Introduction to partially ordered patterns

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Occurrences of the “classical” pattern 132 in 13524:

1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4
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1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4

A generalized pattern is a pattern that allows the requirement that two adjacent letters in the pattern must be adjacent in the permutation.

<table>
<thead>
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<th>Pattern</th>
<th>Occurrences in 13542</th>
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<tr>
<td>1-3-2</td>
<td>1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2</td>
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<td>1-32</td>
<td>1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2</td>
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<tr>
<td>[1-3-2]</td>
<td>1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2</td>
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<tr>
<td>132</td>
<td>1 3 5 4 2</td>
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segmental patterns \(=\) segmented patterns \(=\) subword patterns \\
\(=\) patterns without internal dashes \(=\) patterns with no dashes \(=\)

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segmental patterns = segmented patterns = subword patterns
= patterns without internal dashes = patterns with no dashes =
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There is a number of results on the distribution of several classes
of segmented patterns. In particular we know the exponential
generating functions (EGF) for the number of permutations that
avoid these classes (Elizalde and Noy).
**Theorem** [Elizalde and Noy, 2001] Let $m$ and $a$ be positive integers with $a \leq m$, let $\sigma = 12 \cdots a \tau(a + 1) \in S_{m+2}$, where $\tau$ is any permutation of $\{a + 2, a + 3, \ldots, m + 2\}$, and let

$$P(u, z) = \sum_{\pi} u^{\sigma(\pi)} z^{|\pi|}/|\pi|!.$$  

Then $P(u, z) = 1/w(u, z)$, where $w$ is the solution of

$$w^{a+1} + (1 - u) z^{m-a+1}/(m-a+1)! w' = 0$$

with $w(0) = 1$, $w'(0) = -1$ and $w^{(k)} = 0$ for $2 \leq k \leq a$. In particular, the distribution does not depend on $\tau$. 
Using an inclusion-exclusion argument we get this:

**Theorem** [Goulden and Jackson, 1983] Let

\[ A_k(x) = A_0 + A_1x + \frac{A_2}{2!}x^2 + \cdots \]

be the EGF for the number of permutations avoiding the pattern 123\( \cdots k \). Then

\[
A_k(x) = \frac{1}{\sum_{i \geq 0} \frac{x^{ki}}{(ki)!} - \sum_{i \geq 0} \frac{x^{ki+1}}{(ki + 1)!}}.
\]
**Theorem [2002]** Let $k$ and $a$ be positive integers with $a < k$, let $p = 12 \cdots a \tau(a + 1) \in S_{k+1}$, where $\tau$ is any permutation of the elements $\{a + 2, a + 3, \ldots, k + 1\}$, and let $A_{k,a}(x)$ be the EGF for the number of permutations that avoid $p$. Let

$$F_{k,a}(x) = \sum_{i \geq 1} \frac{(-1)^{i+1} x^{ki+1}}{(ki + 1)!} \prod_{j=2}^{i} \left( \binom{jk - a}{k - a} \right).$$

Then

$$A_{k,a}(x) = \frac{1}{(1 - x + F_{k,a}(x))}.$$
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Then

$$A_{k,a}(x) = \frac{1}{1 - x + F_{k,a}(x)}.$$ 

**Example** If $k = 2$ and $a = 1$ ($p = 132$), then

$$F_{2,1}(x) = \sum_{i \geq 1} \frac{(-1)^i x^{ki+1}}{i!(k!)^i(ki + 1)} = x - \int_{0}^{x} e^{-t^2/2} dt.$$
Let $p = \sigma-k$, where $\sigma$ is an arbitrary segmented pattern on the elements $1, 2, \ldots, k-1$. So the last letter of $p$ is greater than any other letter. Let $A(x)$ (resp. $B(x)$) be the EGF for the number of permutations that avoid $\sigma$ (resp. $p$).

