Uniquely $k$-determined permutations

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Joint work with

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Permutation 253641 contains two occurrences of the consecutive pattern \texttt{132}: \texttt{253164} and \texttt{253641}
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Basic question: How many of $n$-permutations contain $k$ occurrences of a given consecutive pattern? In particular, how many permutations avoid a given pattern.

More general question: Find joint distribution of patterns from a given set of consecutive patterns.
Approaches to study consecutive patterns:

1. Direct combinatorial arguments;

2. Method of inclusion-exclusion;

3. Tree representations of permutations;

4. Spectral theory of integral operators on $L^2([0, 1]^k)$;

\[ \cdots \]

\[ n. \] Considering the graph of patterns overlaps.
1. Direct combinatorial argument: 
\[ A_n(123, 321, 132) = (n - 1)!! + (n - 2)!! \] (SK)

2. Method of inclusion-exclusion: Generating function for \( A_n(12543) \)

\[
1 - x + \sum_{i \geq 1} \frac{(-1)^{i+1} x^{4i+1}}{(4i + 1)!} \prod_{j=2}^{i} \left( \frac{4j - 2}{2} \right)^{-1}
\] (SK)

3. Tree representations of permutations: Bivariate GF for distribution of 132 is 

\[
\left(1 - \int_0^z \exp((u - 1)t^2/2)\,dt\right)^{-1}
\] (Elizalde, Noy)

4. Spectral theory of integral operators on \( L^2([0, 1]^k) \):

\[
\frac{A_n(213)}{n!} = \lambda_0^{n+1} \exp \left( \frac{1}{2\lambda_0^2} \right) + O \left( \left( \frac{1}{\sqrt{2}} \right)^n \right)
\]

where \( \lambda_0 = 0.7839769312 \ldots \) (Ehrenborg, SK, Perry)
The de Bruijn graphs for the alphabet $A = \{0, 1\}$ and $n = 2, 3$:
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Graph of patterns overlaps: permutations instead of binary words.
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**Observation:** For any $n$-permutation, there is a (unique) path in $\mathcal{P}_k$ of length $n - k + 1$ corresponding to it (assuming $n \geq k$).

**Example:** $k = 3$; to $13542$ there corresponds the path $123 \rightarrow 132 \rightarrow 321$ in $\mathcal{P}_3$. 
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Here a verbal description of our approach comes ...
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**Uniquely $k$-determined permutations** are those that can be reconstructed uniquely from the path corresponding to them.

**Example:** 12…$n$ is uniquely $k$-determined for any $k \geq 2$; no $n$-permutation, $n \geq 2$, is uniquely 1-determined; each $n$-permutation is uniquely $n$-determined.
A few questions to ask:

1. Given a permutation, is it uniquely $k$-determined?

2. How many uniquely $k$-determined permutations are there? Is the generating function for the number of these permutations rational?

3. Suppose $k$ is fixed; does there exist a finite set of prohibitions describing the uniquely $k$-determined permutations?

4. What is the structure of the uniquely $k$-determined permutations?
First criterion on unique $k$-determinability

Suppose $\pi = \pi_1\pi_2\ldots\pi_n$ is a permutation and $i < j$. The distance $d_\pi(\pi_i, \pi_j) = d_\pi(\pi_j, \pi_i)$ between $\pi_i$ and $\pi_j$ is $j - i$. For example, $d_{253164}(3, 6) = d_{253164}(6, 3) = 2$. 
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**Theorem.** An $n$-permutation $\pi$ is uniquely $k$-determined if and only if for each $1 \leq x < n$, the distance $d_\pi(x, x + 1) \leq k - 1$. 
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Coming back to 13542 we see why it isn’t uniquely 3-determined: $d_{13542}(2, 3) = 3 = k$. 

Second criterion on unique $k$-determinability

$V = \{1, 2, \ldots, n\}$ and $M$ is a subset of $V$. A path-scheme $P(n, M)$ is a graph $G = (V, E)$, where the edge set $E$ is $\{(x, y) \mid |x - y| \in M\}$. For example, $P(6, \{2, 4\})$ is
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![Path-scheme diagram](image)

Let $G_{k,n} = P(n, \{1, 2, \ldots, k - 1\})$, where $k \leq n$. Clearly, $G_{k,n}$ is a subgraph of $G_{n,n}$. 

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![Graph](image)

Let $G_{k,n} = P(n, \{1, 2, \ldots, k-1\})$, where $k \leq n$. Clearly, $G_{k,n}$ is a subgraph of $G_{n,n}$.

**Theorem.** Let $\Phi$ be a map that sends a uniquely $k$-determined $n$-permutation $\pi$ to the directed hamiltonian path in $G_{n,n}$ corresponding to $\pi^{-1}$. $\Phi$ is a bijection between the set of all uniquely $k$-determined $n$-permutations and the set of all directed hamiltonian paths in $G_{k,n}$. 
A quick checking of whether an $n$-permutation $\pi$ is uniquely $k$-determined or not: consider the $n-1$ differences of the adjacent elements in $\pi^{-1}$ to see whether at least one of those differences exceeds $k-1$ or not.

