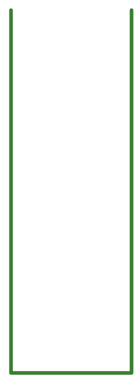
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3 5 6

2 3 1



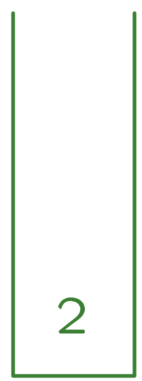
Numbers on stack
must increase
from top

Sorting with a stack

1 4 2 3 5 6

3 1

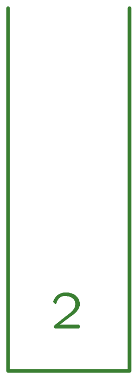
Numbers on stack
must increase
from top



Theorem. [Knuth] A permutation is stack-sortable if and only if it avoids 2-3-1.

1 4 2 3 5 6

3 1



Numbers on stack
must increase
from top

Theorem. [Knuth] A permutation is stack-sortable if and only if it avoids 2-3-1.

The number of such permutations is the n-th Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Theorem. [Knuth] A permutation is stack-sortable if and only if it avoids 2-3-1.

The number of such permutations is the n-th Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

They have the generating function

$$C(x) = \sum_{n=0}^{\infty} C_n x^n \;=\; \frac{1-\sqrt{1-4x}}{2x}$$

1969 D. Knuth: The Art of computer programming, vol. I

1985 R. Simion, F. Schmidt: Restricted permutations, European J. Combin. **6**, no. 4, 383–406.

1969 D. Knuth: The Art of computer programming, vol. I

1985 R. Simion, F. Schmidt: Restricted permutations, European J. Combin. **6**, no. 4, 383–406.

1992 Present: Explosive growth (several hundreds papers appeared)

2002 H. Wilf: The patterns of permutations, DM **257**, 575–583.

2003 S. Kitaev, T. Mansour: Survey of certain pattern problems

2004 M. Bóna: Combinatorics of Permutations, xiv+383 pp.

1969 D. Knuth: The Art of computer programming, vol. I

1985 R. Simion, F. Schmidt: Restricted permutations, European J. Combin. **6**, no. 4, 383–406.

2002 H. Wilf: The patterns of permutations, DM **257**, 575–583.

2003 S. Kitaev, T. Mansour: Survey of certain pattern problems

2004 M. Bóna: Combinatorics of Permutations, xiv+383 pp.

2004 M. Atkinson: Permutation Patterns Home page http://www.cs.otago.ac.nz/staffpriv/mike/PPPages/PPhome.html

Permutation Patterns:
Classical patterns: Knuth, 1969
Generalized patterns: Babson and Steingrímsson, 2000
Partially ordered patterns: Kitaev, 2001

Permutation Patterns:
Classical patterns: Knuth, 1969 Generalized patterns: Babson and Steingrímsson, 2000 Partially ordered patterns: Kitaev, 2001

Word Patterns:
Classical word patterns: Burstein, 1998 Generalized word patterns: Burstein and Mansour, 2002 Partially ordered word patterns: Kitaev and Mansour, 2003

Permutation Patterns:
Classical patterns: Knuth, 1969 Generalized patterns: Babson and Steingrímsson, 2000 Partially ordered patterns: Kitaev, 2001

Word Patterns:
Classical word patterns: Burstein, 1998 Generalized word patterns: Burstein and Mansour, 2002 Partially ordered word patterns: Kitaev and Mansour, 2003

Patterns in matrices:	Kitaev, Mansour and Vella, 2003
Patterns in n-dimensional objects:	Kitaev and Robbins, 2004

Patterns in even (odd) permutations:	Simion and Schmidt, 1985
Colored patterns in colored permutations:	Mansour, 2001
Signed patterns in signed permutations:	Mansour and West, 2002
Patterns with respect to parity:	Kitaev and Remmel, 2005

Let R be a set of patterns.

Let $S_n(p)$ be the set of all permutations in S_n which avoid the pattern p.

Then $S_n(R) = \bigcap_{p \in R} S_n(p)$.

An extreme case is $S_n(\emptyset) = S_n$ for all $n \geqslant 1$.

$N_n(R)$ is the number of elements of $S_n(R)$.