**Theorem [2002]** We have $B(x) = e^{F(x,A(y))}$, where

$$F(x, A(y)) = \int_0^x A(y) \, dy.$$
Example Let $p = 1-2$. Here $\sigma = 1$, whence $A(x) = 1$ since $A_n = 0$ for all $n \geq 1$. So

$$B(x) = e^{F(x,1)} = e^x.$$
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Example Suppose $p = 12-3$. Here $\sigma = 12$, whence $A(x) = e^x$, since there is only one permutation that avoids $\sigma$. So

$$B(x) = e^{F(x,e^y)} = e^{e^x-1}.$$ 

It is known [Claesson, 2001] that the number of $n$-permutations that avoid $p$ is the $n$-th Bell number whose EGF is $B(x)$. 
A partially ordered pattern (POP) is a generalized pattern where some of the letters can be incomparable. (2002)

**Example** The permutation 3142 has two occurrences of the pattern $1'-2-1''$:

$$3 \overline{1} \overline{4} \overline{2}$$
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Example The permutation 3142 has two occurrences of the pattern $1'-2-1''$:

$$3 \ 1 \ 4 \ 2$$

The number of permutations that avoid $1'-2-1''$ is $2^{n-1}$:

Write $\pi = \pi_1 1 \pi_2$

Then $\pi_1$ must be decreasing and $\pi_2$ must be increasing.
A pattern $\sigma = 3-a21''-1'$

Occurrences of $\sigma$: 8 7 1 4 2 3 5 6
A pattern $\sigma = 3-a21''-1'$

Occurrences of $\sigma$: $\bar{8} \ 7 \ 1 \ 4 \ 2 \ 3 \ 5 \ 6$

avoiding a POP = avoiding a set of generalized patterns
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Occurrences of $\sigma$: $\bar{8} \ 7 \ 1 \ 4 \ 2 \ 3 \ 5 \ 6$

avoiding a POP = avoiding a set of generalized patterns

The number of $n$-permutations avoiding $123$, $132$ and $213$ is $\binom{n}{\lfloor n/2 \rfloor}$; a rather complicated argument was used to prove this.

Considering $11'2$ gives a two-lines proof of the same result.
A pattern $\sigma$ is co-unimodal if $\sigma = \sigma_1 > \sigma_2 > \cdots > \sigma_j < \cdots < \sigma_k$ for some $2 \leq j \leq k$. 
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Corresponding poset:
A pattern $\sigma$ is co-unimodal if $\sigma = \sigma_1 > \sigma_2 > \cdots > \sigma_j < \cdots < \sigma_k$ for some $2 \leq j \leq k$.

The inversion index, $\text{inv}(\pi)$, of a permutation $\pi$ is the number of ordered pairs $(i, j)$ such that $i < j$ and $\pi_i > \pi_j$.

**Example** \[ \text{inv}(41352) = 3 + 1 + 1 = 5 \]
A pattern $\sigma$ is co-unimodal if $\sigma = \sigma_1 > \sigma_2 > \cdots > \sigma_j < \cdots < \sigma_k$ for some $2 \leq j \leq k$.

The inversion index, $\text{inv}(\pi)$, of a permutation $\pi$ is the number of ordered pairs $(i, j)$ such that $i < j$ and $\pi_i > \pi_j$.

The major index, $\text{maj}(\pi)$, is the sum of all $i$ such that $\pi_i > \pi_{i+1}$.

**Example** \[ \text{maj}(43152) = 1 + 2 + 4 = 7 \]
A pattern $\sigma$ is co-unimodal if $\sigma = \sigma_1 > \sigma_2 > \cdots > \sigma_j < \cdots < \sigma_k$ for some $2 \leq j \leq k$.

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The major index, $\text{maj}(\pi)$, is the sum of all $i$ such that $\pi_i > \pi_{i+1}$.

Suppose $\sigma$ is a segmental POP and

$$\text{place}_\sigma(\pi) = \{i \mid \pi \text{ has an occurrence of } \sigma \text{ starting at } \pi_i\}.$$ 

Let $\text{maj}_\sigma(\pi)$ be the sum of the elements of $\text{place}_\sigma(\pi)$. 
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Let $\text{maj}_\sigma(\pi)$ be the sum of the elements of $\text{place}_\sigma(\pi)$.

