Introduction to partially ordered patterns

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Occurrences of the "classical" pattern 132 in 13524:

1 3 5 2 4,   1 3 5 2 4,   1 3 5 2 4,   1 3 5 2 4

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A generalized pattern is a pattern that allows the requirement that two adjacent letters in the pattern must be adjacent in the permutation.

Pattern	Occurrences in 13542
1-3-2	1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2
1-32	1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2
[1-3-2	1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2
132	1 3 5 4 2

segmental patterns = segmented patterns = subword patterns = patterns without internal dashes = patterns with no dashes = patterns without hyphens = patterns without gaps segmental patterns = segmented patterns = subword patterns = patterns without internal dashes = patterns with no dashes = patterns without hyphens = patterns without gaps

There is a number of results on the distribution of several classes of segmented patterns. In particular we know the exponential generating functions (EGF) for the number of permutations that avoid these classes (Elizalde and Noy).

Theorem [Elizalde and Noy, 2001] Let m and a be positive integers with $a \leqslant m$, let $\sigma = 12 \cdots a\tau(a+1) \in \mathcal{S}_{m+2}$, where τ is any permutation of $\{a+2, a+3, \ldots, m+2\}$, and let

$$P(u,z) = \sum_{\pi} u^{\sigma(\pi)} \frac{z^{|\pi|}}{|\pi|!}.$$

Then $P(u,z) = 1/w(u,z)$, where w is the solution of

$$w^{a+1} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!} w' = 0$$

with $w(0) = 1$, $w'(0) = -1$ and $w^{(k)} = 0$ for $2 \leqslant k \leqslant a$. In particular, the distribution does not depend on τ.

Using an inclusion-exclusion argument we get this:

Theorem [Goulden and Jackson, 1983] Let

$$A_k(x) = A_0 + A_1 x + \frac{A_2}{2!}x^2 + \cdots$$

be the EGF for the number of permutations avoiding the pattern $123\cdots k$. Then

$$A_k(x) = \frac{1}{\sum_{i \geqslant 0} \frac{x^{ki}}{(ki)!} - \sum_{i \geqslant 0} \frac{x^{ki+1}}{(ki+1)!}}.$$

Theorem [2002] Let k and a be positive integers with $a < k$, let $p = 12\cdots a\tau(a+1) \in \mathcal{S}_{k+1}$, where τ is any permutation of the elements $\{a+2, a+3, \ldots, k+1\}$, and let $A_{k,a}(x)$ be the EGF for the number of permutations that avoid p. Let

$$F_{k,a}(x) = \sum_{i \geqslant 1} \frac{(-1)^{i+1} x^{ki+1}}{(ki+1)!} \prod_{j=2}^{i} \binom{jk-a}{k-a}.$$

Then

$$A_{k,a}(x) = 1/(1 - x + F_{k,a}(x)).$$

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Then

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Example If $k = 2$ and $a = 1$ ($p = 132$), then

$$F_{2,1}(x) = \sum_{i \geqslant 1} \frac{(-1)^{i+1} x^{ki+1}}{i!(k!)^i(ki+1)} = x - \int_0^x e^{-t^2/2} dt.$$

Let $p = \sigma\text{-}k$, where σ is an arbitrary segmented pattern on the elements $1, 2, \ldots, k-1$. So the last letter of p is greater than any other letter. Let $A(x)$ (resp. $B(x)$) be the EGF for the number of permutations that avoid σ (resp. p).

Theorem [2002] We have $B(x) = e^{F(x,A(y))}$, where

$$F(x, A(y)) = \int_0^x A(y)\ dy.$$

Example Let $p = 1\text{-}2$. Here $\sigma = 1$, whence $A(x) = 1$ since $A_n = 0$ for all $n \geqslant 1$. So

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Example Suppose $p = 12\text{-}3$. Here $\sigma = 12$, whence $A(x) = e^x$, since there is only one permutation that avoids σ. So

$$B(x) = e^{F(x,e^y)} = e^{e^x - 1}.$$

It is known [Claesson, 2001] that the number of n-permutations that avoid p is the n-th Bell number whose EGF is $B(x)$.