Questions about $S_n(R)$:

1. Formula for $N_n(R)$;

2. Generating function for $N_n(R)$, that is, $f_R(x) = \sum_i N_i(R)x^i$;

3. Relations to other combinatorial structures;

4. Is $S_n(R) = S_n(R')$ for all n?

In this case R and R' are said to be from the same Wilf class.

5. P-recursiveness of $N_n(R)$;

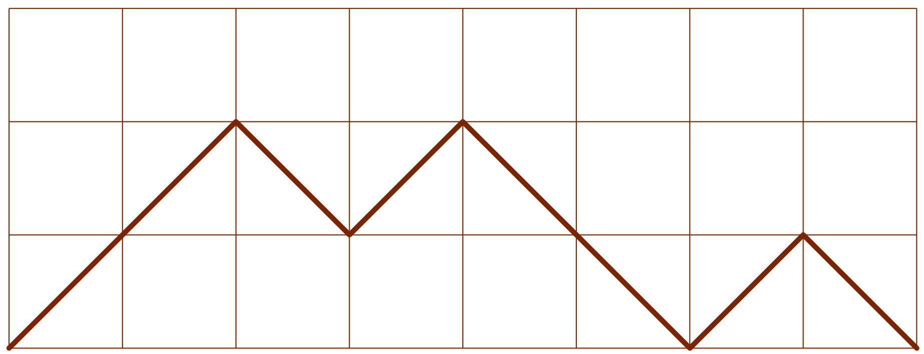
A function $f : \mathbf{N} \to \mathbf{C}$ is called P-recursive if there exist polynomials P_0, $P_1, \ldots, P_k \in \mathbf{C}[n]$, so that for all $n \in \mathbf{N}$

$$P_k(n)f(n+k) + P_{k-1}(n)f(n+k-1) + \cdots + P_0(n)f(n) = 0.$$

Theorem. [Knuth] For all $n \geqslant 1$, and for all classical patterns $p \in S_3$, $N_n(p)$ is given by the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Theorem. [Knuth] For all $n \geqslant 1$, and for all classical patterns $p \in S_3$, $N_n(p)$ is given by the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Dyck paths



pattern p	formula for $N_n(p)$	P-recursive
1-2-3-4 4-3-2-1	$(\star)$ Gessel	yes Zeilberger
1-3-4-2 2-4-3-1 3-1-2-4 4-2-1-3	$(\star\star)$ Bóna	yes Bóna
1-3-2-4 4-2-3-1	open	open

$$(\star) = 2\sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2 \frac{3k^2+2k+1-n-2kn}{(k+1)^2(k+2)(n-k+1)}$$

$$(\star\star) = \tfrac{7n^2-3n-2}{2}\cdot(-1)^{n-1} + 3\sum_{i=2}^{n} 2^{i+1}\cdot\frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2}(-1)^{n-i}$$

Theorem. [Regev] For all n, N_n(1-2-$\cdots$-k) asymptotically equals

$$\lambda_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}.$$

Here

$$\lambda_k = {\gamma_k}^2 \int_{x_1\geqslant} \int_{x_2\geqslant} \cdots \int_{\geqslant x_k} [D(x_1,x_2,\ldots,x_k)\cdot e^{-(k/2)x^2}]^2 \, dx_1 dx_2 \ldots dx_k,$$

where $D(x_1,x_2,\ldots,x_k) = \prod_{i<j}(x_i - x_j)$ and $\gamma_k = (1/\sqrt{2\pi})^{k-1} \cdot k^{k^2/2}$.

Theorem. [Regev] For all n, N_n(1-2-$\cdots$-k) asymptotically equals

$$\lambda_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}.$$

Here

$$\lambda_k = {\gamma_k}^2 \int_{x_1 \geqslant} \int_{x_2 \geqslant} \cdots \int_{\geqslant x_k} [D(x_1, x_2, \ldots, x_k) \cdot e^{-(k/2)x^2}]^2 \, dx_1 dx_2 \ldots dx_k,$$

where $D(x_1, x_2, \ldots, x_k) = \prod_{i<j} (x_i - x_j)$ and $\gamma_k = (1/\sqrt{2\pi})^{k-1} \cdot k^{k^2/2}$.