**Theorem [Björner and Wachs, 1991]** We have

$$\sum_{\pi \in S_n} t^{\text{maj}_\sigma(\pi^{-1})} q^{\text{maj}(\pi)} = \sum_{\pi \in S_n} t^{\text{maj}_\sigma(\pi^{-1})} q^{\text{inv}(\pi)}.$$
**Definition** Suppose \( \{\sigma_0, \sigma_1, \ldots, \sigma_k\} \) is a set of generalized patterns with no dashes and \( p = \sigma_1-\sigma_2-\cdots-\sigma_k \) where each letter of \( \sigma_i \) is incomparable with any letter of \( \sigma_j \) whenever \( i \neq j \). We call such POPs **multi-patterns**.
**Definition** Suppose \( \{\sigma_0, \sigma_1, \ldots, \sigma_k\} \) is a set of generalized patterns with no dashes and \( p = \sigma_1-\sigma_2-\cdots-\sigma_k \) where each letter of \( \sigma_i \) is incomparable with any letter of \( \sigma_j \) whenever \( i \neq j \). We call such POPs multi-patterns.

Corresponding poset:

**Example** The permutation 53142 has two occurrences of the pattern 21-2′1′:

5 3 1 4 2

53142
**Theorem [2002]** There are \((n - 2)2^{n-1} + 2\) permutations in \(S_n\) that avoid the pattern \(p = 12-1'2'\).

**Theorem [2002]** The EGF for the number of permutations that avoid the pattern \(p = 122'1'\) is

\[
\frac{1}{2} + \frac{1}{4} \tan x (1 + e^{2x} + 2e^x \sin x) + \frac{1}{2} e^x \cos x.
\]
The following theorem is the basis for calculating the number of permutations that avoid a multi-pattern.

**Theorem [2002]** Let $p = \sigma_1 \sigma_2 \cdots \sigma_k$ be a multi-pattern and let $A_i(x)$ be the number of permutations that avoid $\sigma_i$. Then the EGF $B(x)$ for the number of permutations that avoid $p$ is

$$B(x) = \sum_{i=1}^{k} A_i(x) \prod_{j=1}^{i-1} ((x - 1)A_j(x) + 1).$$
Corollary Let $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ be a multi-pattern, where $|\sigma_i| = 2$ for all $i$. That is, $\sigma_i$ is either 12 or 21. Then the EGF for the number of permutations that avoid $p$ is given by

$$B(x) = \frac{1 - (1 + (x - 1)e^x)^k}{1 - x}.$$
A descent in a permutation $\pi = a_1a_2\cdots a_n$ is an $i$ such that $a_i > a_{i+1}$. The number of descents is a well-known statistic for a permutation $\pi$.

Two descents $i$ and $j$ overlap if $j = i + 1$.

We define a new statistic, namely the maximum number of non-overlapping descents in a permutation.

<table>
<thead>
<tr>
<th>Permutation</th>
<th>$4\ 3\ 1\ 2$</th>
<th>$2\ 1\ 4\ 3$</th>
<th>$4\ 3\ 2\ 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximal number of non-over. descents</td>
<td>one</td>
<td>two</td>
<td>two</td>
</tr>
</tbody>
</table>

We find the distribution of this new statistic by using the results for multi-patterns.
**Theorem [2002]** Let $p$ be a segmented pattern. Let $A(x)$ be the EGF for the number of permutations that avoid $p$. Let

$$D(x, y) = \sum_{\pi} y^{N(\pi)} \frac{x^{\left|\pi\right|}}{\left|\pi\right|!}$$

where $N(\pi)$ is the maximum number of non-overlapping occurrences of $p$ in $\pi$. Then

$$D(x, y) = \frac{A(x)}{1 - y((x - 1)A(x) + 1)}.$$
Example For descents, \( A(x) = e^x \), hence the distribution of the maximum number of non-overlapping descents is

\[
D(x, y) = \frac{e^x}{1 - y(1 + (x - 1)e^x)}.
\]
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\[
D(x, y) = \frac{e^x}{1 - y(1 + (x - 1)e^x)}.
\]