A partially ordered pattern (POP) is a generalized pattern where some of the letters can be incomparable. (2002)

Example The permutation 3142 has two occurrences of the pattern $1'$-2-$1''$:

$$\underline{3}\;\overline{1}\;\overline{\underline{4}}\;\overline{\underline{2}}$$

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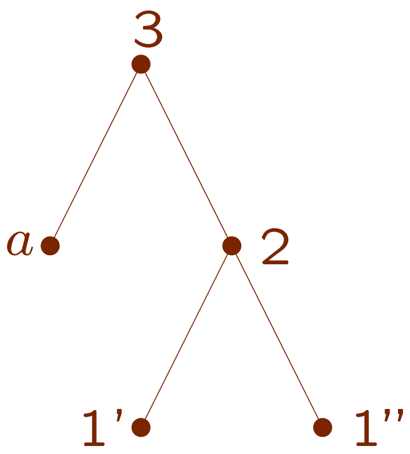
The number of permutations that avoid $1'$-2-$1''$ is 2^{n-1}:

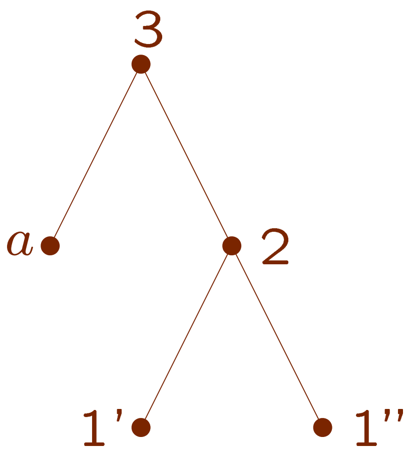
Write $\pi = \pi_1 1 \pi_2$

Then π_1 must be decreasing and π_2 must be increasing.



A pattern σ =3-a21''-1'

Occurrences of σ: $\overline{8}\ \underline{7}\ \underline{\overline{1}}\ \underline{\overline{4}}\ \underline{\overline{2}}\ \underline{\overline{3}}\ 5\ 6$



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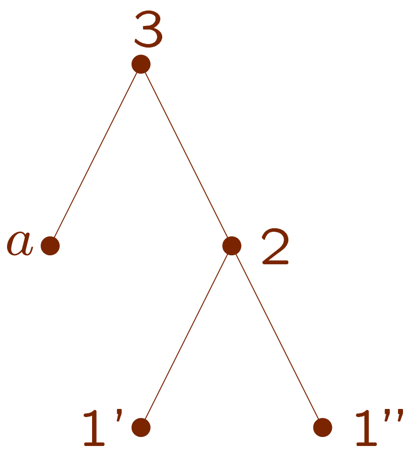

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The number of n-permutations avoiding 123, 132 and 213 is $\binom{n}{\lfloor n/2 \rfloor}$; a rather complicated argument was used to prove this.

Considering $11'2$ gives a two-lines proof of the same result.

A pattern σ is co-unimodal if $\sigma = \sigma_1 > \sigma_2 > \cdots > \sigma_j < \cdots < \sigma_k$ for some $2 \leqslant j \leqslant k$.

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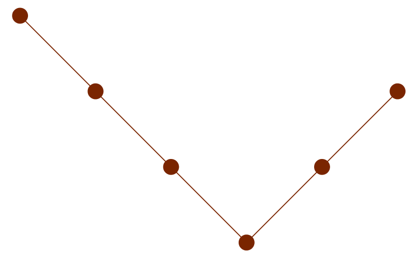
Corresponding poset: 

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The inversion index, inv(π), of a permutation π is the number of ordered pairs (i, j) such that $i < j$ and $\pi_i > \pi_j$.

Example inv(41352)=3+1+1=5

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The major index, maj(π), is the sum of all i such that $\pi_i > \pi_{i+1}$.

Example maj(43152)=1+2+4=7

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The major index, $\mathrm{maj}(\pi)$, is the sum of all i such that $\pi_i > \pi_{i+1}$.

Suppose σ is a segmental POP and

$$\mathrm{place}_\sigma(\pi) = \{i \mid \pi \text{ has an occurrence of } \sigma \text{ starting at } \pi_i\}.$$

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Theorem [Björner and Wachs, 1991] We have

$$\sum_{\pi \in S_n} t^{\mathsf{maj}_\sigma(\pi^{-1})} q^{\mathsf{maj}(\pi)} = \sum_{\pi \in S_n} t^{\mathsf{maj}_\sigma(\pi^{-1})} q^{\mathsf{inv}(\pi)}.$$

Definition Suppose $\{\sigma_0, \sigma_1, \ldots, \sigma_k\}$ is a set of generalized patterns with no dashes and $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ where each letter of σ_i is incomparable with any letter of σ_j whenever $i \neq j$. We call such POPs multi-patterns.