The number of uniquely $k$-determined $n$-permutations, $n \geq 1$:

<table>
<thead>
<tr>
<th>$k \geq 2$</th>
<th>$1, 2, 2, 2, 2, 2, 2, 2, 2, \ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k \geq 3$</td>
<td>$1, 2, 6, 12, 20, 34, 56, 88, 136, \ldots$</td>
</tr>
<tr>
<td>$k \geq 4$</td>
<td>$1, 2, 6, 24, 72, 180, 428, 1042, 2512, \ldots$</td>
</tr>
<tr>
<td>$k \geq 5$</td>
<td>$1, 2, 6, 24, 120, 480, 1632, 5124, 15860, \ldots$</td>
</tr>
<tr>
<td>$k \geq 6$</td>
<td>$1, 2, 6, 24, 120, 720, 3600, 15600, 61872, \ldots$</td>
</tr>
<tr>
<td>$k \geq 7$</td>
<td>$1, 2, 6, 24, 120, 720, 5040, 30240, 159840, \ldots$</td>
</tr>
<tr>
<td>$k \geq 8$</td>
<td>$1, 2, 6, 24, 120, 720, 5040, 40320, 282240, \ldots$</td>
</tr>
</tbody>
</table>

The sequence corresponding to the case $k = 3$ appears in Sloane, where we learn that the inverses to the uniquely 3-determined permutations are called key permutations.
Theorem. We have, for the number $A_{k,n}$ of uniquely $k$-determined $n$-permutations,

$$2((k - 1)!)^\left\lfloor \frac{n}{k} \right\rfloor < A_{k,n} < 2(2(k - 1))^n.$$
Prohibitions giving uniquely $k$-determined permutations

Let $|X|$ be the number of elements in $X$.

The set of uniquely $k$-determined $n$-permutations can be described by prohibiting patterns $xX(x + 1)$ and $(x + 1)Xx$, where $X$ is a permutation on $\{1, 2, \ldots, |X| + 2\} - \{x, x + 1\}$, $|X| \geq k - 1$, and $1 \leq x < n$.

We collect all such patterns in $\mathcal{L}_{k,n}$; also, let $\mathcal{L}_k = \cup_{n \geq 0} \mathcal{L}_{k,n}$. 
Prohibitions giving uniquely $k$-determined permutations

A prohibited pattern $X = aYb$ from $\mathcal{L}_k$, where $a$ and $b$ are some consecutive elements, is called irreducible if the patterns of $Yb$ and $aY$ are not prohibited, that is, the patterns of $Yb$ and $aY$ are uniquely $k$-determined permutations.

Let $\mathcal{L}_k$ consists only of irreducible prohibited patterns.
Prohibitions giving uniquely $k$-determined permutations

**Theorem.** Suppose $k$ is fixed. The number of (irreducible) prohibitions in $\mathcal{L}_k$ is finite. Moreover, the longest prohibited patterns in $\mathcal{L}_k$ are of length $2k - 1$.

Here it comes a verbal description of how we use the theorem above and the graph of patterns overlaps $\mathcal{P}_{2k-1}$ to apply the transfer matrix method ...

**Theorem.** The generating function $A_k(x) = \sum_{n \geq 0} A_{k,n} x^n$ for the number of uniquely $k$-determined permutations is rational.
An $n$-permutation is crucial if it is uniquely $k$-determined, but adjoining any letter to the right of it, and thus creating an $(n+1)$-permutation, leads to a non-uniquely $k$-determined permutation.
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**Theorem.** There are no crucial permutations.
The case $k = 3$

Suppose $w'$ denotes the complement to an $n$-permutation $w$. All uniquely 3-determined 4-permutations:

\[
\begin{align*}
& a = 1234 \quad a' = 4321 \\
& b = 1324 \quad b' = 4231 \\
& c = 1243 \quad c' = 4312 \\
& d = 3421 \quad d' = 2134 \\
& e = 1423 \quad e' = 4132 \\
& f = 3241 \quad f' = 2314
\end{align*}
\]
The case \( k = 3 \)

\[
A_3(x) = \sum_{n \geq 0} A_{3,n} x^n = \frac{1 - 2x + 2x^2 + x^3 - x^5 + x^6}{(1 - x - x^3)(1 - x)^2}.
\]
Open problems

Any $n$-permutation is uniquely $n$-determined, whereas for $n \geq 2$ no $n$-permutation is uniquely 1-determined. Moreover, for any $n \geq 2$ there are exactly two uniquely 2-determined permutations, namely the monotone permutations.

**Index $IR(\pi)$ of reconstructibility** is the minimal integer $k$ such that the permutation $\pi$ is uniquely $k$-determined.

**Problem 1.** Describe the distribution of $IR(\pi)$ among all $n$-permutations.
Open problems

Problem 2. Study the set of uniquely $k$-determined permutations in the case when a set of nodes is removed from $\mathcal{P}_k$, that is, when some of patterns of length $k$ are prohibited.
Open problems

An $n$-permutation $\pi$ is $m$-$k$-determined, $m, k \geq 1$, if there are exactly $m$ (different) $n$-permutations having the same path in $\mathcal{P}_k$ as $\pi$ has. In particular, the uniquely $k$-determined permutations correspond to the case $m = 1$.

**Problem 3.** Find the number of $m$-$k$-determined $n$-permutations.
Open problems

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Problem 3. Find the number of $m$-$k$-determined $n$-permutations.

Problem 3 is directly related to finding the number of linear extensions of a poset. Indeed, to any path $w$ in $\mathcal{P}_k$ there naturally corresponds a poset $\mathcal{W}$. In particular, any factor of length $k$ in $w$ consists of comparable to each other elements in $\mathcal{W}$. If $k = 3$ and $w = 134265$ (7-3-determined) then $\mathcal{W}$ is the following poset:

```
       6
      /|
     / \
    4   5
   / \
  3   2
 / 
1   2
```
Open problems

Recall that $\mathcal{L}_k$ is a set of irreducible prohibited patterns giving all uniquely $k$-determined permutations.

**Problem 4.** Describe the structure of $\mathcal{L}_k$. Is there a nice way to generate $\mathcal{L}_k$? How many elements does $\mathcal{L}_k$ have?
Thank you for your attention!

Questions?