Theorem. [Marcus and Tardos] For every permutation pattern p, there is a constant $c = c(p) < \infty$ such that for all n $N_n(p) < c^n$. [This was the famous Stanley-Wilf Conjecture]

Multi-avoidance of classical patterns

For avoiding a pair of classical 3-patterns, we have 3 Wilf classes with $N_n(p)$ given by 2^{n-1}, $\binom{n}{2}+1$ and 0 (Simion and Schmidt).

Multi-avoidance of classical patterns

For avoiding a pair of classical 3-patterns, we have 3 Wilf classes with $N_n(p)$ given by 2^{n-1}, $\binom{n}{2}+1$ and 0 (Simion and Schmidt).

restrictions	formula	author
1-2-3,4-3-2-1	0	West
1-2-3,3-4-2-1	$\binom{n}{4}+2\binom{n}{3}+n$	West
1-3-2,4-3-2-1	$\binom{n}{4}+\binom{n+1}{4}+\binom{n}{2}+1$	West
1-2-3,4-2-3-1	$\binom{n}{5}+2\binom{n}{4}+\binom{n}{3}+\binom{n}{2}+1$	West
1-2-3,3-2-4-1	$3\cdot 2^{n-1}-\binom{n+1}{2}-1$	West
1-2-3,3-4-1-2	$2^{n+1}-\binom{n+1}{3}-2n-1$	Stanley
1-3-2,4-2-3-1	$1+(n-1)2^{n-2}$	Guibert
1-3-2,3-4-2-1	$1+(n-1)2^{n-2}$	West
1-3-2,3-2-1-4	GF: $\frac{(1-x)^3}{1-4x+5x^2-3x^3}$	West

The following were given by West:

restrictions	restrictions	formula
1-2-3,2-1-4-3 1-2-3,2-4-1-3 1-3-2,2-3-1-4 1-3-2,2-3-4-1 3-1-2,2-3-1-4 1-3-2,3-4-1-2 3-1-2,1-4-3-2	3-1-2,1-3-4-2 3-1-2,3-2-4-1 3-1-2,3-2-1-4 1-2-3,3-2-1-4 3-1-2,4-3-2-1 3-1-2,3-4-2-1 1-3-2,3-2-4-1	F_{2n} (Fibonacci number)
3-1-4-2,2-4-1-3	4-1-3-2,4-2-3-1	GF: $\frac{1-x-\sqrt{1-6x+x^2}}{2x}$

The following were given by West:

restrictions	restrictions	formula
1-2-3,2-1-4-3 1-2-3,2-4-1-3 1-3-2,2-3-1-4 1-3-2,2-3-4-1 3-1-2,2-3-1-4 1-3-2,3-4-1-2 3-1-2,1-4-3-2	3-1-2,1-3-4-2 3-1-2,3-2-4-1 3-1-2,3-2-1-4 1-2-3,3-2-1-4 3-1-2,4-3-2-1 3-1-2,3-4-2-1 1-3-2,3-2-4-1	F_{2n} (Fibonacci number)
3-1-4-2,2-4-1-3	4-1-3-2,4-2-3-1	GF: $\frac{1-x-\sqrt{1-6x+x^2}}{2x}$

Theorem. [Simion and Schmidt] For every $n \geqslant 1$,

$$N_n(1\text{-}2\text{-}3, 1\text{-}3\text{-}2, 2\text{-}1\text{-}3) = F_{n+1},$$

where F_n is the n-th Fibonacci number.

Generalized patterns

The following were given by Claesson

Generalized patterns	Related combinatorial objects
2-31	Dyck paths (Catalan numbers)
1-23	Partitions (Bell numbers)
1-23, 12-3	Non-overlapping partitions (Bessel numbers)
1-23, 1-32	Involutions
1-23, 13-2	Motzkin paths

Claesson and Mansour provided complete solution for the number of permutations avoiding a pair of type x-yz or xy-z. Out of $\binom{12}{2} = 66$ pairs there are 21 symmetry classes and 10 Wilf classes.