Example If we consider the maximum number of non-overlapping occurrences of the pattern 132 then the distribution of these numbers is

\[
D(x, y) = \frac{1}{1 - yx + (y - 1) \int_0^x e^{-t^2/2} \, dt}.
\]
The result of the previous theorem, as well as the results on the multi-patterns, were extended to the case of words.

**Theorem** [Kitaev and Mansour, 2003] Let $\tau$ be a segmented pattern and $A_\tau(x; k) = \sum_{n \geq 0} a_\tau(n; k)x^n$ is the generating function for the numbers $a_\tau(n; k)$ of words in $[k]^n$ avoiding the pattern $\tau$. Then for all $k \geq 1$,

$$
\sum_{n \geq 0} \sum_{\sigma \in [k]^n} y^{N_\tau(\sigma)} x^n = \frac{A_\tau(x; k)}{1 - y((kx - 1)A_\tau(x; k) + 1)},
$$

where $N_\tau(\sigma)$ is the maximum number of non-overlapping occurrences of $\tau$ in $\sigma$. 
Example For descents $A_{12}(x; k) = (1 - x)^{-k}$, hence the distribution of the maximum number of non-overlapping descents is

$$
\sum_{n \geq 0} \sum_{\sigma \in [k]^n} y^{N_{12}(\sigma)} x^n = \frac{1}{(1 - x)^k + y(1 - kx - (1 - x)^k)}.
$$
**Example** For descents \( A_{12}(x; k) = (1 - x)^{-k} \), hence the distribution of the maximum number of non-overlapping descents is

\[
\sum_{n \geq 0} \sum_{\sigma \in [k]^n} y^{N_{12}(\sigma)} x^n = \frac{1}{(1 - x)^k + y(1 - kx - (1 - x)^k)}.
\]

**Example** The distribution of the maximum number of non-overlapping occurrences of the pattern \( 122 \) is given by the formula:

\[
\sum_{n \geq 0} \sum_{\sigma \in [k]^n} y^{N_{122}(\sigma)} x^n = \frac{x}{(1 - x^2)^k + x - 1 + y(1 - kx^2 - (1 - x^2)^k)},
\]

since

\[
A_{122}(x; k) = \frac{x}{(1 - x^2)^k - (1 - x)}.
\]
$q$-analogues of $n$ and $n!$ are

$$[n]_q = \frac{1-q^n}{1-q} = q^0 + \cdots + q^{n-1} \quad \text{and} \quad [n]_q! = [n]_q \cdots [1]_q$$

$inv(\pi) - \text{number of inversions in a permutations } \pi = \pi_1 \cdots \pi_n.$
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$inv(\pi) -$ number of inversions in a permutations $\pi = \pi_1 \cdots \pi_n$.

The following theorem is a $q$-analogue of a theorem above.

**Theorem** [Mendes, 2004] Let $p$ be a segmented pattern. Then

$$\sum_\pi y^{N(\pi)} q^{inv(\pi)} \frac{x^{|\pi|}}{[|\pi|]_q!} = \frac{A_q(x)}{1 - y((x - 1)A_q(x) + 1)}$$

where $N(\pi)$ is the maximum number of non-overlapping occurrences of $p$, and

$$A_q(x) = \sum_{\pi \text{ avoids } p} q^{inv(\pi)} \frac{x^{|\pi|}}{[|\pi|]_q!}$$
Patterns built on the letters $a, a_1, \ldots, a_k$ with the only relations $a < a_i$ for all $i$.