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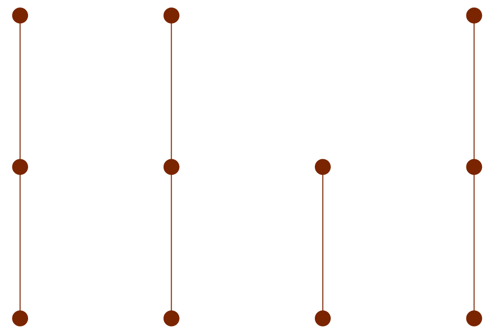
Corresponding poset: 

Example The permutation 53142 has two occurrences of the pattern 21-2′1′:

$$\underline{5}\;\underline{\overline{3}}\;\overline{1}\;\underline{\overline{4}}\;\underline{\overline{2}}$$

Theorem [2002] There are $(n-2)2^{n-1}+2$ permutations in $\mathcal{S}_n$ that avoid the pattern $p = 12\text{-}1'2'$.

Theorem [2002] The EGF for the number of permutations that avoid the pattern $p = 122'1'$ is

$$\frac{1}{2} + \frac{1}{4}\tan x(1 + e^{2x} + 2e^{x}\sin x) + \frac{1}{2}e^{x}\cos x.$$

The following theorem is the basis for calculating the number of permutations that avoid a multi-pattern.

Theorem [2002] Let $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ be a multi-pattern and let $A_i(x)$ be the number of permutations that avoid σ_i. Then the EGF $B(x)$ for the number of permutations that avoid p is

$$B(x) = \sum_{i=1}^{k} A_i(x) \prod_{j=1}^{i-1} ((x-1)A_j(x) + 1).$$

Corollary Let $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ be a multi-pattern, where $|\sigma_i| = 2$ for all i. That is, σ_i is either 12 or 21. Then the EGF for the number of permutations that avoid p is given by

$$B(x) = \frac{1 - (1 + (x-1)e^x)^k}{1 - x}.$$

A descent in a permutation $\pi = a_1 a_2 \cdots a_n$ is an i such that $a_i > a_{i+1}$. The number of descents is a well-known statistic for a permutation π.

Two descents i and j overlap if $j = i + 1$.

We define a new statistic, namely the maximum number of non-overlapping descents in a permutation.

Permutation	4 3 1 2	2 1 4 3	4 3 2 1
Maximal number of non-over. descents	one	two	two

We find the distribution of this new statistic by using the results for multi-patterns.

Theorem [2002] Let p be a segmented pattern. Let $A(x)$ be the EGF for the number of permutations that avoid p. Let

$$D(x,y) = \sum_{\pi} y^{N(\pi)} \frac{x^{|\pi|}}{|\pi|!}$$

where $N(\pi)$ is the maximum number of non-overlapping occurrences of p in π. Then

$$D(x,y) = \frac{A(x)}{1 - y((x-1)A(x) + 1)}.$$

Example For descents, $A(x) = e^x$, hence the distribution of the maximum number of non-overlapping descents is

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Example If we consider the maximum number of non-overlapping occurrences of the pattern 132 then the distribution of these numbers is

$$D(x, y) = \frac{1}{1 - yx + (y - 1)\int_0^x e^{-t^2/2}\, dt}.$$

The result of the previous theorem, as well as the results on the multi-patterns, were extended to the case of words.

Theorem [Kitaev and Mansour, 2003] Let τ be a segmented pattern and $A_\tau(x;k) = \sum_{n\geqslant 0} a_\tau(n;k)x^n$ is the generating function for the numbers $a_\tau(n;k)$ of words in $[k]^n$ avoiding the pattern τ. Then for all $k \geqslant 1$,

$$\sum_{n\geqslant 0}\sum_{\sigma\in[k]^n} y^{N_\tau(\sigma)}x^n = \frac{A_\tau(x;k)}{1-y((kx-1)A_\tau(x;k)+1)},$$

where $N_\tau(\sigma)$ is the maximum number of non-overlapping occurrences of τ in σ.

Example For descents $A_{12}(x;k) = (1-x)^{-k}$, hence the distribution of the maximum number of non-overlapping descents is

$$\sum_{n\geqslant 0} \sum_{\sigma\in[k]^n} y^{N_{12}(\sigma)} x^n = \frac{1}{(1-x)^k + y(1-kx-(1-x)^k)}.$$

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Example The distribution of the maximum number of non-overlapping occurrences of the pattern 122 is given by the formula:

$$\sum_{n\geqslant 0}\sum_{\sigma\in[k]^n} y^{N_{122}(\sigma)}x^n = \frac{x}{(1-x^2)^k + x - 1 + y(1-kx^2-(1-x^2)^k)},$$

since

$$A_{122}(x;k) = \frac{x}{(1-x^2)^k - (1-x)}.$$

q-analogues of n and $n!$ are

$$[n]_q = \frac{1-q^n}{1-q} = q^0 + \cdots + q^{n-1} \quad \text{and} \quad [n]_q! = [n]_q \cdots [1]_q$$

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The following theorem is a q-analogue of a theorem above.