The following were given by Kitaev:

Restrictions	**Formula**
123, 321, 132, 213	$2C_k$, if $n = 2k+1$ $C_k + C_{k-1}$, if $n = 2k$ (C_k – Catalan number)
123, 132, 213	$\binom{n}{\lfloor n/2 \rfloor}$
123, 132, 231	n
132, 213, 312	$1 + 2^{n-2}$
123, 132, 312	Recursive Formula
123, 321, 231	$(n-1)!! + (n-2)!!$
123, 231, 312	EGF: $1 + x(\sec(x) + \tan(x))$ (with Mansour)
132, 213	Recursive Formula (with Mansour)
123, 321	$2E_n$, where E_n is the n-th Euler number
132, 231	2^{n-1}

Theorem. [Elizalde and Noy, 2001] Let m and a be positive integers with $a \leqslant m$, let $\sigma = 12 \cdots a\tau(a+1) \in \mathcal{S}_{m+2}$, where τ is any permutation of $\{a+2, a+3, \ldots, m+2\}$, and let

$$P(u,z) = \sum_{\pi} u^{\sigma(\pi)} \frac{z^{|\pi|}}{|\pi|!}.$$

Then $P(u,z) = 1/w(u,z)$, where w is the solution of

$$w^{a+1} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!}w' = 0$$

with $w(0) = 1$, $w'(0) = -1$ and $w^{(k)} = 0$ for $2 \leqslant k \leqslant a$. In particular, the distribution does not depend on τ.

Using an inclusion-exclusion argument we get this:

Theorem. [Goulden and Jackson, 1983] Let

$$A_k(x) = A_0 + A_1 x + \frac{A_2}{2!}x^2 + \cdots$$

be the EGF for the number of permutations avoiding the pattern $123\cdots k$. Then

$$A_k(x) = \frac{1}{\displaystyle\sum_{i\geqslant 0} \frac{x^{ki}}{(ki)!} - \sum_{i\geqslant 0} \frac{x^{ki+1}}{(ki+1)!}}.$$

Theorem. [2002] Let k and a be positive integers with $a < k$, let $p = 12 \cdots a\tau(a+1) \in \mathcal{S}_{k+1}$, where τ is any permutation of the elements $\{a+2, a+3, \ldots, k+1\}$, and let $A_{k,a}(x)$ be the EGF for the number of permutations that avoid p. Let

$$F_{k,a}(x) = \sum_{i \geqslant 1} \frac{(-1)^{i+1} x^{ki+1}}{(ki+1)!} \prod_{j=2}^{i} \binom{jk-a}{k-a}.$$

Then

$$A_{k,a}(x) = 1/(1 - x + F_{k,a}(x)).$$

Theorem. [2002] Let k and a be positive integers with $a < k$, let $p = 12\cdots a\tau(a+1) \in \mathcal{S}_{k+1}$, where τ is any permutation of the elements $\{a+2, a+3, \ldots, k+1\}$, and let $A_{k,a}(x)$ be the EGF for the number of permutations that avoid p. Let

$$F_{k,a}(x) = \sum_{i \geqslant 1} \frac{(-1)^{i+1} x^{ki+1}}{(ki+1)!} \prod_{j=2}^{i} \binom{jk-a}{k-a}.$$

Then

$$A_{k,a}(x) = 1/(1 - x + F_{k,a}(x)).$$

Example. If $k = 2$ and $a = 1$ ($p = 132$), then

$$F_{2,1}(x) = \sum_{i \geqslant 1} \frac{(-1)^{i+1} x^{ki+1}}{i!(k!)^i(ki+1)} = x - \int_0^x e^{-t^2/2} dt.$$

Let $p = \sigma\text{-}k$, where σ is an arbitrary segmented pattern on the elements $1, 2, \ldots, k-1$. So the last letter of p is greater than any other letter. Let $A(x)$ (resp. $B(x)$) be the EGF for the number of permutations that avoid σ (resp. p).

Theorem. [2002] We have $B(x) = e^{F(x,A(y))}$, where

$$F(x, A(y)) = \int_0^x A(y)\ dy.$$

Example. Let $p = 1\text{-}2$. Here $\sigma = 1$, whence $A(x) = 1$ since $A_n = 0$ for all $n \geqslant 1$. So

$$B(x) = e^{F(x,1)} = e^{x}.$$

Example. Let $p = 1\text{-}2$. Here $\sigma = 1$, whence $A(x) = 1$ since $A_n = 0$ for all $n \geqslant 1$. So

$$B(x) = e^{F(x,1)} = e^x.$$

Example. Suppose $p = 12\text{-}3$. Here $\sigma = 12$, whence $A(x) = e^x$, since there is only one permutation that avoids σ. So

$$B(x) = e^{F(x,e^y)} = e^{e^x - 1}.$$

It is known [Claesson, 2001] that the number of n-permutations that avoid p is the n-th Bell number whose EGF is $B(x)$.