Corresponding poset:
[Distribution of $a_1 \cdots a_k aa_{k+1} \cdots a_{k+\ell}$] Let

$$P := P(x, y) = \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} y^{e(\pi)} x^n / n!$$

be the BGF for permutations where $e(\pi)$ is the number of occurrences of the segmental POP $a_1 \cdots a_k aa_{k+1} \cdots a_{k+\ell}$ in $\pi$. Then $P$ is the solution to

$$\frac{\partial P}{\partial x} = y \left( P - \frac{1 - x^k}{1 - x} \right) \left( P - \frac{1 - x^\ell}{1 - x} \right) + \frac{2 - x^k - x^\ell}{1 - x} P - \frac{1 - x^k - x^\ell + x^{k+\ell}}{(1 - x)^2}$$

with the initial condition $P(0, y) = 1$. 
Peaks in a permutation: 63427519

**Corollary [2005]** The BGF for the peaks distribution in permutations is

\[ 1 - \frac{1}{y} + \frac{1}{y} \sqrt{y-1} \cdot \tan \left( x \sqrt{y-1} + \arctan \left( \frac{1}{\sqrt{y-1}} \right) \right). \]
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**Proposition [Claesson, 2001]** The involutions in \( S_n \) are in one-to-one correspondence with permutations in \( S_n \) avoiding 1-23 and 1-32 (that is, avoiding \( a-a_1 a_2 \) in the notation above).

**Generalization [2005]** The permutations in \( S_n \) having cycles of length at most \( k \) are in one-to-one correspondence with permutations in \( S_n \) that avoid \( a-a_1 \cdots a_k \). Thus, the EGF for the number of permutations avoiding \( a-a_1 \cdots a_k \) is given by \( \exp \left( \sum_{i=1}^{k} x^i / i \right) \).
$$P_k = \sum_{n=0}^{k-1} \frac{1}{n+1} \binom{2n}{n} x^n. \text{ So } P_k \text{ is the } k \text{ initial terms in the expansion of the generating function } \frac{1-\sqrt{1-4x}}{2x} \text{ for the Catalan numbers.}$$

**[Distribution of } a_1 \cdots a_k aa_{k+1} \cdots a_{k+\ell} \text{ on } S_n(2\cdots1\cdots3)\text{]** Let

$$P := P(x, y) = \sum_{n \geq 0} \sum_{\pi \in S_n(2\cdots1\cdots3)} y^{e(\pi)} x^n$$

be the BGF for 2-1-3-avoiding permutations where $e(\pi)$ is the number of occurrences of $a_1 \cdots a_k aa_{k+1} \cdots a_{k+\ell}$ in $\pi$. Then

$$P = \frac{1 - x(1 - y)(P_k + P_\ell) - \sqrt{(x(1 - y)(P_k + P_\ell) - 1)^2 - 4xy(x(y - 1)P_k P_\ell + 1)}}{2xy}.$$
\[ P_k = \sum_{n=0}^{k-1} \frac{1}{n+1} \binom{2n}{n} x^n. \] So \( P_k \) is the \( k \) initial terms in the expansion of the generating function \( \frac{1-\sqrt{1-4x}}{2x} \) for the Catalan numbers.

**[Distribution of \( a_1 \cdots a_k a a_k+1 \cdots a_k+\ell \) on \( S_n(2\text{-}1\text{-}3) \)]** Let

\[ P := P(x, y) = \sum_{n \geq 0} \sum_{\pi \in S_n(2\text{-}1\text{-}3)} y^{e(\pi)} x^n \]

be the BGF for 2-1-3-avoiding permutations where \( e(\pi) \) is the number of occurrences of \( a_1 \cdots a_k a a_k+1 \cdots a_k+\ell \) in \( \pi \). Then

\[ P = \frac{1 - x(1 - y)(P_k + P_\ell) - \sqrt{(x(1 - y)(P_k + P_\ell) - 1)^2 - 4xy(x(y - 1)P_k P_\ell + 1)}}{2xy}. \]

For certain choices of \( k, \ell, \) and \( y \) in the theorem above one gets Catalan numbers, Pell numbers, and the triangle of Narayana numbers.
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<td>1-23, 1-32</td>
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