Theorem [Mendes, 2004] Let p be a segmented pattern. Then

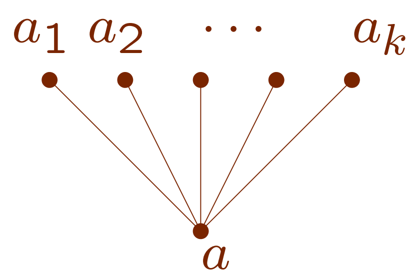
$$\sum_{\pi} y^{N(\pi)} q^{inv(\pi)} \frac{x^{|\pi|}}{[|\pi|]_q!} = \frac{A_q(x)}{1 - y((x-1)A_q(x) + 1)}$$

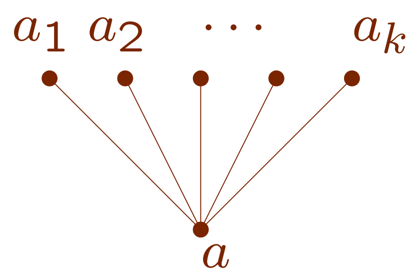
where $N(\pi)$ is the maximum number of non-overlapping occurrences of p, and

$$A_q(x) = \sum_{\pi \text{ avoids } p} q^{inv(\pi)} \frac{x^{|\pi|}}{[|\pi|]_q!}$$

Patterns built on the letters $a, a_1, \ldots, a_k$ with the only relations $a < a_i$ for all i.

Corresponding poset:



Corresponding poset: 

[**Distribution of** $a_1 \cdots a_k a a_{k+1} \cdots a_{k+\ell}$] Let

$$P := P(x,y) = \sum_{n \geqslant 0} \sum_{\pi \in_n} y^{e(\pi)} x^n / n!$$

be the BGF for permutations where $e(\pi)$ is the number of occurrences of the segmental POP $a_1 \cdots a_k a a_{k+1} \cdots a_{k+\ell}$ in π. Then P is the solution to

$$\frac{\partial P}{\partial x} = y\left(P - \frac{1-x^k}{1-x}\right)\left(P - \frac{1-x^\ell}{1-x}\right) + \frac{2-x^k-x^\ell}{1-x}P - \frac{1-x^k-x^\ell+x^{k+\ell}}{(1-x)^2}$$

with the initial condition $P(0,y) = 1$.

Peaks in a permutation: 63427519

Corollary [2005] The BGF for the peaks distribution in permutations is

$$1 - \frac{1}{y} + \frac{1}{y}\sqrt{y-1} \cdot \tan\left(x\sqrt{y-1} + \arctan\left(\frac{1}{\sqrt{y-1}}\right)\right).$$

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Proposition [Claesson, 2001] The involutions in S_n are in one-to-one correspondence with permutations in S_n avoiding 1-23 and 1-32 (that is, avoiding a-a_1a_2 in the notation above).

Generalization [2005] The permutations in S_n having cycles of length at most k are in one-to-one correspondence with permutations in S_n that avoid a-$a_1\cdots a_k$. Thus, the EGF for the number of permutations avoiding a-$a_1\cdots a_k$ is given by $\exp(\sum_{i=1}^{k} x^i/i)$.

$P_k = \sum_{n=0}^{k-1} \frac{1}{n+1}\binom{2n}{n} x^n$. So P_k is the k initial terms in the expansion of the generating function $\frac{1-\sqrt{1-4x}}{2x}$ for the Catalan numbers.

[**Distribution of** $a_1 \cdots a_k a a_{k+1} \cdots a_{k+\ell}$ **on** S_n(2-1-3)] Let

$$P := P(x,y) = \sum_{n\geqslant 0} \sum_{\pi \in S_n(\text{2-1-3})} y^{e(\pi)} x^n$$

be the BGF for 2-1-3-avoiding permutations where $e(\pi)$ is the number of occurrences of $a_1 \cdots a_k a a_{k+1} \cdots a_{k+\ell}$ in π. Then

$$P = \frac{1 - x(1-y)(P_k + P_\ell) - \sqrt{(x(1-y)(P_k+P_\ell)-1)^2 - 4xy(x(y-1)P_kP_\ell + 1)}}{2xy}.$$

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For certain choices of k, ℓ, and y in the theorem above one gets Catalan numbers, Pell numbers, and the triangle of Narayna numbers.

Patterns	Related objects
no restrictions	Increasing binary trees
1-2-3	Dyck paths
1-23	Partitions
1-23, 12-3	Non-overlapping partitions
1-23, 1-32	Involutions
1-23, 13-2	Motzkin paths
132, [21	Increasing rooted trimmed trees
$aa_1 \cdots a_k$	Permutations with cycles of length at most k
12'21'	Lattice walks in N, S, E, W
11'22', 22'11'	Certain walks on the x-axis

The End! ;-)