A descent in a permutation $\pi = a_1 a_2 \cdots a_n$ is an i such that $a_i > a_{i+1}$. The number of descents is a well-known statistic for a permutation π.

Two descents i and j overlap if $j = i + 1$.

We define a new statistic, namely the maximum number of non-overlapping descents in a permutation.

Permutation	4 3 1 2	2 1 4 3	4 3 2 1
Maximal number of non-over. descents	one	two	two

Theorem. [2002] Let p be a segmented pattern. Let $A(x)$ be the EGF for the number of permutations that avoid p. Let

$$D(x,y) = \sum_{\pi} y^{N(\pi)} \frac{x^{|\pi|}}{|\pi|!}$$

where $N(\pi)$ is the maximum number of non-overlapping occurrences of p in π. Then

$$D(x,y) = \frac{A(x)}{1 - y((x-1)A(x) + 1)}.$$

Example. For descents, $A(x) = e^x$, hence the distribution of the maximum number of non-overlapping descents is

$$D(x, y) = \frac{e^x}{1 - y(1 + (x - 1)e^x)}.$$

Example. For descents, $A(x) = e^x$, hence the distribution of the maximum number of non-overlapping descents is

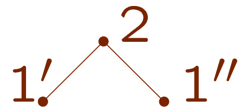
$$D(x,y) = \frac{e^x}{1 - y(1 + (x-1)e^x)}.$$

Example. If we consider the maximum number of non-overlapping occurrences of the pattern 132 then the distribution of these numbers is

$$D(x,y) = \frac{1}{1 - yx + (y-1)\int_0^x e^{-t^2/2}\, dt}.$$

A partially ordered pattern (POP) is a generalized pattern where some of the letters can be incomparable. (2002)

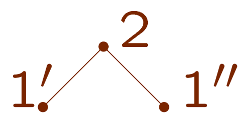
A partially ordered pattern (POP) is a generalized pattern where some of the letters can be incomparable. (2002)



The permutation 3142 has two occurrences of the POP $1'$-2-$1''$:

$$\underline{3}\ \overline{1}\ \underline{\overline{4}}\ \underline{\overline{2}}$$

A partially ordered pattern (POP) is a generalized pattern where some of the letters can be incomparable. (2002)



The permutation 3142 has two occurrences of the POP $1'$-2-$1''$:

$$\underline{3}\;\overline{1}\;\overline{\underline{4}}\;\overline{\underline{2}}$$

The number of permutations that avoid $1'$-2-$1''$ is 2^{n-1}:

Write $\pi = \pi_1 1 \pi_2$

Then π_1 must be decreasing and π_2 must be increasing.

avoiding a POP = avoiding a set of generalized patterns

avoiding a POP = avoiding a set of generalized patterns

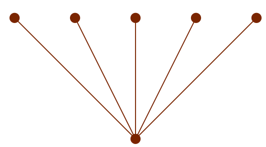
The number of n-permutations avoiding 123, 132 and 213 is $\binom{n}{\lfloor n/2 \rfloor}$; a rather complicated argument was used to prove this.

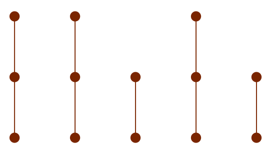
Considering $11'2$ gives a two-lines proof of the same result.

avoiding a POP = avoiding a set of generalized patterns

The number of n-permutations avoiding 123, 132 and 213 is $\binom{n}{\lfloor n/2 \rfloor}$; a rather complicated argument was used to prove this.

Considering $11'2$ gives a two-lines proof of the same result.





There are so many things to discover about patterns ...

There are so many things to discover about patterns ...

What are you doing tonight?

Fourth annual conference on

Permutation patterns

Reykjavík University

June 12-16, 2006

The End